J. Phys. A: Math. Theor. **46** (2013) 250301 (5pp) doi:10.1088/1751-8113/46/25/250301

PREFACE

Lyapunov analysis: from dynamical systems theory to applications

Massimo Cencini[1] **and Francesco Ginelli**[2]

[1] Istituto dei Sistemi Complessi, Consiglio Nazionale delle Ricerche, Via dei Taurini 19, I-00185 Rome, Italy
[2] SUPA, Institute for Complex Systems and Mathematical Biology, Kings College, University of Aberdeen, Aberdeen AB24 3UE, UK

E-mail: massimo.cencini@cnr.it and francesco.ginelli@abdn.ac.uk

Received 17 April 2013
Published 4 June 2013
Online at stacks.iop.org/JPhysA/46/250301

The study of deterministic laws of evolution has characterized the development of science since Newton's times. Chaos, namely the manifestation of irregular and unpredictable dynamics (*not random but look random* [1]), entered the debate on determinism at the end of the 19th century with the discovery of sensitivity to initial conditions, meaning that small infinitesimal differences in the initial state might lead to dramatic differences at later times. Poincaré [2, 3] was the first to realize that solutions of the three-body problem are generically highly sensitive to initial conditions. At about the same time, this property was recognized in geodesic flows with negative curvature by Hadamard [4]. One of the first experimental observations of chaos, as understood much later, was when irregular noise was heard by Van der Pol in 1927 [5] while studying a periodically forced nonlinear oscillator. Nevertheless, it was only with the advent of digital computing that chaos started to attract the interest of the wider scientific community. After the pioneering investigation of ergodicity in a chain of nonlinear oscillators by Fermi, Pasta and Ulam in 1955 [6], it was in the early 1960s that the numerical studies of Lorenz [7] and Hénon and Heiles [8] revealed that irregular and unpredictable motions are a generic feature of low-dimensional nonlinear deterministic systems. The existence and onset of chaos was then rigorously analyzed in several systems. While an exhaustive list of such mathematical proofs is beyond the scope of this preface, one should mention the contributions of Kolmogorov [9, 10], Chirikov [11], Smale [12], Ruelle and Takens [13], Li and Yorke [14] and Feigenbaum [15].

The characteristic Lyapunov exponents introduced by Oseledets in 1968 [16] are the fundamental quantities for measuring the sensitivity to initial conditions. Oseledets' work generalized the concept of Lyapunov stability to irregular trajectories building upon earlier studies of Birkhoff [17], von Neumann [18], Krylov [19][3] and Asonov and Sinai [20] on ergodic theory. Lyapunov exponents quantify exponential sensitivity to initial conditions and provide direct access to the entropy production in ergodic systems via the Pesin theory [21].

[3] Prior to their publication in the West at the end of the 1970s, Krylov's results appeared in his PhD dissertation, published posthumously in 1950.

Further advances have been made possible by the introduction of proper physical invariant measures for certain dissipative systems due to Sinai [22], Ruelle [23] and Bowen [24, 25].

However, it was necessary to wait until the end of the 1970s before the independent works of Shimada and Nagashima [26] and Benettin *et al* [27] introduced the numerical algorithms required to compute Lyapunov exponents beyond the largest one. The availability of such algorithms and also, at about the same time, of those necessary for the computation of fractal dimensions and entropies by Grassberger and Procaccia [28], made possible the study of chaotic behavior in physically relevant models. Lyapunov analysis, applied to experimental systems [29], was also made possible by a combination of these numerical methods with ideas from nonlinear time series analysis [30]. As a result, it is nowadays widely recognized that Lyapunov exponents are a central tool of chaos theory, crucial for characterizing a number of interesting physical properties including dynamical entropies and fractal dimensions [31]. Their pivotal role in modern dynamical systems theory has been established by a fruitful exchange between a rigorous (and beautiful) mathematical theory and the algorithmic approaches essential for understanding many physical phenomena.

From the 1990s to the present, with the concomitant progress in both theoretical understanding and computer capabilities, there has been a progressive shift of interest from low dimensional towards high dimensional systems. This shift towards dynamics characterized by many degrees of freedom, possibly spatially organized and/or with several characteristic temporal scales, has been accompanied by the need for generalizations of the Lyapunov exponents. Moreover, further attention has also been paid to the so-called Lyapunov vectors, which provide information on geometrical properties of the tangent space characterizing stable and unstable directions [23]. This approach could prove extremely useful in systems with many degrees of freedom such as the models used, for instance, in atmospheric physics, where different forms of Lyapunov vectors (both infinitesimal and of finite size) have been proposed to generate suitable perturbations to be used for optimizing ensemble forecasting methods [32]. While some of these tools are routinely used in standard applications, a comprehensive understanding of Lyapunov vectors and their relation to different properties of spatiotemporal chaos is still lacking.

From this brief and largely incomplete historical tour it should be clear that Lyapunov analysis has had and still has a great impact on both fundamental and applied approaches to deterministic dynamical systems. Furthermore, some recent results have generated renewed attention on certain aspects of Lyapunov analysis and related applications in a number of different research communities. For instance, different types of Lyapunov vectors are regularly employed in the most diverse applications from applied mathematics and physics to atmospheric sciences. Depending on the context, researchers use covariant Lyapunov vectors (CLVs), singular Lyapunov vectors, bred vectors, finite-time Lyapunov vectors, etc. Similarly, for the Lyapunov exponents several generalizations are possible to account for, e.g., the multifractal nature of the invariant measure, the local fluctuations of stretching rates (finite-time Lyapunov exponents), to characterize the nonlinear regime of perturbation evolution (finite-size Lyapunov exponents) or their spatiotemporal dynamics (co-moving Lyapunov exponents). All these quantities often have different physical interpretations, are characterized by different levels of mathematical development and only provide access to partial pieces of information. Moreover, the scattered state of the present literature, with key contributions published in journals read by different communities (mathematicians, nonlinear and statistical physicists, fluid dynamicists and geophysicists), makes it difficult to develop a general picture. This special issue aims to offer an up-to-date view of current research on Lyapunov analysis, discussing both its mathematical theory and its applications to a number of different problems. Moreover, in order to facilitate the comparison and exchange of ideas and tools among different fields of research, contributions (either original or topical reviews) from researchers working in different disciplines have been selected for this issue.

After the compact review of the basic mathematical results on Lyapunov exponents by Lai-Sang Young, the special issue is organized into nine sections broadly focused on the following topics:

Large deviations and rare trajectories. Lyapunov exponents are mean quantities which characterize the sensitivity to initial conditions of typical trajectories. A large deviation theory of their finite time fluctuations, however, is relevant for the construction of a thermodynamic formalism of deterministic chaos. Moreover, the weighted sampling of extreme fluctuations allows one to access rare trajectories and phase–space topological structures.

Random matrices. Lyapunov exponents are suitable quantities to statistically characterize products of random matrices, with a number of applications to transfer matrix methods and, more generally, to the statistical mechanics of disordered systems. In particular, Lyapunov exponents have long played a central role in the theory of Anderson localization. These aspects are reviewed here, together with an original application to the transfer matrix.

Covariant Lyapunov vectors: theory and applications. CLVs constitute an intrinsic tangent space decomposition into stable and unstable directions associated with Lyapunov exponents. Recently, novel and efficient algorithms for computing CLVs have become available, turning them into a practical tool for the study and characterization of chaotic dynamical systems, especially when many degrees of freedom are considered. Applications include—but are not limited to—the characterization of collective chaotic modes and the study of intermittency in shell models for turbulence. This section also includes a pedagogical discussion of CLVs in low dimensional Hamiltonian systems.

Time series analysis. The computation of Lyapunov exponents from time series data is a very active field of research playing a crucial role in experimental studies, with applications ranging from nonlinear optics to neuroscience. The need to reconstruct a higher dimensional phase space from scalar or lower dimensional data poses fundamental issues, introducing spurious Lyapunov exponents and limiting the accuracy when quantities such as the Kaplan–Yorke-dimension or the Kolmogorov–Sinai entropy are concerned. Recent advances based on CLVs, however, may help in identifying such spurious exponents, increasing the accuracy of embedding techniques.

Many-body systems. The study of many-body dynamical systems plays a fundamental role in the debate on the foundations of statistical physics. In this respect, the discovery of long wavelength tangent space modes associated with close-to-zero Lyapunov exponents, the so-called hydrodynamic Lyapunov modes, has raised hopes that a clear link could be established between hydrodynamic transport coefficients and microscopic dynamical properties. In this section, three contributions discuss Lyapunov analysis applied to many-body systems such as hard disks and lattices of interacting classical spins.

Spatiotemporal chaos and hyperbolicity. Spatially extended chaotic systems pose a number of new issues when spatiotemporal chaos is considered, ranging from questions on its extensivity properties in the thermodynamic limit to the need for new tools to describe the spatial propagation of infinitesimal perturbations. Related questions are concerned with the geometrical structure of tangent space and the study of (local) violations of hyperbolicity, a key property for establishing rigorous mathematical results. This section, while far from being exhaustive, collects four different contributions representative of the current status of research in spatiotemporal chaos.

Weakly chaotic systems. Hamiltonian systems typically display a complex interplay of chaotic and regular orbits. Moreover, for many orbits chaos can be very 'weak', characterized by small Lyapunov exponents and/or very slow decay of correlation functions. In such cases even the proper identification of the orbit character constitutes an issue. Additionally, in time-dependent Hamiltonian systems, orbit character may change with time. In all such situations Lyapunov analysis is severely challenged and requires suitable modifications, discussed here by means of an example from celestial mechanics.

Predictability in geophysical and multi-scale systems. Multi-scale systems pose further challenges due to the interplay between different spatial and/or temporal scales. Many phenomena of interest in multi-scale systems involve finite amplitude perturbations, whose nonlinear evolution outside of tangent space may be described in terms of the finite-size Lyapunov exponent and bred vectors. The issue of predictability in these systems is of fundamental importance in geophysical applications, where ensemble forecasting and data assimilation techniques based on different kinds of Lyapunov vectors, both infinitesimal and of finite size, are routinely used. This section reviews some basic results in this field and presents two original contributions on Lyapunov vectors applied to atmospheric problems.

Lagrangian coherent structures and transport in fluid flows. Lagrangian coherent structures are topological structures playing a prominent role in the studies, both theoretical and applied, of transport, stirring and mixing properties in fluid flows. This section contains two contributions, one more theoretical in nature and a second with biological applications, where Lagrangian coherent structures and their effect on transport and mixing are analyzed through finite-size Lyapunov exponents.

While this special issue, in its very nature, cannot be fully exhaustive, we hope that it will provide a clear and up-to-date picture of the theory and applications of Lyapunov analysis, further stimulating fruitful debate across a number of related research fields.

Acknowledgments

We would like to acknowledge the *Journal of Physics A* editorial staff, in particular A Haywood, R Gillan and E O'Callaghan for their continuous assistance which made this special issue possible.

References

[1] Lorenz E 1993 *The Essence of Chaos* (Seattle, WA: University of Washington)
[2] Poincaré H 1890 *Acta Math.* **13** 1
[3] Poincaré H 1892 *Les méthods nouvelles de la mécanique céleste* vol 1 (Paris: Gauthier-Villars)
Poincaré H 1893 *Les méthods nouvelles de la mécanique céleste* vol 2 (Paris: Gauthier-Villars)
Poincaré H 1899 *Les méthods nouvelles de la mécanique céleste* vol 3 (Paris: Gauthier-Villars)
Poincaré H 1992 *New Methods of Celestial Mechanics* (New York: American Institute of Physics) (Engl. transl.)
[4] Hadamard J 1898 *J. Math. Pures Appl.* **4** 27
[5] van der Pol B 1927 *Phil. Mag.* **3** 65
[6] Fermi E, Pasta J and Ulam S 1955 Studies of nonlinear problems *Technical Report LA-1940* (Los Alamos, NM: Los Alamos Scientific Laboratory)
[7] Lorenz E N 1963 *J. Atmos. Sci.* **20** 130
[8] Hénon M and Heiles C 1964 *Astron. J.* **69** 73
[9] Kolmogorov A N 1958 *Dokl. Akad. Nauk SSSR* **119** 861
[10] Kolmogorov A 1959 *Dokl. Akad. Nauk SSSR* **124** 754
[11] Chirikov B 1959 *At. Energ.* **6** 630
[12] Smale S 1967 *Bull. Am. Math. Soc.* **73** 747
[13] Ruelle D and Takens F 1971 *Commun. Math. Phys.* **20** 167
[14] Li T Y and Yorke J A 1975 *Am. Math. Mon.* **82** 985
[15] Feigenbaum M J 1978 *J. Stat. Phys.* **19** 25
[16] Oseledets V I 1968 *Trans. Moscow Math. Soc.* **19** 197
[17] Birkhoff G D 1931 *Proc. Natl Acad. Sci. USA* **17** 656

[18] von Neumann J 1932 *Proc. Natl Acad. Sci. USA* **18** 70
[19] Krylov N S 1979 *Works on the Foundations of Statistical Physics* (Princeton, NJ: Princeton University Press)
[20] Anosov D V and Sinai Y G 1967 *Russ. Math. Surv.* **22** 103
[21] Pesin Y B 1976 *Sov. Math. Dokl.* **17** 196
[22] Sinai Y G 1972 *Russ. Math. Surv.* **27** 21
[23] Ruelle D 1979 *Publ. Math. l'IHES* **50** 27
[24] Bowen R 1975 *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms (Lecture Notes in Mathematics* vol 470*)* (Berlin: Springer)
[25] Bowen R and Ruelle D 1975 *Invent. Math.* **29** 181
[26] Shimada I and Nagashima T 1979 *Prog. Theor. Phys.* **61** 1605
[27] Benettin G, Galgani L, Giorgilli A and Strelcyn J M 1980 *Meccanica* **15** 9
[28] Grassberger P and Procaccia I 1984 *Physica* D **13** 34
[29] Wolf A, Swift J B, Swinney H L and Vastano J A 1985 *Physica* D **16** 285
[30] Takens F 1981 Detecting strange attractors in turbulence *Dynamical Systems and Turbulence* (*Lecture Notes in Mathematics* vol 898) ed D A Rand and L S Young (Berlin: Springer) p 366
[31] Eckmann J P and Ruelle D 1985 *Rev. Mod. Phys.* **57** 617
[32] Legras B and Vautard R 1996 A guide to Lyapunov vectors *Predictability (ECWF Seminar* vol 1*)* ed T Palmer (Reading: ECMWF) p 141