Tumbling of small non-spherical particles in a shear flow

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based on joint work with J. Einarsson[1)], F. Candélier[2)], F. Lundell[3)] & J. R. Angilella[4)]

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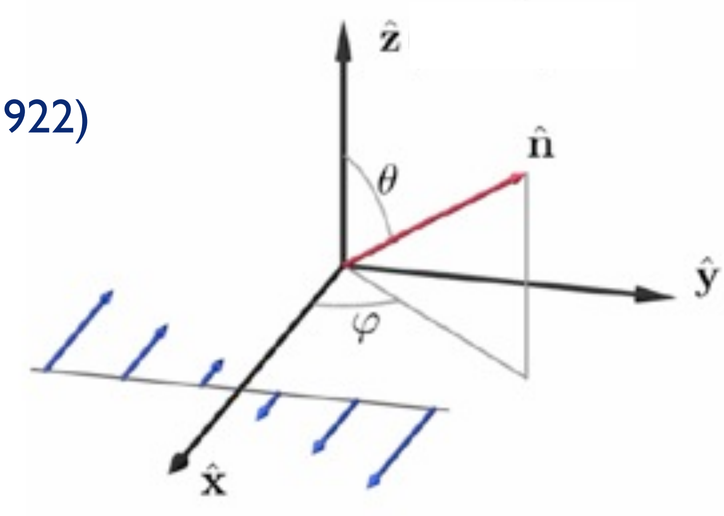
Effect of weak fluid inertia on Jeffery orbits,
J. Einarsson, F. Candélier, F. Lundell, J. R. Angilella & B. Mehlig, submitted to Phys. Rev. E

Shear flow

Shear flow $\boldsymbol{u}(y) = sy\hat{\boldsymbol{x}}$, flow-gradient matrix $\mathbf{A} = \begin{bmatrix} 0 & s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Degenerate orientational dynamics of small axisymmetric particles: Jeffery orbits (tumbling).

Jeffery, Proc. Roy. Soc. London Ser. A **102**, 161 (1922)



Expect that perturbations have strong effect.

Although this limit may seem to be of limited interest, ...

... these small changes can have a strong (or even dominant) cumulative effect on the particle's position or orientation. This occurs for the class of so-called "indeterminate" particle motions, in which *no* position or orientation is intrinsically favored under "standard conditions."

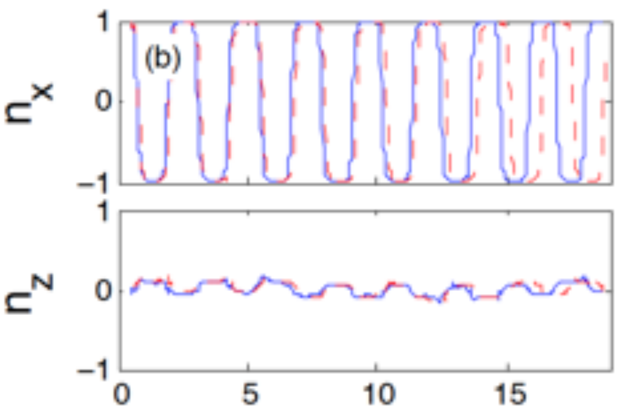
L. G. Leal
Ann. Rev. Fluid Mech. 1980.

Tumbling in simple shear

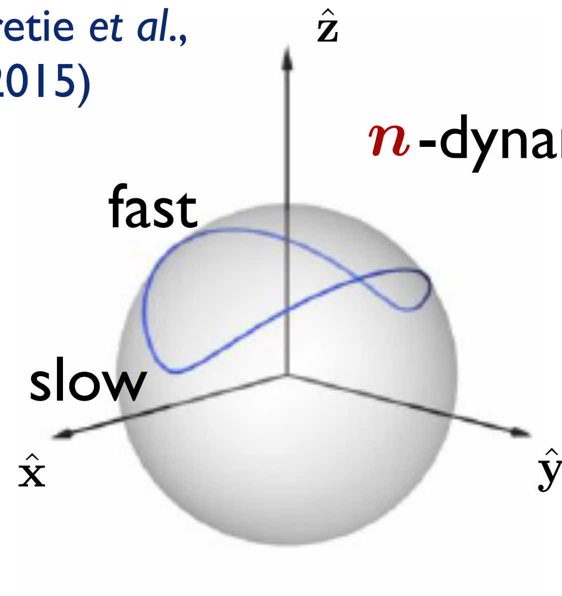
Axisymmetric particles in shear flow rotate periodically (Jeffery orbits).
Infinitely many degenerate periodic orbits.

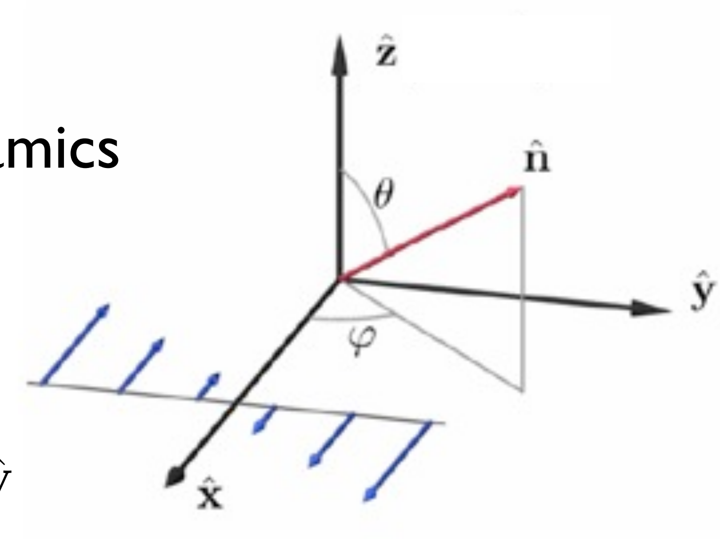
Jeffery, Proc. Roy. Soc. London Ser. A **102,** 161 (1922)

J. Einarsson, B. Mihiretie *et al.*, arxiv:1503.03023 (2015)



$\boldsymbol{n}$-dynamics



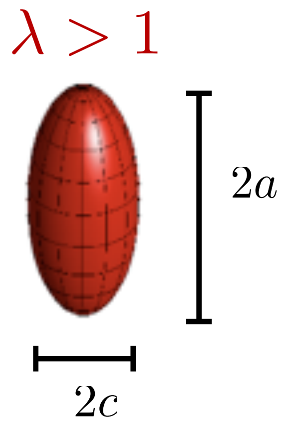


Micron-sized glass rods in a micro-channel flow.

forward, - - - - - - backwards.

$n_z = 1$: log-rolling,
$n_z = 0$: tumbling. Symmetry axis $\boldsymbol{n}$ spends a long time aligned with flow-direction $\hat{\mathbf{x}}$.

Dimensionless parameters



Axisymmetric particles: shape parameter $\Lambda = (\lambda^2 - 1)/(\lambda^2 + 1)$ (aspect ratio $\lambda = a/c$).

Fluid inertia: $\mathrm{Re_s} = a^2 s/\nu$
shear Reynolds number

ν kinematic viscosity
s shear rate

Particle inertia: $\mathrm{St} = \frac{\rho_\mathrm{p}}{\rho_\mathrm{f}} \mathrm{Re_s}$
Stokes number

ρ_p particle density
ρ_f fluid density

Brownian rotation: $\mathrm{Pe} = s/\mathcal{D}$ where $\mathcal{D}$ rotational diffusion constant
Péclet number

Jeffery equation obtained for small particles for $\mathrm{Re_s} = 0$, $\mathrm{St} = 0$, $\mathrm{Pe} = \infty$.

Degeneracy: must consider effect of perturbations. Here: $\mathrm{Re_s} > 0$.

Inertial effects - particles in flows

Particle- and Navier Stokes equations coupled by boundary conditions.

Phys. Fluids 26, 883 (1983); doi: 10.1063/1.864230

Equation of motion for a small rigid sphere in a nonuniform flow

Martin R. Maxey[a]
Department of Chemical Engineering, The Johns Hopkins University, Baltimore, Maryland 21218
James J. Riley
Flow Research Company, Kent, Washington 98031

Maxey-Riley equation,
correction to Stokes law

J. Fluid Mech. (1965), *vol.* 22, *part* 2, *pp.* 385–400

The lift on a small sphere in a slow shear flow

By P. G. SAFFMAN

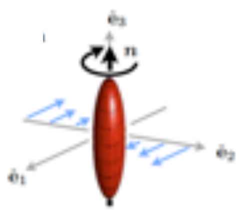
Saffman lift on small sphere
due to shear.

J. Fluid Mech. (2005), *vol.* 535, *pp.* 383–414.

Inertial effects on fibre motion in simple shear flow

By G. SUBRAMANIAN AND DONALD L. KOCH

Tumbling of a neutrally buoyant
fibre in shear flow,
slender-body limit.



Log-rolling unstable.

P. G. Saffman, J. Fluid Mech. **1**, 540 (1956)

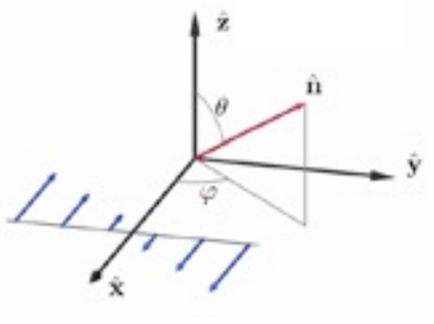
Tumbling of a nearly spherical
neutrally buoyant particle in shear.

Log-rolling stable.

Effective equation of motion

Neutrally buoyant spheroid.

Find effective vector field, correction to Jeffery's equation (caveat: caustics)



$$\dot{\boldsymbol{n}} = \underbrace{\boldsymbol{F}_0(\boldsymbol{n})}_{\text{Jeffery}} + \underbrace{\mathrm{St}\,\boldsymbol{F}_1(\boldsymbol{n})}_{\text{particle inertia}} + \underbrace{\mathrm{Re}_\mathrm{s}\boldsymbol{F}_2(\boldsymbol{n})}_{\text{fluid inertia}} + \ldots$$

$\boldsymbol{F}_1(\boldsymbol{n})$ known. Einarsson, Angilella & Mehlig, Physica D **278-279**, 79 (2014)

Now $\boldsymbol{F}_2(\boldsymbol{n})$. Difficulty: need to calculate torque on particle.

Requires solving Navier Stokes equations.

Perturbation theory in Re_s, St, neglect terms of order $\mathrm{St}\,\mathrm{Re}_\mathrm{s}$, St^2, Re_s^2,... .

Still very difficult problem. Solve it by exploiting the symmetries of problem.

Equations of motion

Orientational motion of neutrally buoyant spheroid

$$\dot{n}_i = \varepsilon_{ijk}\omega_j n_k\,, \quad \mathrm{St}(I_{ij}\dot{\omega}_j + \dot{I}_{ij}\omega_j) = T_i$$

Summation convention.

angular velocity ω_j , particle inertia tensor $I_{ij} = A^I(\delta_{ij} - P_{ij}) + B^I P_{ij}$ with $P_{ij} = \delta_{ij} - n_i n_j$, A^I and B^I are moments of inertia along and $\perp$ to $\boldsymbol{n}$.

Hydrodynamic torque T_i determined by integrating fluid stresses over particle surface $\mathcal{S}$. Requires solving Navier Stokes equations. Dimensionless variables (scales: time $T = s^{-1}$, length $L = a$, velocity $U = sa$, pressure $P = \mu s$):

$$\mathrm{Re_s}(\partial_t u_i + u_j \partial_j u_i) = -\partial_i p + \partial_j \partial_j u_i$$

$$\partial_i u_i = 0, \quad u_i = \varepsilon_{ijk}\omega_j r_k \text{ for } \boldsymbol{r} \in \mathcal{S},$$

$$u_i = u_i^{(\infty)} \text{ as } \boldsymbol{r} \to \infty .$$

p pressure, $\boldsymbol{\omega}$ particle angular velocity, $u_i^{(\infty)}$ shear in dimensionless variables.

Perturbation theory

Use `reciprocal theorem´ to compute hydrodynamic torque.

Lovalenti & Brady, J. Fluid Mech. **256**, 561 (1993)
Subramanian & Koch, J. Fluid Mech. **535**, 383 (2005)

Result (simple shear flow)

$$T_k = \underbrace{T_k^{(0)}}_{\text{Jeffery}} - \mathrm{Re_s} \int_{\mathcal{V}} \mathrm{d}v\, \tilde{U}_{ik} (\underbrace{\partial_t u_i}_{\text{unsteady fluid inertia}} + \underbrace{u_j \partial_j u_i}_{\text{convective fluid inertia}})$$

Integral over volume $\mathcal{V}$ outside particle.

The coefficients $\tilde{U}_{ik}$ are obtained by solving auxiliary Stokes problem. Above relation is exact. But unknown u_i, solution of Navier Stokes equations. Perturbation theory: $\mathrm{St} = \mathrm{Re_s} = 0$ - solutions to evaluate T_k to leading order.

Then expand:

$$\omega_i = \omega_i^{(0)} + \mathrm{St}\, \omega_i^{(\mathrm{St})} + \mathrm{Re_s}\, \omega_i^{(\mathrm{Re_s})} + \ldots .$$

Insert into angular-momentum equation. Find: $\omega_i^{(0)}$ Jeffery angular velocity, $\omega_i^{(\mathrm{St})}$ particle-inertia contribution, $\omega_i^{(\mathrm{Re_s})}$ fluid-inertia contribution, depends on volume integral. Difficult to calculate. Integrand depends non-linearly on $\boldsymbol{n}$.

Symmetries

Make use of the symmetries of the problem. Simple shear flow $\mathbb{A}^\infty = \mathbb{S}^\infty + \mathbb{O}^\infty$.

incompressibility:	$S^\infty_{ii} = 0$
symmetry of $\mathbb{S}^\infty$:	$S^\infty_{ij} = S^\infty_{ji}$
antisymmetry of $\mathbb{O}^\infty$:	$O^\infty_{ij} = -O^\infty_{ji}$
steady shear:	$O^\infty_{ij} O^\infty_{jk} = -S^\infty_{ij} S^\infty_{jk}\,,\; O^\infty_{ij} S^\infty_{jk} = -S^\infty_{ij} O^\infty_{jk}$
normalisation of $\boldsymbol{n}$:	$n_i \dot{n}_i = 0$
inversion symmetry:	invariance under $n_i \to -n_i\,,\; \dot{n}_i \to -\dot{n}_i$

Symmetries constrain form of equation of motion.

$$
\begin{aligned}
\dot{n}_i \;&=\; O^\infty_{iq} n_q + \Lambda[S^\infty_{ip} n_p - (n_p S^\infty_{pq} n_q) n_i] \\
&+\; \beta_1 (n_p S^\infty_{pq} n_q) P_{ij} S^\infty_{jk} n_k + \beta_2 (n_p S^\infty_{pq} n_q) O^\infty_{ij} n_j \\
&+\; \beta_3\, P_{ij} O^\infty_{jk} S^\infty_{kl} n_l + \beta_4\, P_{ij} S^\infty_{jk} S^\infty_{kl} n_l\,.
\end{aligned}
$$

First row: Jeffery equation. Remainder: St- and $\mathrm{Re_s}$-corrections, determined by only four scalar functions

$$\beta_\alpha = \mathrm{St}\,\beta_\alpha^{(\mathrm{St})}(\lambda) + \mathrm{Re_s}\,\beta_\alpha^{(\mathrm{Re_s})}(\lambda) \quad \text{for} \quad \alpha = 1,\ldots,4\,.$$

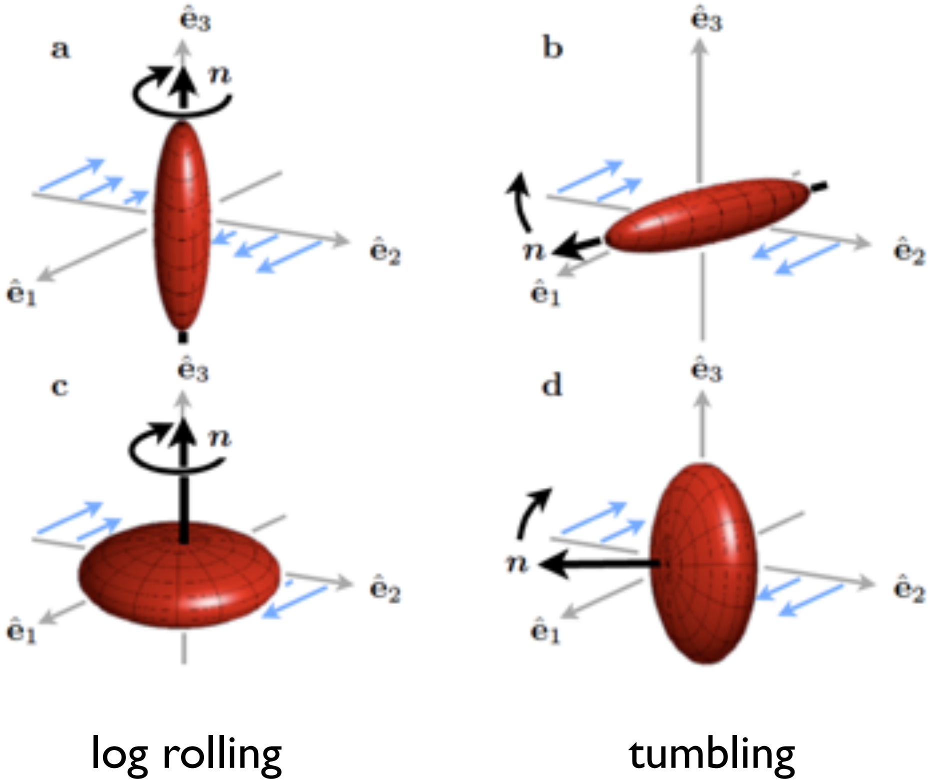
Can be calculated.

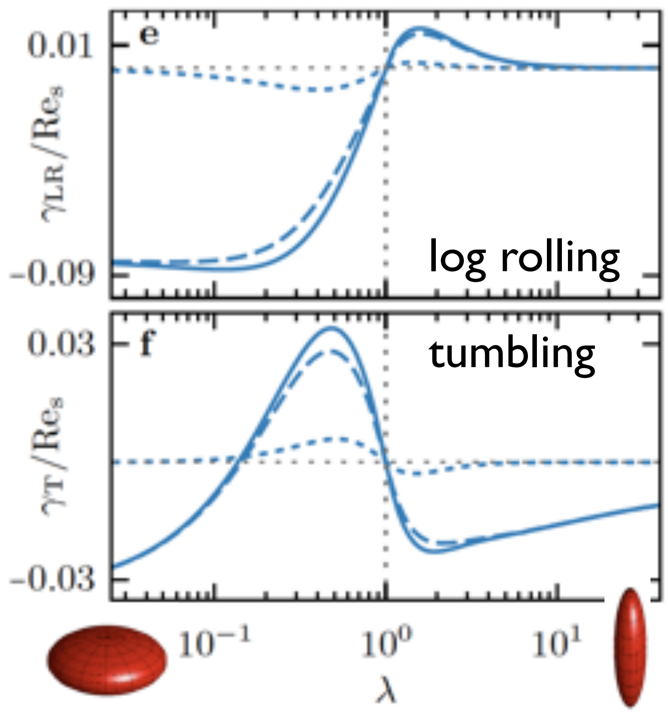
Results

Stability of log-rolling and tumbling orbits.

Linear stability analysis at small Re_s.

Stability exponents $\gamma_{\mathrm{LR}}, \gamma_{\mathrm{T}}$





stability exponent
contribution from fluid inertia
contribution from particle inertia

Conclusions

Orientational dynamics of neutrally buoyant axisymmetric particle in shear.
Stability analysis of Jeffery orbits at infinitesimal Re_s . Results:

-log-rolling unstable for prolate particles, tumbling in shear plane stable.
-for oblate particles (but not too disk-like) stabilities are reversed
-fluid inertia contributes more strongly than particle inertia
-both unsteady and convective fluid inertia matter. It would be qualitatively wrong to neglect either.

To do:
-analyse orientational motion for small but finite Re_s . Rosén, Lundell & Aidun (2014)
-wall effects
-settling $\rho_\mathrm{p} \neq \rho_\mathrm{f}$ (more difficult)
-unsteady flows (more difficult)
-turbulence (much more difficult)

Lovalenti & Brady, J. Fluid Mech. **256**, 561 (1993)