Fluctuations in a diffusive medium with gain

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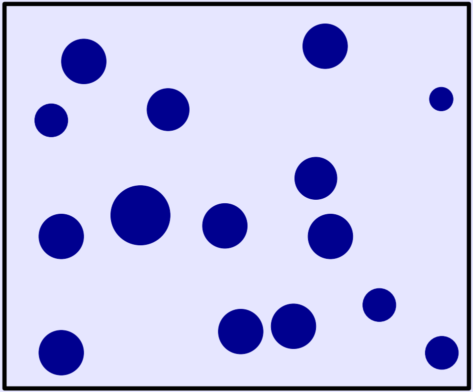
An amplifying random medium



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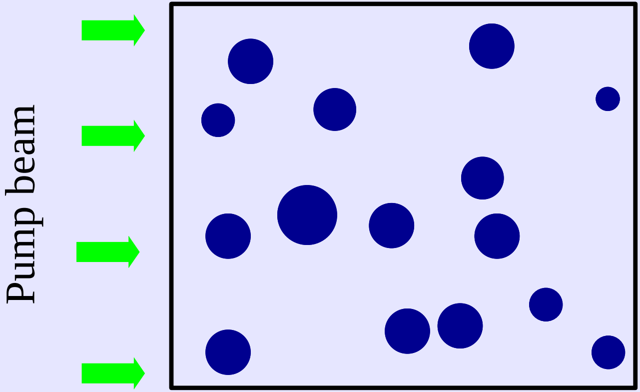

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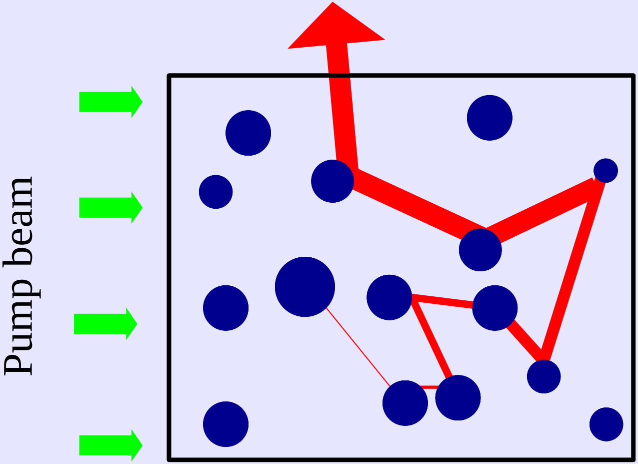
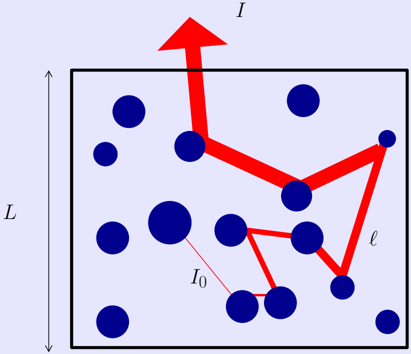


Length scales



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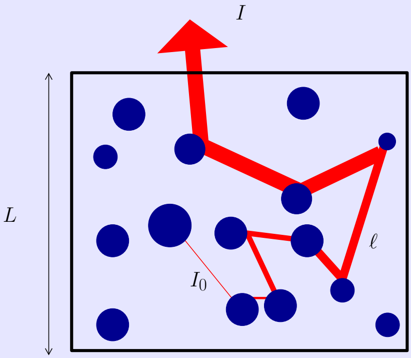

Sample size L
Mean free path ℓ
Path length l
Gain length ℓ_G

$$I(l) = I_0 \exp(l/\ell_G)$$

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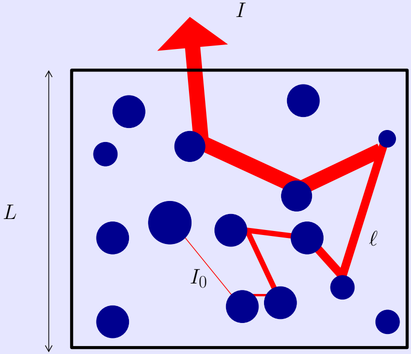

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Mean free path	ℓ
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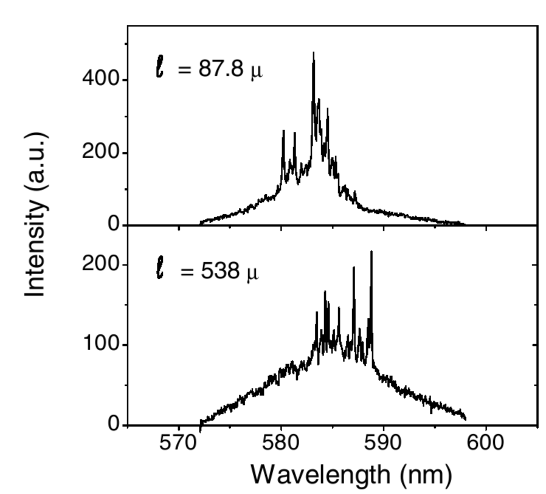
$$I(l) = I_0 \exp(l/\ell_G)$$

- Diffusive regime, $\lambda \ll \ell$: $D = v\ell/3$
- If the path is long enough amplification occurs: "random laser"

Fluctuations in emission spectra



S. Mujumdar, M. Ricci, R. Torre, and D. S. Wiersma Phys. Rev. Lett. **93**, 053903 (2004).

A statistical effect?

Diffusive estimates

- Distribution of path lengths: $p(l) \ = \ \frac{\exp(-l/\langle l \rangle)}{\langle l \rangle}$
- First-passage time estimate: $\langle l \rangle \ = \ \frac{v}{D\Lambda}$
  Λ smallest eigenvalue of ∇^2 (e.g $\Lambda = q^2$, $q = \pi/L$ for slab of height L)
- Amplification: $I(l) \ = \ I_0 \exp(l/\ell_G)$

Distribution of emitted intensities:

$$p(I) \ = \ \frac{\ell_G}{\langle l \rangle} I^{-(1+\alpha)} \quad , \qquad \alpha \ = \ \frac{\ell_G}{\langle l \rangle} \quad .$$

for $I > I_{cutoff}$.

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Gaussian versus Lèvy statistics

Detection of the sum of many events, each with distribution $p(I) \propto I^{-(1+\alpha)}$:

- for $\alpha > 2$: standard central limit theorem, Gaussian statistics
- for $0 < \alpha \leq 2$, $\langle I^2 \rangle = \infty$! Lèvy statistics;
  - a single rare event dominates the overall emission ("lucky photons")
  - fluctuations do not average out: "irreproducible" results.....
  - dependence on the size, nr. of measurements etc.

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Model

Ensemble of M random walkers with positions x_i and energy E_i

- **Spontaneous emission:** At each t start a new walker with probability $\gamma \Delta t$ ($\gamma =$ sp. em. rate of the single atom) and energy ε, $M \longrightarrow M+1$
- **Diffusion:** Parallel and asynchronous update of the walkers positions: $x_i \rightarrow x_i \pm a_i$
- **Absorption:** remove walkers at the boundaries $x = 0, L$, record E_i and let $M \rightarrow M-1$.
- **Stimulated emission:** For $i = 1, \ldots, M$ $E \rightarrow E + \varepsilon$ ($\varepsilon = 1$), with a gain rate $\Gamma(E)$ e.g

$$\Gamma(E) = \gamma E \qquad \text{"linear gain" or}$$
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Single walker

Langevin limit:

$$\dot{x} = \sqrt{2D}\,\xi, \quad \dot{E} = \Gamma(E) + \sqrt{\Gamma(E)}\,\eta$$

ξ, η are δ-correlated Gaussian, $\langle \xi \rangle = 0$ $\langle \xi^2 \rangle = \langle \eta^2 \rangle = 1$
Fokker-Planck eq. for $P(x, E, t)$

$$\dot{P} = D\frac{\partial^2 P}{\partial x^2} - \frac{\partial}{\partial E}\left(\Gamma P - \frac{1}{2}\frac{\partial \Gamma P}{\partial E}\right)$$

Boundary conditions on $[0, L] \times [1, \infty)$:

- Absorbtion: $P(0, E, t) = P(L, E, t) = 0$
- Coupling of diffusion and gain processes : eq for $P(x, 1, t) \equiv f(x, t)$

$$\dot{f} = Df'' - \gamma f + \gamma$$

Fat tails

- Solve by separation of variables $P(x,E) = Q(x)W(E)$ $Q \sim \sin(kx)$ with $k = m\pi/L$ (m integer)
- Approximate solution (single mode $m = 1$); for linear gain

$$P(x,E) \ \propto \ \frac{\sin\left(\frac{\pi x}{L}\right)}{E^{1+\alpha}}$$

  where $\gamma_c \equiv D(\pi/L)^2$ and $\alpha \equiv \gamma_c/\gamma$
- "Laser threshold" $\gamma = \gamma_c$: Cauchy-like tail $\alpha = 1$, $P(x,E) \propto E^{-2}$,

Energy field

For many walkers x_i, E_i let $\phi(x,t) = \sum_i^{M(t)} E_i \, \delta\left(x - x_i(t)\right)$, stochastic calculus ....

Langevin equation

$$\frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2} + \Gamma(\phi) + \frac{\partial}{\partial x}(v\phi) + s$$

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Diffusion and gain: Lethokov equation for random lasers (1968)
Threshold at γ_c, absorbing b. $\phi(0,t) = \phi(L,t) = 0$

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Additive noise from spontaneous emission, Poisson process

$$\phi \to \phi + \varepsilon$$

at random times, $\langle s \rangle = \gamma$.

Energy field

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Random advection

$$\langle v(x,t) v(x',t') \rangle = D\ell\Delta(x-x')\delta(t-t')$$

Δ short-range correlated in space

Remarks

Linear gain:

$$\frac{\partial \phi}{\partial t} = D\frac{\partial^2 \phi}{\partial x^2} + \gamma\phi + \frac{\partial}{\partial x}(v\phi) + s$$

- Model for active scalar [Deutsch 1993]
- Random advection + additive noise $\longrightarrow$ power-laws
- Mean-field solution $\bar{\phi}(x)$ unstable at $\gamma = \gamma_c$
- Important differences:
  1. Finite system, dissipation
  2. Two sources of add. noise: s, $\partial(v\bar{\phi})/\partial x$
  3. Advective term is a finite-size effect! From dimensional analysis (in d dimensions):

$$\lambda(L)\nabla \cdot (\mathbf{u}\phi) \quad \lambda \propto (\ell/L)^{\frac{d}{2}}$$

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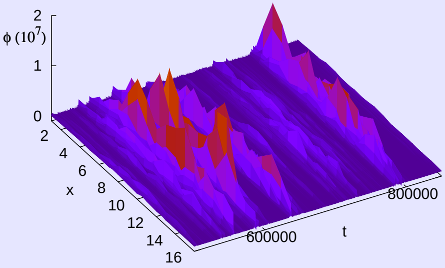
Intermittency

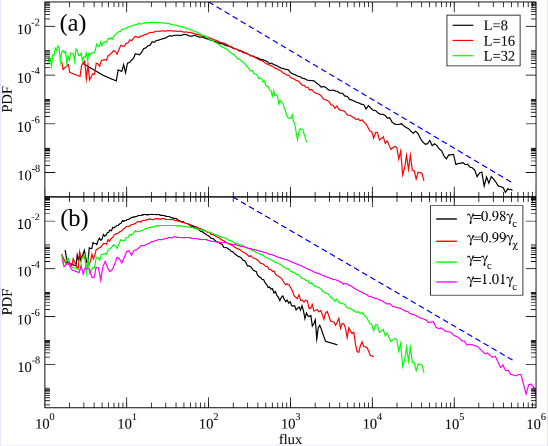
Discretization on the lattice:

$$\dot{\phi}_i = D\left[\phi_{i+1} + \phi_{i-1} - 2\phi_i\right] + \Gamma(\phi_i) + \frac{1}{2}(v_{i+1}\phi_{i+1} - v_{i-1}\phi_{i-1}) + s_i$$

$\langle v_i^2 \rangle = D\ell$ $\phi_0 = \phi_{N+1} = 0$, $v_1 = v_N = 0$: $\sum_i \phi_i$ conserved.



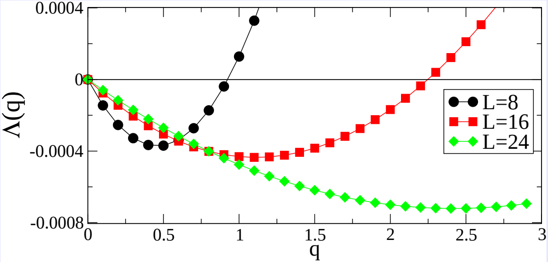
Statistics



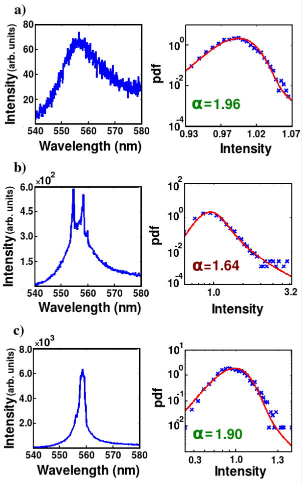
Distributions of flux $D[\phi_1(t) + \phi_N(t)]/2$

Finite-time Lyapunov

$$R(\tau) = \frac{\|\delta\phi(t+\tau)\|}{\|\delta\phi(t)\|}, \quad \overline{R^q(\tau)} \sim \exp(\Lambda(q)\tau)$$

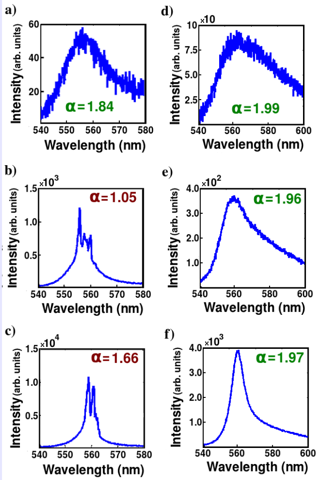


Comparison with experiments



Levý statistics compatible with data, different regimes

Comparison with experiments



Left: $\ell_{mfp} = 3.9mm$, right $\ell_{mfp} = 0.4mm$

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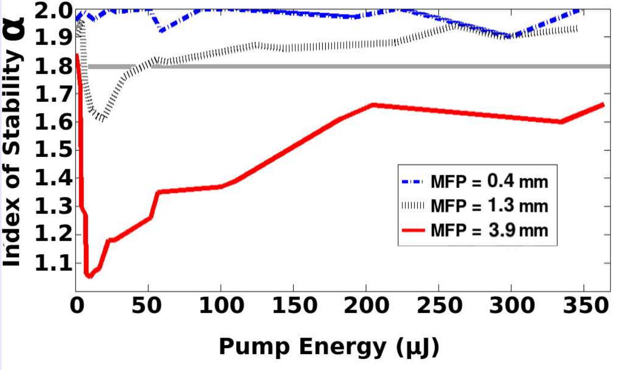

Range of Levý statistics reduces decreasing the mean-free-path

Outlook

- Random amplifying media;
- A statistical explanation for fluctuations in the emission spectra;
- Langevin equation for the energy density with multiplicative random advection term
- Gaussian and Lèvy regimes
- Effective noise strength $\lambda(L) \sim (\ell/L)^{\frac{d}{2}}$: criterion for different statistical regimes
- Comparison with experiments

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