

Diffusion, Peer Pressure, and Tailed Distributions

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We present a general, physically motivated nonlinear and nonlocal advection equation in which the diffusion of interacting random walkers competes with a local drift arising from a kind of peer pressure. We show, using a mapping to an integrable dynamical system, that on varying a parameter the steady-state behavior undergoes a transition from the standard diffusive behavior to a localized stationary state characterized by a tailed distribution. Finally, we show that recent empirical laws on economic growth can be explained as a collective phenomenon due to peer pressure interaction.

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Fluctuations—measured by deviations from the mean value of an observable—of large systems often satisfy a Gaussian distribution. A classic example is a linear diffusion process [1–4] which has numerous applications in many branches of science [5]. However, there are many situations in Nature in which the probability of occurrence of a fluctuation of size $|\Delta|$ is proportional to $\exp(-|\Delta|^p)$, with p taking on a value greater than or equal to 1 (exponential tails), but smaller than 2 [6]. Examples include the temperature distribution of a Rayleigh-Benard system [7,8], disordered systems such as foams [9], and glasses or granular materials [10], with even fatter tails in financial data [11,12].

We present here a physically motivated advection equation and its exact steady-state solution. The equation has a drift term, originating from a kind of peer pressure, of the same nature as that due to mechanisms of chemotactic signaling by microorganisms [13,14] or by the onset of cooperation in social groups [15,16], or by competition between economic units [17]. We show that, on varying a parameter, there is a transition from diffusive behavior to a localized stationary state characterized by an exponential distribution.

Consider a diffusional process in which random walkers move either to the right or to the left randomly and with no bias. In the long time limit, the distribution of walkers becomes flat and infinitely spread out. The mechanism for collective self-organization in our model is a kind of peer pressure. The nonlinearity arises from an interaction (of spatial range ξ) between the walkers, which leads to a drift or a bias term which opposes the diffusional spreading and promotes aggregation. The basic idea is one in which a walker perceives the populations of other walkers over a range ξ both right and left of her own location and has a drift in the more crowded direction. In the limit of small ξ , one obtains regular diffusive behavior, whereas a new class of steady-state behavior characterized by non-Gaussian distributions is obtained when ξ is sufficiently large.

This idea may be encapsulated in a nonlinear equation,

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}[v(x, t)P] + D\frac{\partial^2 P}{\partial x^2}, \quad (1)$$

where $P(x, t)$ is the distribution function of the locations of the random walkers at time t with a drift velocity $v(x, t)$. The nonlinearity arises because v is itself a nonlocal function of P and is given by

$$v(x, t) = \frac{\lambda}{2} \frac{\Phi_+(x, t) - \Phi_-(x, t)}{\Phi_+(x, t) + \Phi_-(x, t)}, \quad (2)$$

where

$$\Phi_+(x, t) = \int_x^\infty dy e^{(x-y)/\xi} P(y, t), \quad (3)$$

$$\Phi_-(x, t) = \int_{-\infty}^x dy e^{(y-x)/\xi} P(y, t), \quad (4)$$

and λ sets the scale for the drift velocity. Physically, $\Phi_+(x, t)$ and $\Phi_-(x, t)$ are measures of the population, within a range ξ , to the right and the left of location x , and the drift velocity is then a normalized imbalance between these two populations.

In colonies of social individuals, each member can release chemical substances in the environment (for example, pheromones) revealing the presence of food sources in a given area. The individuals in the colony are able to detect higher concentrations of the chemical signal and are attracted toward the food and move accordingly. The plausible assumption that the concentration of the chemicals in a given region is proportional to the local density of individuals translates into an effective interaction between members of the colony, and leads to the nonlocal drift term in Eq. (1). The interaction causes the net migration of an individual in the direction of the higher local density. The length scale, ξ , is a measure of the range of the biological sensory system of individuals. Thus, Eq. (1) describes

Brownian motion with a bias that mimics attractive chemotactic signaling [13] and promotes aggregation leading to a drift controlled by the population difference in localized regions on the right and the left of x . Physically, the standard Gaussian solution is obtained when there is no drift ($\lambda = 0$) or when the interaction length scale goes to zero ($\xi \rightarrow 0$).

The existence of a transition in the steady-state behavior is revealed by a linear stability analysis of the homogeneous state. One can look for solutions of Eq. (1) in the form $P \sim \exp[ikx - \gamma(k)t]$ representing spatially periodic perturbations. A direct substitution into Eq. (1) provides the exponential growth rate,

$$\gamma(k) = D(1 - r)k^2, \quad (5)$$

where

$$r = \frac{\lambda \xi}{2D}. \quad (6)$$

Hence, the homogeneous state is stable [$\gamma(k) > 0$] when $r < 1$. For $r > 1$, on the contrary, small perturbations grow with time and the homogeneous state becomes unstable. Thus, r plays the role of a control parameter of a phase transition.

A mapping to a Hamiltonian dynamical systems allows us to derive the stationary solution of Eq. (1). Indeed the steady-state distribution ($\dot{P}_s = 0$) satisfies the ordinary differential equation,

$$P'_s = \frac{\lambda}{2D} \frac{\Phi_+ - \Phi_-}{\Phi_+ + \Phi_-} P_s, \quad (7)$$

where the prime denotes the derivative d/dx . The steady-state values of Φ_+ and Φ_- , in turn, satisfy

$$\Phi'_+ = \Phi_+/\xi - P_s, \quad (8)$$

$$\Phi'_- = -\Phi_-/\xi + P_s, \quad (9)$$

obtained by taking derivatives of Eqs. (3) and (4), respectively. It is convenient to introduce the new variables $Q = \Phi_+ + \Phi_-$ and $\Pi = \Phi_+ - \Phi_-$, so that Eqs. (8) and (9) may be written as

$$Q' = \Pi/\xi, \quad (10)$$

$$\Pi' = Q/\xi - 2P_s. \quad (11)$$

Then, from Eq. (7) we have

$$\frac{P'_s}{P_s} = \frac{\lambda}{2D} \frac{\Pi}{Q}, \quad (12)$$

which, with the aid of Eq. (10), can be integrated with respect to x , yielding

$$\ln P_s - r \ln Q = \text{const.} \quad (13)$$

Thus, the steady distribution is $P_s(x) = CQ(x)^r$, where C is a normalization factor.

The system of ordinary differential equations [Eqs. (10) and (11)] can be regarded as an integrable dynamical system (whose energy is conserved):

$$Q' = \Pi/\xi, \quad (14)$$

$$\Pi' = Q/\xi - 2CQ^r, \quad (15)$$

describing the motion of a particle in a one-dimensional potential,

$$V(Q) = -\frac{Q^2}{2\xi} + \frac{2C}{1+r} Q^{r+1}. \quad (16)$$

The spatial coordinate in the original system, x , plays the role of time in the dynamical system. The equation of motion can be derived from the Hamiltonian,

$$H(\Pi, Q) = \frac{\Pi^2}{2\xi} + V(Q), \quad (17)$$

which is a constant of motion, $H(\Pi, Q) = E$ in the dynamical system. Its constant value E is fixed by the boundary conditions on $P_s(x)$. For instance, the requirement that P_s be normalized for infinite systems implies that $Q(x = \pm\infty) = 0$ and $\Pi(x = \pm\infty) = 0$, so that E is automatically set to zero.

This mechanical analogy allows one to obtain a complete understanding of the model with the key factor being the shape of the potential energy [Eq. (16)]. A qualitative change in the nature of $V(Q)$ occurs (see Fig. 1) when r crosses the value 1, i.e., for $\lambda\xi = 2D$, signaling a transition in the system's behavior. When $r < 1$ (dashed curve of Fig. 1), only the solution $Q(x) = 0$ is physically acceptable and, thence, $P_s(x) = 0$. This corresponds to a probability distribution which spreads out and vanishes as in standard diffusion. When $r > 1$, a nontrivial trajectory of the Hamiltonian system is possible (solid line of Fig. 1) with Q vanishing when the fictitious time $x \rightarrow \pm\infty$. This

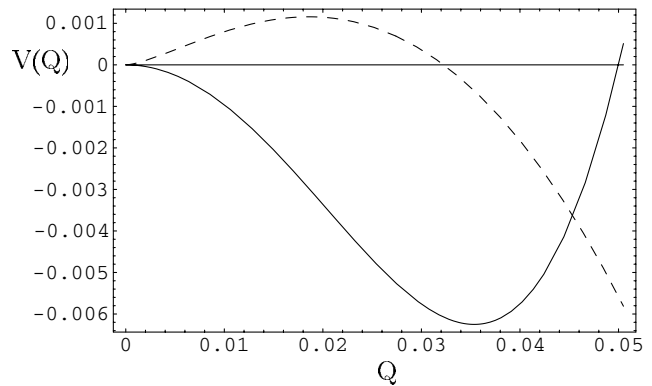


FIG. 1. Plot of the potential $V(Q)$ for $\lambda = 10$, $D = 1/4$, and $\xi = 1/31$ ($r = 20/31$ —dashed line) and $\lambda = 30$, $D = 1/4$, and $\xi = 1/20$ ($r = 3$ —solid line).

trajectory corresponds to a nontrivial asymptotic distribution (see Fig. 2). The solution for $r > 1$ is obtained by integrating the differential equation,

$$Q' = 1/\xi \sqrt{Q^2 - gQ^{r+1}}, \quad (18)$$

which is obtained by solving for Π as a function of Q (for $E = 0$) and substituting the result into Eq. (14), with $g = 4C\xi/(1+r)$. This leads to

$$P_s(x) = A(r) \cosh\left[\frac{r-1}{2\xi}(x-x_0)\right]^{-2r/(r-1)}, \quad (19)$$

where $A(r) = (r-1)\Gamma[(3r-1)/(2r-2)]/\{2\xi\sqrt{\pi}[r/(2r-2)]\}$ enforces normalization and x_0 is a constant which arises from the integration of Eq. (18).

The presence of a single control parameter in the model can be deduced by a dimensional argument. Indeed, through the rescaling, $X = x/\xi$ and $\tau = Dt/\xi^2$, the coefficient $\lambda/2$ in Eq. (2) becomes r , and the dependence on ξ disappears. Indeed, when expressed in terms of X , the stationary state distribution depends only on r .

The change of behavior at $r = 1$ is a proper phase transition. Indeed, it is accompanied by *spontaneous breaking* of the translation symmetry $x \rightarrow x + a$ of Eq. (1) for $r > 1$. The location, x_0 , of the distribution's center is dynamically selected and it depends, in a complex manner, on the initial conditions. Furthermore, one may define a localization length, ℓ , which is the spread of the distribution $P_s(x)$, and which diverges as $\ell = |1-r|^{-\nu}$ with a critical exponent, $\nu = 1$, signaling the transition to the delocalized, translationally invariant steady state. The dynamics of relaxation to the steady state is characterized by an exponential decay with a characteristic time diverging as one approaches the transition.

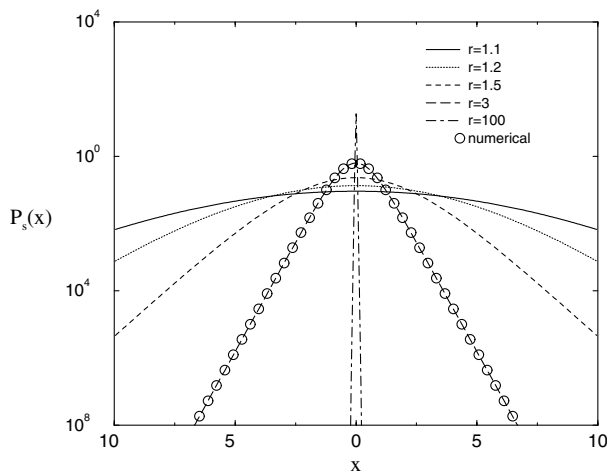


FIG. 2. The scaled asymptotic distribution P_s for $\lambda = 1$ and several values of r . Solid lines refer to the exact solution, Eq. (19). The open circles are obtained on numerically integrating the differential Eq. (1) in the large t limit for $r = 3$.

Another interesting feature of the system is the possibility of spatially periodic steady-state solutions of Eq. (1) for suitable boundary conditions, corresponding to periodic orbits of the associated Hamiltonian systems with negative energy. However, unlike Eq. (19), such solutions are expected to be unstable to perturbations.

Many examples in which aggregation processes compete with diffusional tendencies can also be found in economics. Competition between economic units may lead to effective “peer” pressure captured by the drift in our model, whereas the diffusion term accounts for individual idiosyncratic behavior. Such a situation could arise in charitable giving, when it is not anonymous. A given individual would, in an attempt to keep up with the Joneses, adjust his contribution in the direction of higher or lower giving, depending on what his peers are doing. Of course, in this instance, the range of interactions, ξ , is limited because it is not feasible for an ordinary individual to mimic the behavior of a Bill Gates. In other contexts, tent-shaped distributions, which deviate sharply from a Gaussian, have been observed for the growth rates of firms, nations’ gross domestic product (GDP) and complex organizations ([17] and references therein). We have analyzed the time series of the growth rates of commodity prices across different Italian cities (see Fig. 3), and we find similar exponential tails.

The growth rate of an economic indicator b —such as the turnover of an organization, a nation’s GDP, or the price of

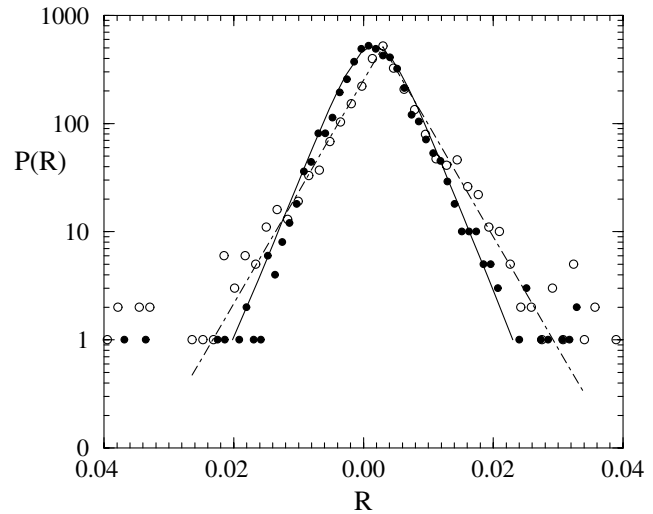


FIG. 3. Histograms of the monthly growth rate R of two consumer price indices collected among 78 Italian Provinces within the period January 1995–June 2001. The data are from “Italy’s National Statistical Institute” (ISTAT). The filled circles refer to expenditures for food and nonalcoholic beverages, whereas the open circles show the data for miscellaneous goods and services [18]. Both sets of data show the typical “tent-shaped” distribution in sharp contrast with the prediction of a Gaussian shape of Gibrat theory. The solid and dot-dashed lines indicate a fit with Eq. (19).

a commodity in a city—is defined as

$$R(t) = \ln \frac{b(t+1)}{b(t)},$$

where t and $t+1$ represent two consecutive time units (days, months, years, etc.). The classical theory, due to Gibrat [19], predicts that $R(t)$ has a Gaussian distribution. Empirical data, however, shows a tent-shaped distribution on a logarithmic scale. In the case of firms, Stanley *et al.* [17,20] have proposed an explanation in terms of the internal hierarchical organization of noninteracting firms.

Our model provides a different explanation in which it is the interaction among firms which is responsible for the deviation from a Gaussian distribution. In a way, the peer pressure mechanism can be regarded as a *regression towards the mean* effect [21]. This term usually refers to an explicit “attraction” of $R(t)$ towards a slowly varying moving average. Such an effect, however, is unable to reproduce tent-shaped distributions without *ad hoc* assumptions on the precise nature of the attraction potential.

Here, instead, we refer to a collective phenomenon of “regression towards the population mean”: Each unit is attracted towards the instantaneous population average. It is the effective interaction among agents that generates a sort of “feedback” that induces the regression towards the mean mechanism. From a microeconomics point of view, it is reasonable to assume that the behavior of an economic agent is influenced by other agents. For example, a firm will strive to increase its growth rate if other peer firms grow at a higher rate. This interaction introduces correlations in $R(t)$ that destroy the diffusive features and produces the characteristic exponential tails.

In summary, we have introduced and studied, both analytically and numerically, a one-dimensional diffusion equation with nonlinear and nonlocal features. The asymptotic evolution of the equation leads to the walkers being attracted toward a state which exhibits a well-defined non-Gaussian distribution characterized by exponential tails independent of the nature of the initial condition. This mechanism of peer pressure may provide the basis for the development of more realistic models of self-aggrega-

tion and self-organization in cooperative states of populations of interacting individuals.

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