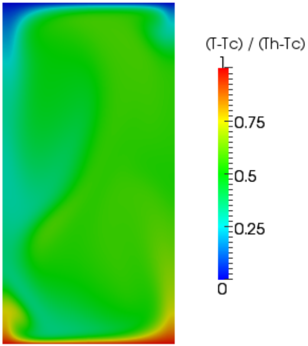
Particles and Bubbles in confined thermal convection

Paolo Oresta[1], Francesco Fornarelli[1], Roberto Verzicco [2,3],
Detlef Lohse [3], Andrea Prosperetti [3,4]

[1] Polytechnic of Bari, and INFN sez. Lecce, Italy.
[2] University of Tor Vergata. Rome, Italy.
[3] University of Twente, The Netherlands.
[4] Johns Hopkins University, Baltimore, USA.

Introduction



Fluid temperature
$Ra = 2 \cdot 10^6$

We investigated:

1. the settling of heavy particles in a gas phase
2. the growth of vapor bubbles in a liquid phase

- **Conclusions:** particles and bubbles ehnance the convective flow and finally the heat transfer even though with different mechanisms.
- **Numerical method:** Eulerian-Lagrangian point particle model two-way coupling

Governing equations

- **The gas/liquid phases - Eulerian frame**
- Navier-Stokes equations with Boussinesq approximation
  - continuity
  - momentum single phase + f
  - energy single phase + Q

- **The particles - Lagrangian point particle**
  - thermal balance at the particle surface →New temperature, T_p
  - position
  - momentum: drag, gravity

- **The bubbles - Lagrangian point particle**
  - thermal balance at the particle surface →New volume, V_b
  - position
  - momentum: drag, lift, buoyancy, added mass, virtual buoyancy

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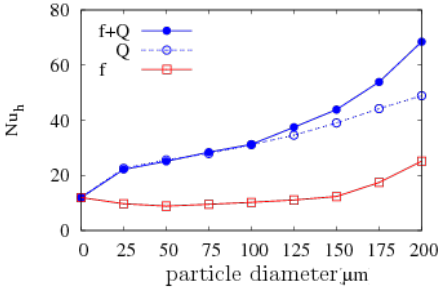
Heat flux at the lower hot plate, Nu_h

Coupling:

- f → mechanical
- Q → thermal
- $f + Q$ → mechanical + thermal



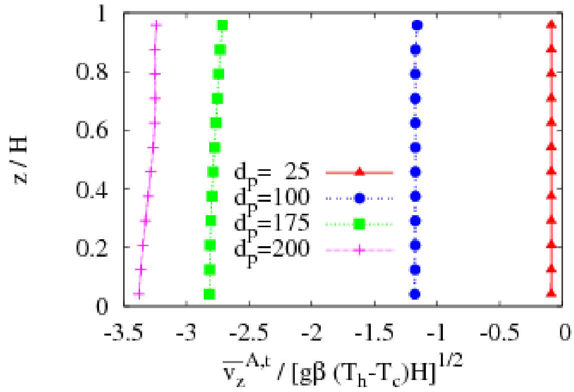
- Particle temperature fixed at T_{cold}

25, 000 particles

4.27×10^{-5} volume fraction (particle diameter 100 μm)

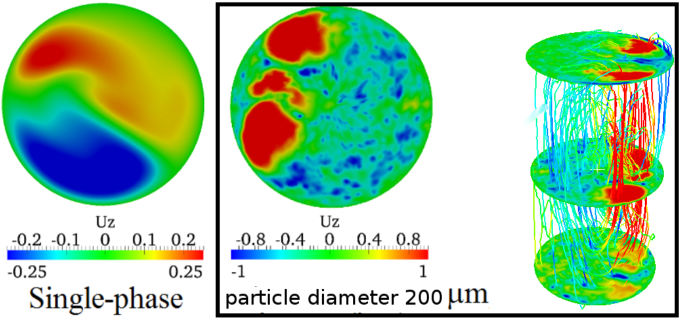
Particle vertical velocity, V_z as function of height, z

- Particle temperature fixed at T_{cold}



Vertical fluid velocity on the midplane cross section

- Particle temperature fixed at T_{cold}

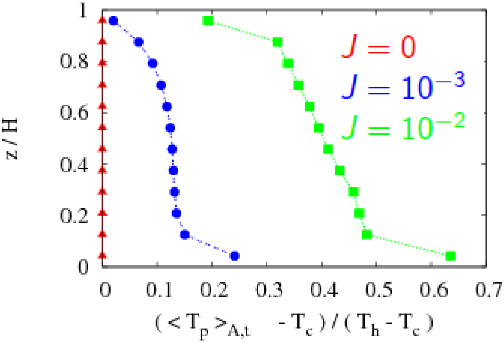


Particle temperature averaged over the cross section

- Variable particle temperature

$$J = \frac{\rho c}{\rho_p c_p}$$

- For increasing J, particles heat up more and more



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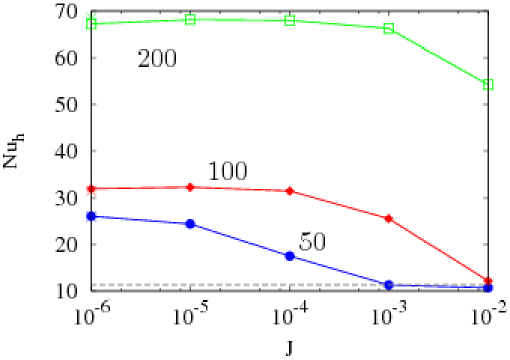
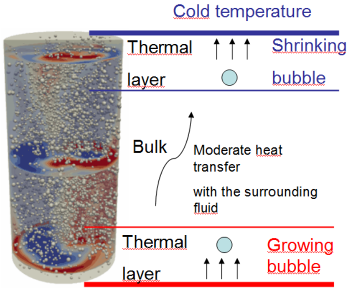
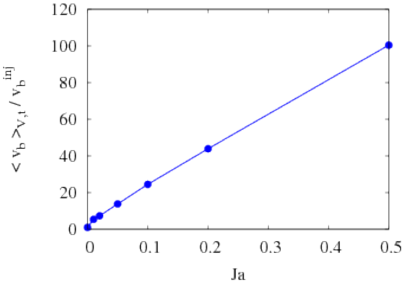


Bubbly convection

- The more Jakob number increases the more bubbles grow

$$Ja = \frac{\rho c (T_h - T_c)}{\rho_b L}$$





Vertical fluid velocity, Uz

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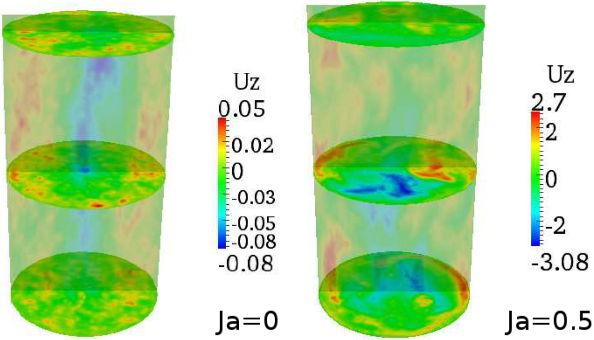
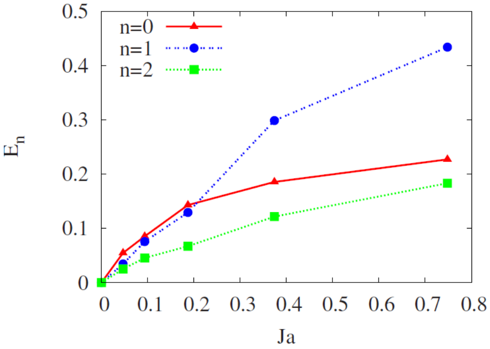
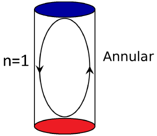


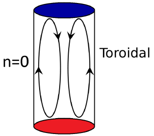
Bubbly convection

- Fluid kinetic energy distribution among azimuthal modes

$$E_n = \frac{\pi}{\beta g H^4 (T_h - T_c)} \int_0^R rdr \int_0^H dz \langle |\mathbf{u}_n|^2 \rangle_t$$







Nusselt number at the hot, Nu_h and at the cold plates Nu_c

$$Nu_{h,c} = \frac{H}{T_h - T_c} \left\langle \partial_z T \right\rangle_{A,t} \Big|_{z=0,H} \tag{1}$$

$$Nu_h = 1 + \frac{H}{\kappa (T_h - T_c)} \left\langle u_z (T - T_c) \right\rangle \tag{2}$$

$$- \frac{1}{\pi R^2 \kappa (T_h - T_c)} \left\langle \sum_i (H - z_{p,b}) Q_{p,b} \right\rangle \tag{3}$$

$$Nu_c = 1 + \frac{H}{\kappa (T_h - T_c)} \left\langle u_z (T - T_c) \right\rangle \tag{4}$$

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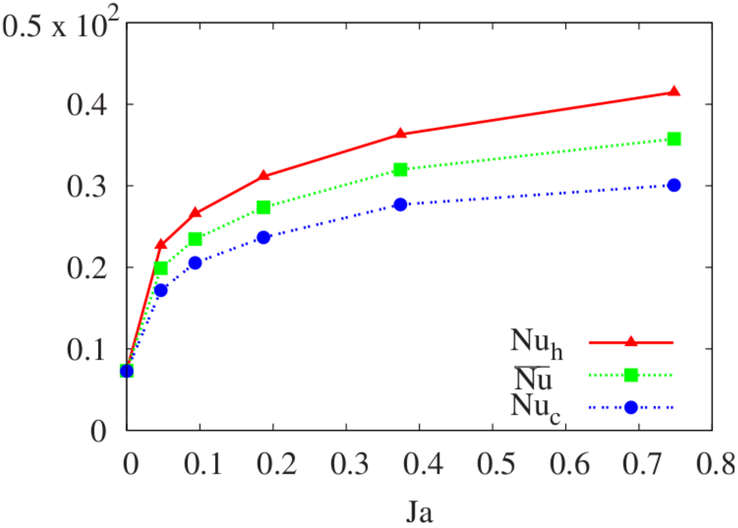
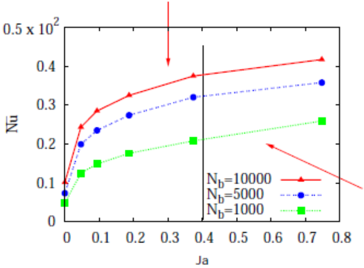


Nusselt number

- Bubbles promote strong convective currents in the liquid thus helping to remove the heated layer near the hot wall.

Rising bubbles
drive convection



Bubble too large:

lower residence time,

less enhancement.

Conclusions

- Particle settling in a gas-phase
  - The particles introduced at the cold temperature cool the entire system and therefore the heat extract from the cold plate is increased.
  - The falling particles tend to drag the cold fluid with them. By continuity the upward flow stream is enhanced which increases the convective heat transfer.
- Vapor bubbles in a liquid-phase
  - The growing bubbles are swept into the upward stream of the annular mode.
  - The more bubbles grow the more they reinforce the buoyancy of the liquid phase making the annular mode stronger.

Fluid momentum and energy balance

$$\nabla \cdot \mathbf{u} = 0 \tag{7}$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} - \beta \rho (T - T_c)\mathbf{g} + \sum \mathbf{F}_{p,b} \delta(\mathbf{x} - \mathbf{x}_{p,b}); \tag{8}$$

$$\mathbf{F}_{p,b} = v_{p,b} \left[\rho \left(\left. \frac{D\mathbf{u}}{Dt} \right|_{\mathbf{x}_{p,b}} - \mathbf{g} \right) - \rho_{p,b} \left(\left. \frac{d\mathbf{v}}{dt} \right|_{\mathbf{x}_{p,b}} - \mathbf{g} \right) \right]$$

- Mazzitelli, Lohse, Toschi, J. Fluid Mech., 488 (2003)

$$\rho c \frac{DT}{Dt} = k \nabla^2 T + \sum Q_{p,b} \delta(\mathbf{x} - \mathbf{x}_{p,b}) \tag{9}$$

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The particle model - momentum equation

- **Particle position**

$$\frac{d\mathbf{x}}{dt} = \mathbf{v} \qquad (10)$$

- **Forces balance**: drag and gravity

$$\frac{d\mathbf{v}}{dt} = \frac{f}{\tau_p}(\mathbf{u} - \mathbf{v}) + \mathbf{g} \qquad (11)$$

$$\tau_p = \frac{\rho_p d_p^2}{18\mu}$$

$$f = 1 + 0.15 Re_p^{0.687}; \qquad Re_p = \frac{d_p|\mathbf{u} - \mathbf{v}|}{\nu};$$

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The particle model - thermal balance

- **Thermal balance at particle surface**

$$m_p c_p \frac{dT_p}{dt} = -Q_p = S_p h_p (T - T_p) \qquad (12)$$

$$Nu_p = \frac{d_p h_p}{\rho c \kappa}; \qquad Nu_p = 2 + 0.6 Re_p^{1/2} Pr^{1/3}$$

- Bergman et al. 2011

$$\frac{dT_p}{dt} = \frac{Nu_p}{2} \frac{1}{\tau_{th,p}} (T - T_p) \qquad (13)$$

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- **Particle position** $d\mathbf{x}/dt = \mathbf{v}$
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$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= 3(\mathbf{u}-\mathbf{v})\frac{1}{d_b}\frac{d}{dt}(d_b) + \left(1+\frac{1}{C_A}\right)\frac{D\mathbf{u}}{Dt} && (14)\\ &+ \frac{f}{\tau_b}(\mathbf{u}-\mathbf{v}) && (15)\\ &+ \frac{C_L}{C_A}(\mathbf{u}-\mathbf{v})\times\mathbf{\Omega} && (16)\\ &- \frac{1}{C_A}\mathbf{g} && (17) \end{aligned}$$

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$$Nu_b = Nu_{b0}\left[1+\left(\frac{4Pe_b}{\pi Nu_{b0}^2}\right)^{n/2}\right]^{1/n} \tag{20}$$

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