Stochastic perturbation of integrable systems

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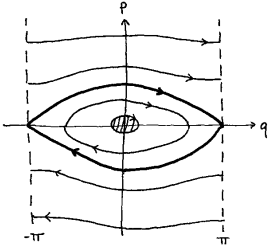
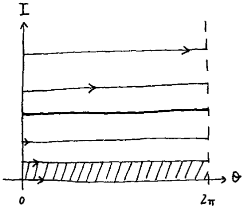
Integrable systems

N independent constants of motion





Action $(I_1, ...I_N)$ and Angle $(\theta_1, ..., \theta_N)$ variables

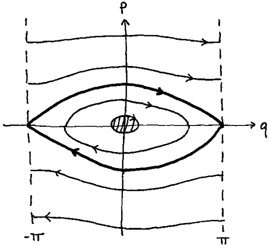
Action-angle representation

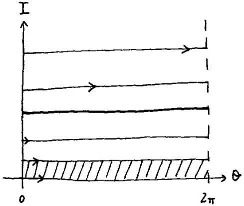
$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned}$$

$$\dot{I}_i = -\frac{\partial H}{\partial \theta_i}$$

$$\dot{\theta}_i = \frac{\partial H}{\partial I_i} = \omega_i(\mathbf{I})$$

Flow is *laminar*, restricted to tori $I_i = const$, $\theta_i = \omega_i t$





Topology: stationary points and separatrices

We perturb with weak, additive noise

$$\begin{aligned}\dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} + \varepsilon^{\frac{1}{2}} \xi_i(t)\end{aligned}$$

mostly consider the case in which the $\xi(t)$ are white noises:

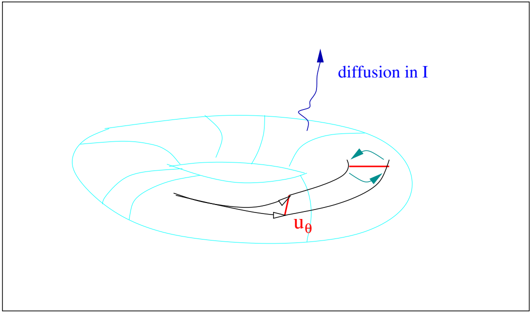
$$\langle \xi(t) \rangle = 0, \qquad \text{and} \qquad \langle \xi(t)\xi(t') \rangle = 2\delta(t - t').$$

In the action-angle variables, the noise is no longer additive, and reads:

$$\dot{I}_i = \varepsilon^{\frac{1}{2}} \sum_k \{I_i, q_k\} \xi_k(t)$$

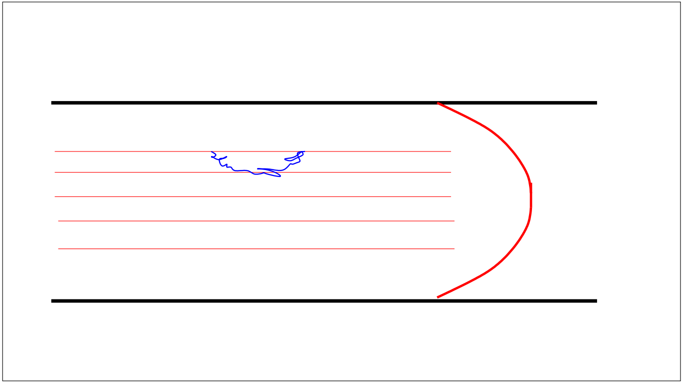
$$\dot{\theta}_i = \omega_i + \varepsilon^{\frac{1}{2}} \sum_k \{\theta_i, q_k\} \xi_k(t)$$

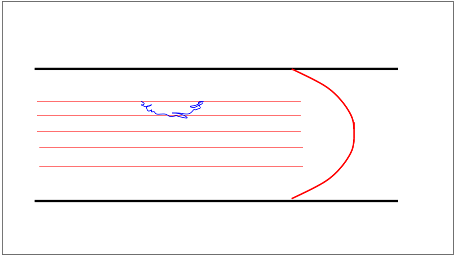
Surprise: a Lyapunov instability appears



two trajectories subjected to the *same* noise diverge exponentially

Hydrodynamic analogy: Taylor's diffusion





$$H = I - \frac{1}{6} I^3 \quad ; \quad \omega(I) = 1 - \tfrac{1}{2} I^2,$$

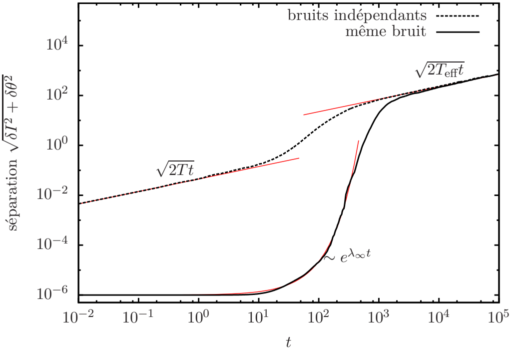
and let us choose:

$$\{\theta, q\} = \sqrt{2} \cos\theta.$$

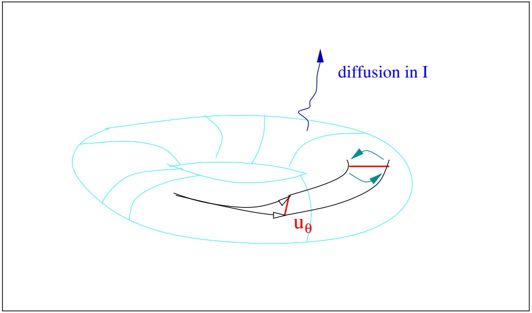
so that:

$$\dot{\theta} = \omega(I)$$
$$\dot{I} = -(2\varepsilon)^{\frac{1}{2}} \sin\theta \; \xi(t)$$

Ordinary diffusion, Taylor diffusion and Lyapunov regimes



To leading order, everything happens



along the flow (on the tori)

$$\dot{u}_{\theta i} = \sum_j \frac{\partial^2 H}{\partial I_j \partial I_i}\, u_{Ij}$$

$$\dot{u}_{Ii} = \sum_{kj} \frac{\partial^2 q_k}{\partial \theta_i \partial \theta_j}(\varepsilon^{-\alpha} t)\, \xi_k(t)\, u_{\theta j}$$

$$\dot{u}_\theta = \frac{d^2 H}{dI^2}\, u_I$$

$$\dot{u}_I = \rho(t) u_\theta$$

with $\overline{\rho(t)\rho(t')} = \delta(t-t')\Lambda_{II\theta\theta}$.

$$\Lambda_{II\theta\theta} = \overline{\left(\frac{\partial^2 q}{\partial \theta^2}\right)^2} = \overline{\left(\frac{\ddot{q}}{\omega(I)^2}\right)^2}$$

$$\begin{aligned}\dot{u}_\theta &= \frac{d^2H}{dI^2}\, u_I \\ \dot{u}_I &= \rho(t) u_\theta\end{aligned}$$

$$\ddot{u}_\theta = \frac{d^2H}{dI^2}\rho(t)u_\theta$$

put $z = \frac{\dot{u}_\theta}{u_\theta}$

$$\ddot{u}_\theta - \frac{d^2H}{dI^2}\rho(t)\, u_\theta \qquad ; \qquad \dot{z} - z^2 = \rho(t)$$

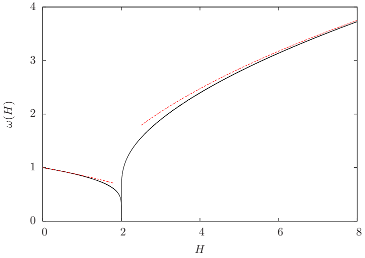
we connect with Halperin, Gardner-Derrida, Mallick-Marcq, ...

And we get, for the Lyapunov exponent

$$\lambda = \omega'(I)\langle z\rangle = \left(\frac{3}{2}\right)^{1/3} \frac{\sqrt{\pi}}{\Gamma(\frac{1}{6})} \left(\varepsilon \overline{(\ddot{q})^2} \left(\frac{1}{\omega} \frac{d\omega}{dH} \right)^2 \right)^{1/3} .$$

in terms of the average of $\ddot{q}$ over a cycle

Separatrices: the pendulum $H = \frac{1}{2}\,p^2 + 1 - \cos q$



$$\lambda = \omega'(I)\langle z\rangle = \left(\frac{3}{2}\right)^{1/3} \frac{\sqrt{\pi}}{\Gamma(\frac{1}{6})} \left(\varepsilon\overline{(\ddot{q})^2} \left(\frac{1}{\omega}\frac{d\omega}{dH}\right)^2\right)^{1/3}.$$

Separatrices: the pendulum

$$H = \frac{1}{2}\, p^2 + 1 - \cos q$$

For $\delta \equiv |H-2|$, one may compute

$$\omega(\delta \to 0) \simeq \frac{\pi}{|\log \delta|} \to 0$$

$$\frac{1}{\omega}\frac{d\omega}{dH} \sim \frac{1}{\delta |\log \delta|} \to \infty.$$

$$\varepsilon^{-\frac{1}{3}} \lambda \to \infty.$$

The angle α of the Lyapunov vector with the torus

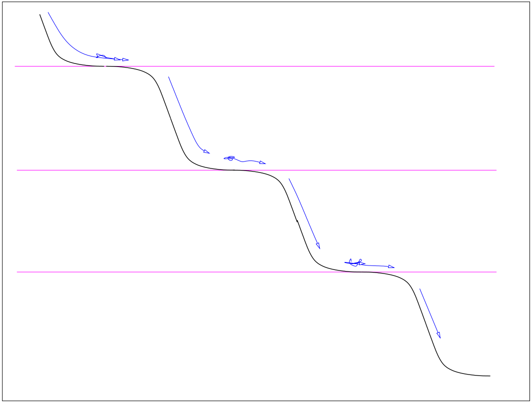
$$\alpha = \arctan z = \arctan\left(\frac{u_I}{u_\theta}\right)$$

The angle α follows a Langevin equation:

$$\dot{\alpha} \approx -\frac{dV}{d\alpha} + \frac{1}{\tau\omega'}\,\xi(t)$$

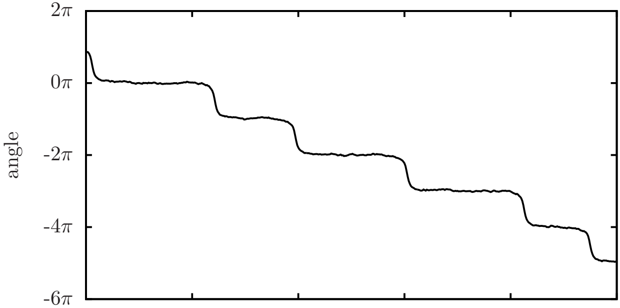
$$V(\alpha) = \frac{\omega'}{2}\left(\alpha - \frac{1}{2}\sin 2\alpha - \ small\right)$$

Evolution of the angle

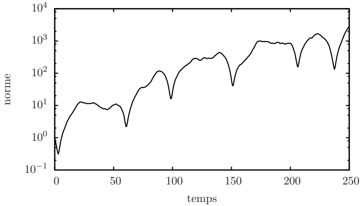


is punctuated by fast phase-slips

A numerical example



Meanwhile, the modulus grows steadily *between slips*



One finds a universal result: $\langle t_{slip} \rangle = 1.81\ \tau_\lambda$

Localization: band-edge phenomenology

$$\ddot{u}_{\theta i} + \sum_j u_{\theta j} = \ddot{u}_{\theta i} + \sum_j \hat{H}_{ij} u_{\theta j}$$

$$\hat{H}_{ij}(t) \equiv - \sum_{kl} \frac{\partial^2 H}{\partial I_l \partial I_i} \frac{\partial^2 G_k}{\partial \theta_l \partial \theta_j} (\varepsilon^{-\alpha} t)\ \xi_k(t)$$

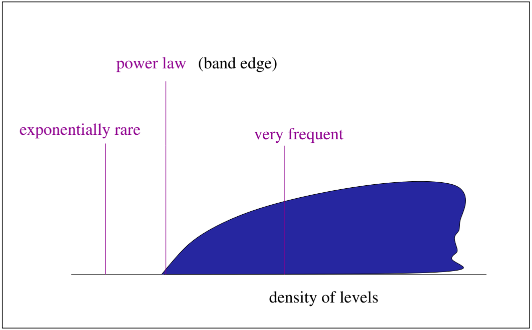
Shrödinger eigenvalue equation

$u_{\theta i} \to \psi_i$ **and** $t \to x$,

$$\nabla^2 \psi + \hat{\mathbf{H}} \psi = e\psi$$

density of zeroes $e < 0 \to$ number of phase-slips per unit time

Gardner-Derrida



... many things to learn from this vast literature

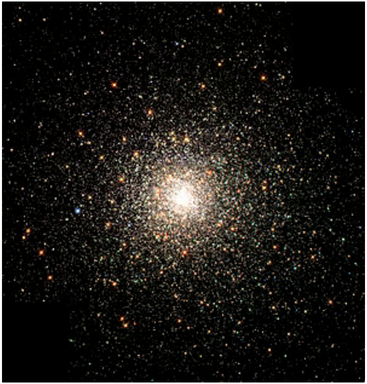
Weakly perturbed integrable models: mimicking complicated perturbations with stochastic ones

1. An integrable mean-field

2. Perturbation in planetary systems

A regime beyond KAM, and beyond the Nekhoroshev, for which there is no theory (?)

A spherical mean field, time-dependent granularity is the nonintegrable perturbation



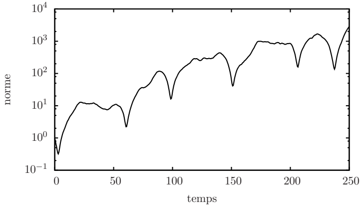
Integrable Kepler trajectories, perturbed by other planets

Lyapunov time (million Years) Moser

Mercury	1.4M
Venus	7.2 M
Earth	4.8M
Mars	4.5M
Jupiter	8.4 M
Saturn	6.4M
Uranus	7.5M
Neptune	6.7M

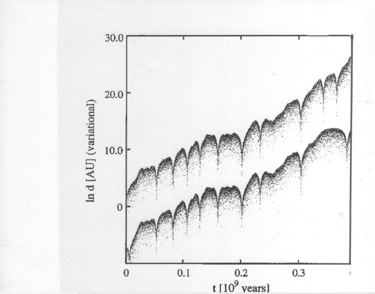
with some questions that I have...

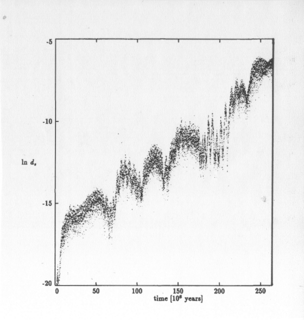
A simple system perturbed by noise



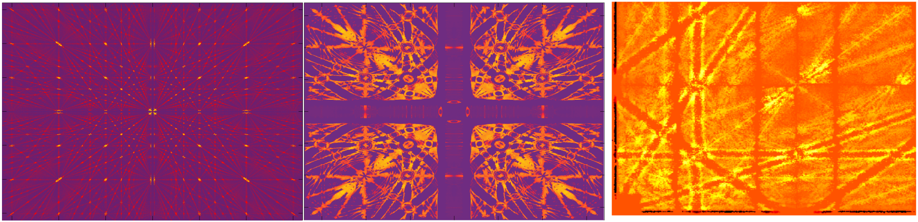
$$\langle t_{slip} \rangle = 1.81\ \tau_\lambda$$

Pluto and Saturn *phase-slips* **J. Wisdom**





Froeschlé model without and with modification



$$H = \sum_{i=1}^{N} \frac{I_i^2}{2} + I_0 + \frac{\epsilon(N+2)}{1 + \frac{1}{N+2}\sum_{i=0}^{N}\cos\theta_i} + \sum_i \cos\theta_i$$

Froeschlé model without and with modification

$$
\begin{aligned}
\dot{\theta}_0 &= 1, \\
\dot{\theta}_i &= I_i \\
\dot{I}_i &= -\cancel{\epsilon \sin \theta_i} - \epsilon \sin \theta_i \xi(t),
\end{aligned}
$$

with

$$
\xi(t) = \left(1 + \frac{1}{N+2} \sum_i \cos \theta_i \right)^{-2} - 1.
$$

Either perturbed pendula, or free rotors

Random I_i with variance β

$$\xi(t) = -\frac{2}{N+2}\sum_{i=0}^{N}\cos\theta_i + \frac{3}{(N+2)^2}\left(\sum_{i=0}^{N}\cos\theta_i\right)^2 + \mathcal{O}(N^{-3}).$$

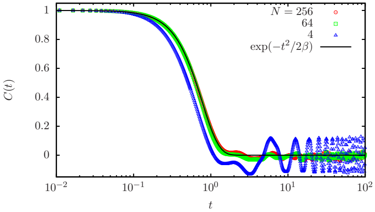
$$\langle\xi\rangle = \frac{3}{2}\frac{N+1}{(N+2)^2} \simeq \frac{3}{2N},$$

$$\langle\xi^2\rangle = 2\frac{N+1}{(N+2)^2} \simeq \frac{2}{N}$$

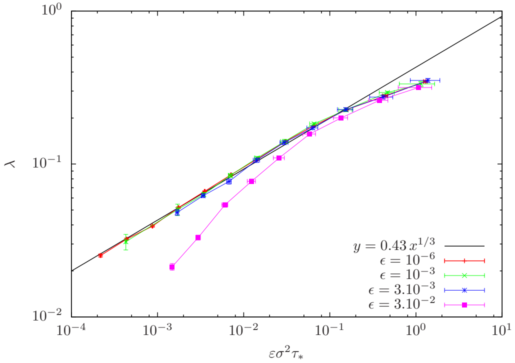
$$\sigma^2 = \langle\xi^2\rangle - \langle\xi\rangle^2 \simeq \frac{2}{N}$$

Random I_i with variance β

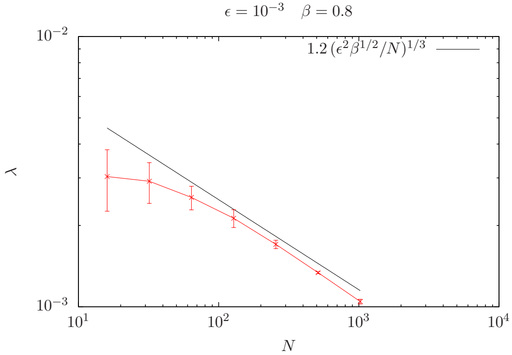
Autocorrelation



Using the true $\xi(t)$ as a noise on a separate system



The modified Froeschle' model



$$4 < N < 8192$$

The case of the unmodified Froeschle' model is interesting

The Lyapunov exponent is dominated by the few I_i that are close to the (pendulum) separatrix

higher Lyapunov exponents should be considered ... and perhaps for planets as well...

Diffusion of the eccentricity of Mercury, slightly different runs

Laskar

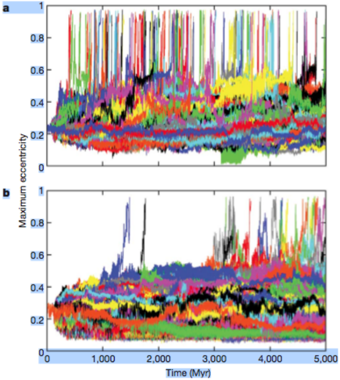


Figure 1 | Mercury's eccentricity over 5 Gyr. Evolution of the maximum

- **Suggests a statistical treatment is in order**
- **assuming partial ergodicity on the torus , etc**