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Approximation of chaotic systems in terms of Markovian processes

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Abstract

We mimic the deterministic chaos appearing in low dimensional systems in terms of a product of suitable random matrices by considering Markov trials. From the high order Markovian processes it is possible to obtain a good approximation for the Lyapunov exponent and the decay rate of the correlation functions.

1. Introduction

The fact that experimental or numerical measurements of the state of a physical system cannot be determined with infinite accuracy and the sensitive dependence on initial conditions in chaotic systems [1], implies the possibility and necessity to introduce probabilistic methods in the study of chaotic dynamical systems. The idea of using stochastic processes to describe chaotic behaviour is rather old. Moreover, many authors have studied the effects of a noisy term on chaotic dynamics [2].

We shall discuss in this paper how a treatment of a chaotic dynamical system in terms of a Markov process allows one to describe some features of the chaotic behaviour (namely Lyapunov exponents and correlation decay rate).

Let us consider a nonlinear deterministic evolution law,

$$\dot{x}(t) = F(x(t)) \tag{1}$$

or in the discrete time case,

$$x(t+1) = G(x(t)). \tag{2}$$

The sensitive dependence on initial conditions can be quantitatively measured by a set of numbers called Lyapunov exponents [3]. The maximum Lyapunov exponent (MLE) of a trajectory is defined as

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \log \|A^t w\| \tag{3}$$

where A^t is the linear evolution operator of tangent vectors, and w is a generic tangent vector. The inverse of the MLE gives the characteristic time scale in which forecasts are still efficient.

For the discrete time systems that we consider, A^t is a product of matrices,

$$A^t = \prod_{s=1}^{t} A(s), \tag{4}$$

where the $A(s)$ are the Jacobian matrices of the transformation (2),

$$A_{ij}(s) = \frac{\partial G_i(x(s))}{\partial x_j}. \tag{5}$$

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So the evaluation of Lyapunov exponents for deterministic maps requires products of the Jacobian matrices computed at different times.

The similarities between the behaviour of a strongly chaotic dynamical system and the behaviour of a stochastic process, suggest using a random process to mimic some aspects of the chaotic dynamics. For example in the computation of the MLE one can try to replace the product of matrices $A(s)$ with a product of matrices extracted by a suitable probabilistic law; basically the randomness of $A(s)$ mimics the chaoticity of the system. Let us stress that this approximation is rather crude but not trivial at all.

In strong chaos cases it is possible to take products of independent matrices to reproduce values of the MLE in good agreement with those directly calculated by the real dynamics of the system. Chirikov [4] has used this approximation for the MLE of the standard map, when the nonlinear parameter K is large enough. Benettin [5] has found the scaling laws of the MLE in terms of the perturbation parameter for billiard systems. Paladin et al. [6] have shown that products of independent RM provide a satisfactory result for the whole Lyapunov spectrum in symplectic systems.

In all these examples strong chaoticity makes the effects of correlations irrelevant, but generally these correlations do not decay very fast, so a more detailed approach is needed. In order to include correlations effects, it is convenient to introduce Markovian products of the RM. Crisanti et al. [9], by taking a one-step Markovian RM to estimate the MLE for the Lozi map, have shown that it is possible to obtain a good improvement compared to the independent RM approximation.

In this paper we discuss a generalization of this method by considering higher order Markovian processes [8] in which the memory is extended to the k last trials. In Section 2 we discuss the general Markovian RM approximation and how it is possible to estimate relevant quantities from a k-order Markovian process.

In Section 3 we show the application of the method to the concrete cases of the Lozi and Hénon maps.

Section 4 is devoted to the approximation of correlation function decay directly from the Markovian transition matrix of the process used to mimic the chaotical behaviour of the system.

Section 5 contains discussions and conclusions.

2. Markovian approach in the RM approximation

In the Markovian RM approximation the tangent matrices $A(s)$ in the product (4) are extracted according to a Markovian process.

A Markovian stochastic process with discrete states is completely characterized by the transition matrix whose elements $P_{i,j}$ represent the probability to observe a transition from the state i to the state j. Markovian processes of order k are a straightforward extension in which the probability to have a trial at time t depends on the last $t-k$ trials only. So a k-order Markovian process is characterized by

$$P\{\sigma|\sigma_k,...,\sigma_1\} = \frac{P\{\sigma,\sigma_k,...,\sigma_1\}}{P\{\sigma_k,...,\sigma_1\}}, \tag{6}$$

which represents the probability of a transition from the sequence $(\sigma_1,...,\sigma_k)$, to the state σ. By introducing the space of all sequences of length k $\alpha = (\sigma_1,...,\sigma_k)$ it is possible to treat these processes as one-step Markovian processes. In particular if we define $\alpha = (\sigma_1,...,\sigma_k)$ and $\beta = (\sigma_2,...,\sigma)$ one can rearrange the transition probabilities in matrix form, $P^{(k)}_{\alpha,\beta}$, like in the one-step case.

The estimation of the MLE in the k-order Markovian RM approximation is obtained by replacing the deterministic product of tangent matrices with a product of matrices having the same form of tangent matrices but extracted according to the k-order Markovian rule. To perform these products the knowledge of the whole set of transition probabilities is needed. Generally explicit expressions for these quantities are not accessible so their computation must be numerically done, by observing the deterministic dynamics of the system. Of course the proposed method is not useful from a practical point of view but it can be conceptually relevant to understand the important role of correlations.

From the generalized transition matrix, associated to the dynamical system, we can obtain further information. For example we can compute the Shannon entropy [10] of the k-order Markovian process which can be considered an approximation of the Kolmogorov-Sinai (K-S) entropy of the dynamical system,

$$h(k) = -\sum_{\alpha,\beta} p^{(k)}_{\alpha} P^{(k)}_{\alpha,\beta} \log P^{(k)}_{\alpha,\beta}, \tag{7}$$

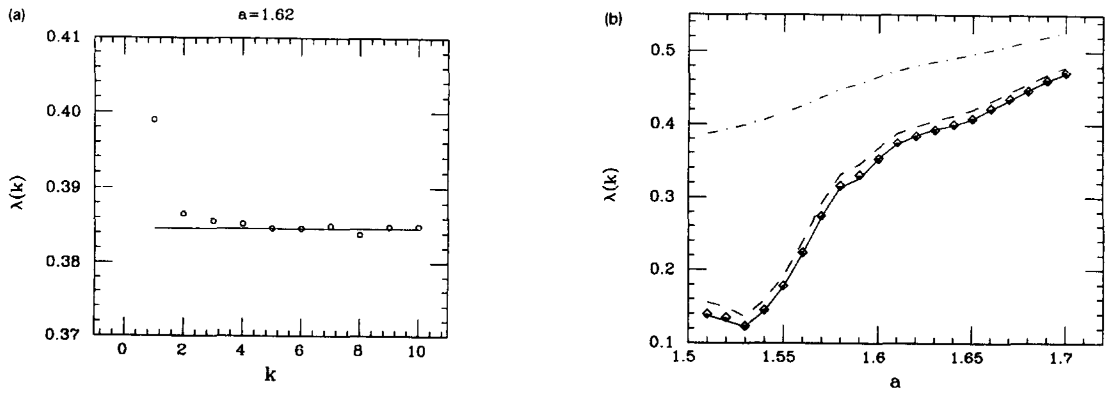

Fig. 1. (a) MLE of the Lozi map (horizontal line) and its Markovian approximations $\lambda(k)$ (○) versus k for $a = 1.62$. (b) Markovian RM approximation for the Lozi map of order $k = 0$ (dash-dotted line), $k = 1$ (dashed line), $k = 2$ (◇) and $k = 3$ (•) compared with the numerically computed MLE (solid line) at various values of the parameter a.

where $\{p_1^{(k)}, p_2^{(k)}, ..., p_r^{(k)}\}$ are the probabilities of the states $\{\alpha_1, ..., \alpha_r\}$ and they are related to the transition matrix $P_{\alpha,\beta}^{(k)}$ by the relation

$$p_\beta^{(k)} = \sum_\alpha p_\alpha^{(k)} P_{\alpha,\beta}^{(k)}. \tag{8}$$

By the number $N(k)$ of all the strings of length k that appear in a typical symbolic sequence generated by the dynamics, one has a k-order estimation of the topological entropy [1] of the system through the quantity

$$h_{\text{top}}(k) = \log \frac{N(k+1)}{N(k)}. \tag{9}$$

Finally the formal similarity between the correlation functions of a Markov process and of a chaotic dynamical system, suggests that Markovian correlation decay (related to the second eigenvalues of the stochastic matrix) is a good approximation, as the memory degree k increases, of the decay of the dynamical system correlation functions.

3. Two applications

Let us now apply the previous general considerations to two concrete cases: the Lozi and Hénon systems.

3.1. The Lozi map

The Lozi map [11,12] is a transformation of the plane on itself defined by the equations

$$x(t+1) = -a|x(t)| + y(t) + 1, \tag{10}$$

$$y(t+1) = bx(t). \tag{11}$$

In the parameter region $1.5 < a \leqslant 1.7$ and $b = 0.5$, the map shows a chaotic behaviour, i.e. a strange attractor and a positive maximum Lyapunov exponent. The map (11) is a rather natural candidate for the approach discussed in Section 3, since the tangent map is defined by only two matrices,

$$A_\pm = \begin{pmatrix} \mp a & 1 \\ b & 0 \end{pmatrix}, \tag{12}$$

where A_+ occurs when $x \geqslant 0$, and A_- when $x < 0$. Since the matrices are not commuting, it is not easy to apply an analytical treatment, so the product (4) must be numerically performed.

The k-order transition probabilities (6) for A_- and A_+ are numerically estimated by the frequencies of the sequences of length k of these matrices. This is equivalent to introducing the natural binary encoding of the system trajectories by sequences of symbols $+$ $(x(t) \geqslant 0)$ and $-$ $(x(t) < 0)$.

Then we carry out the product of matrices (12) according to the k-order Markovian stochastic dynamics defined by the transition probabilities obtained.

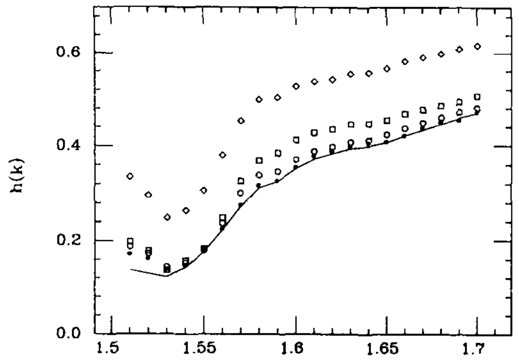

Fig. 2. Shannon entropy of the Markovian process used to estimate the MLE for the Lozi map, of orders: $k = 1$ (◇), $k = 4$ (□), $k = 7$ (○), $k = 10$ (•), compared to the numerically computed MLE (solid line).

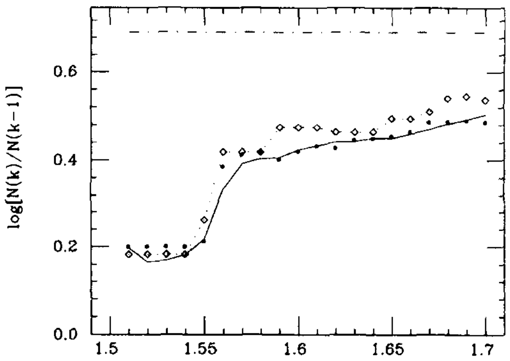

Fig. 3. Topological entropy estimation for the Lozi map (Eq. (9)) for $k = 1$ (dash-dotted line), $k = 7$ (◇), $k = 10$ (•) and the generalized Lyapunov exponent computed at $q = 1$ (solid line), as functions of the parameter a.

For $k = 0$ the method reproduces the independent RM approximation and for $k = 1$ we obtain the results of Crisanti et al. [9]. By increasing the memory of the Markov process used a better approximation is achieved (see Figs. 1a and 1b).

In addition the knowledge of the k-order transition matrix allows a k-order evaluation of the metric and topological entropy (7), (9) (see Figs. 2 and 3 respectively).

3.2. *The Hénon map*

The random matrix approximation for the Hénon map [13,14],

$$x(t+1) = -ax(t)^2 + y(t) + 1, \tag{13}$$

$$y(t+1) = bx(t), \tag{14}$$

is more delicate compared to the Lozi case. In fact the linearized tangent map defined by the matrices

$$A(x) = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix} \tag{15}$$

is a continuous function of the coordinate x. Nevertheless the treatment performed for the Lozi map can be repeated introducing a suitable discretization of the tangent map. We have taken a simple partition of the Hénon attractor by dividing it in r elements,

$$B_i = \{(x,y) \in \mathbb{R}^2 | x_i \leqslant x < x_{i+1}\}, \tag{16}$$

where the set $\{x_i\}$ realizes a uniform partition, in r elements, of the segment $[-1.34, 1.34]$ on the x axis; we assign the constant tangent matrix

$$A(\overline{x}_i) = \begin{pmatrix} -2a\overline{x}_i & 1 \\ b & 0 \end{pmatrix} \tag{17}$$

to each element of the partition, where $\overline{x}_i$ indicates the center of the interval $[x_i, x_{i+1}]$. Typically we use $r = 10$ and $r = 20$ elements. With this set of matrices we perform the independent and the one-step Markovian approximation for the MLE. As in the Lozi map case, the determination of the transition probabilities for matrices (17) must be numerically done. The results in Fig. 4 show that trivial refinements of the partition (i.e. changing r from 10 to 20) do not give an effective improvement in the independent RM approximation.

Let us note that even for the Hénon map using a trivial partition ($x < 0, x > 0$) but a rather large value of k one has a reasonable estimation of the MLE from $h(k)$. In Fig. 5 we can observe that the convergence of $h(k)$ to the MLE is rather good, apart from some values of a.

4. Correlation decay

The k-order transition matrix $P^{(k)}_{\alpha,\beta}$ of the process used to perform the RM approximation should represent a rough approximation of the Perron–Frobenius

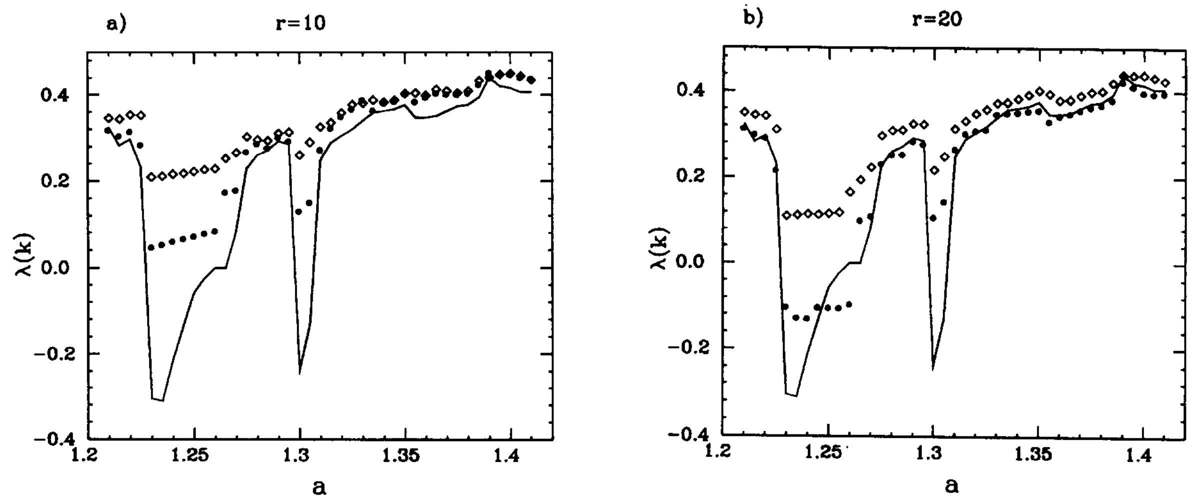

Fig. 4. Numerical computation of the MLE for the Hénon map (solid line), its independent RM approximation (○) and one-step Markovian approximation (•); the partition (16) is taken with $r = 10$ elements (a) and with $r = 20$ elements (b).

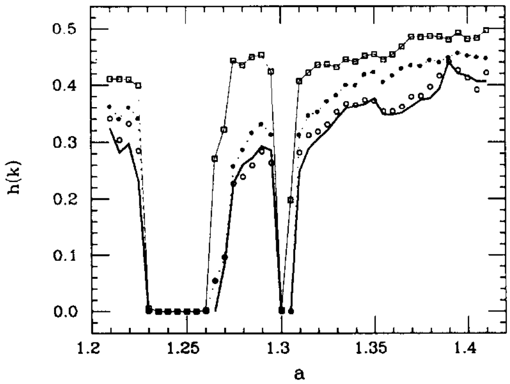

Fig. 5. Entropy estimation (Eq. (7)) of the Hénon map for: $k = 4$ (□), $k = 7$ (•) and $k = 10$ (○), compared to the positive part of the MLE (fat solid line).

(P–F) operator [15] as the analogy between properties of this operator and the transition matrix suggests. Since the P–F operator rules the correlation function behaviour [16,17], the matrix $P^{(k)}_{\alpha,\beta}$ should contain information concerning correlation functions of the system. To be more explicit, one expects that the asymptotic decay rates of the correlation functions for a Markovian stochastic process and for a "generic" dynamical system have the same form,

$$|C(\tau)| \sim |\nu_1|^\tau \sim \exp(\log|\nu_1|\tau), \tag{18}$$

where in the former case ν_1 represents the second eigenvalue of the transition matrix, in the latter the second eigenvalue of the P–F operator (in a dynamical system the P–F operator assumes the role of transition matrix).

According to these considerations, we expect that the matrix $P^{(k)}_{\alpha,\beta}$ provides at least the correlation function decay $\log|\nu_1^{(k)}|$.

We can evaluate the correlation function of coordinate x using standard algorithms,

$$C_x(\tau) = \frac{1}{T-\tau}\sum_{s=1}^{T-\tau}[x(s)-\overline{x}]\,[x(s+\tau)-\overline{x}]. \tag{19}$$

For large τ the slope of the linear asymptotic behaviour of $\log|C(\tau)|$ provides the decay coefficient γ.

We have compared this decay rate with $\log|\nu_1^{(k)}|$ obtained directly from the second eigenvalue of the matrix $P^{(k)}_{\alpha,\beta}$; typical results for some values of the parameter a are shown in Figs. 6a and 6b for the Lozi system and in Figs. 7a and 7b for the Hénon system.

5. Remarks and conclusions

We have investigated the random matrix approximation for two typical low dimensional dynamical systems, i.e. the Lozi and the Hénon maps, in which

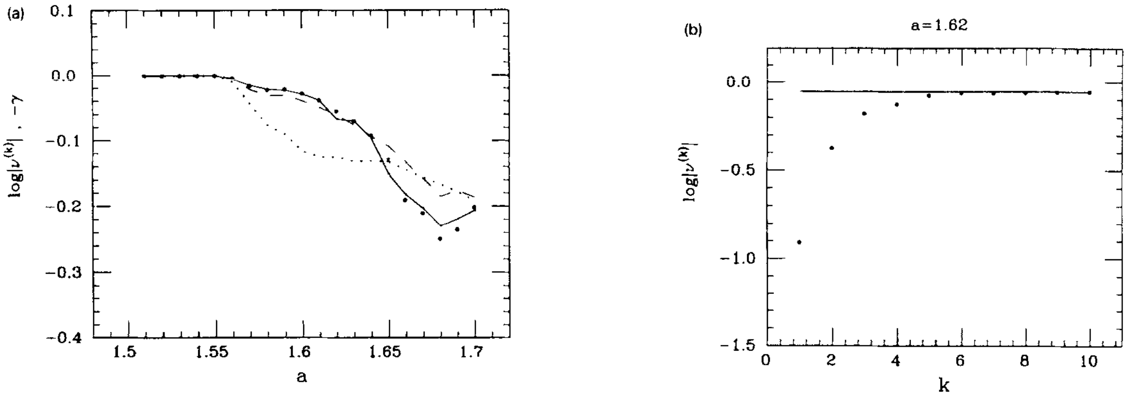

Fig. 6. (a) Lozi map: decay coefficient $-\gamma$ (solid line) and its approximations ($\log|\nu^{(k)}|$) for: $k = 4$ (dotted line) $k = 7$ (dashed line) and $k = 10$ (•) versus parameter a . (b) Lozi map: approximation of decay coefficient $-\gamma$ (horizontal line) with the second eigenvalue of the matrix $P_{\alpha,\beta}$ (•) for $a = 1.62$.

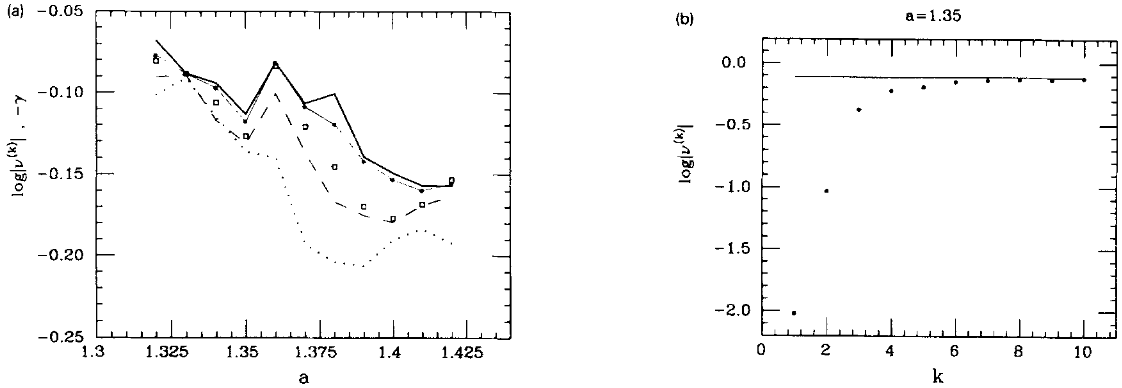

Fig. 7. (a) Hénon map: decay coefficient $-\gamma$ (fat solid line) and its approximations ($\log|\nu^{(k)}|$) for: $k = 7$ (dotted line), $k = 8$ (dash-dotted line), $k = 9$ (□) and $k = 10$ (•) at various values of the parameter a . (b) Hénon map: approximation of the decay coefficient $-\gamma$ (horizontal line) with the second eigenvalue of the matrix $P_{\alpha,\beta}$ (•) for $a = 1.35$.

the simple independent RM approximation does not work since it neglects correlations. We have that the "correct" RM approach should be of Markovian type. Since the transition probabilities are not a priori known the method is not useful for practical purposes. Nevertheless it shows in a simple way that for the systems considered a Markovian approach is needed to reproduce good results. The application to the Hénon map shows that the order of the Markovian process used (i.e. the memory) is more relevant than the structure of the partition. In other words taking processes with greater memory is more convenient than refining discretization. Another interesting outcome of this Markovian analysis is the information contained in the transition matrix about the metric and topological entropies and correlation function decay of the system.

The method described in this paper can be used also for the analysis of experimental signals if it is possible to identify a criterion which allows one to introduce a symbolic encoding of the signals in a natural way. Under this condition the method could be more powerful than the standard embedding technique [1]. For example Shimada [19] computed with high accuracy the K–S entropy of the Lorenz model by studying the strings of symbols $\{-1, 1\}$ generated by the dynamics. Since the trajectory revolves around two instable fixed points C_+ and C_-, he used the following criterion: the symbol $+1$ is associated to each circulation around C_+ and -1 to each circulation around C_-. From this time series of symbols ± 1 and the mean circulation time it is possible to extract the correct value of the K–S entropy of the Lorenz system.

We conclude noting that generally the convergence for the K–S entropy is faster than the convergence for the decay coefficient. This is a clear indication that the K–S entropy h is not the unique indicator of the "complexity" of a sequence. Let us mention [20,21] that the past–future mutual information C defined as

$$C = \lim_{n\to\infty} \left(\sum_{k=1}^{n} h(k) - nh \right) \tag{20}$$

contains important information on the structure of the sequence and, even for Markov processes, has not a simple relation with h.

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References

[1] J.P. Eckmann and D. Ruelle, Rev. Mod. Phys. 57 (1985) 617.
[2] J.P. Crutchfield, J.D. Farmer and B.A. Huberman, Phys. Rep. 92 (1982) 45;
Y. Ueda, Int. J. Nonlin. Mech. 20 (1985) 481;
T. Kapitaniak, Phys. Lett. A 116 (1986) 251.
[3] G. Benettin, L. Galgani, A. Giorgilli and J.M. Strelcyn, Meccanica 15 (1980) 9, 21;
A.J. Lichtenberg and M.A. Liebermann, Regular and stochastic motion (Springer, Berlin, 1983).
[4] B.V. Chirikov, Phys. Rep. 52 (1979) 263.
[5] G. Benettin, Physica D 13 (1984) 211.
[6] G. Paladin and A. Vulpiani, J. Phys. A 19 (1986) 1881.
[7] A. Crisanti, G. Paladin and A. Vulpiani, Products of random matrices in statistical physics (Springer, Berlin, 1993).
[8] R. Kluiving, H.W. Capel and R.A. Pasmanter, Physica A 183 (1992) 67, 96; 186 (1992) 405.
[9] A. Crisanti, G. Paladin and A. Vulpiani, Phys. Rev. A 39 (1989) 6491.
[10] P. Billingsley, Ergodic theory and information (Wiley, New York, 1965).
[11] R. Lozi, J. Phys. (Paris) Colloq. 39 (1978) 9.
[12] P. Collet and Y. Levy, Commun. Math. Phys. 93 (1984) 461.
[13] M. Hénon, Commun. Math. Phys. 50 (1976) 69.
[14] J.H. Curry, Commun. Math. Phys. 68 (1979) 129.
[15] A. Lasota and M.C. Mackey, Probabilistic properties of deterministic systems (Cambridge Univ. Press, Cambridge, 1985).
[16] S. Grossmann and S. Thomae, Z. Naturforsch. 32a (1977) 1353.
[17] H. Mori, B.-C. So and T. Ose, Prog. Theor. Phys. 66 (1981) 4.
[18] F. Christiansen, S. Isola, G. Paladin and H.H. Rugh, J. Phys. A 23 (1990) L1301.
[19] I. Shimada, Prog. Theor. Phys. 62 (1979) 61.
[20] P. Grassberger, Int. J. Theor. Phys. 25 (1986) 907.
[21] W. Li, Complex Syst. 5 (1991) 381.