Diffusion, super-diffusion and coalescence from a single step

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Abstract. From the exact single step evolution equation of the two-point correlation function of a particle distribution subjected to a stochastic displacement field $\mathbf{u}(\mathbf{x})$, we derive different dynamical regimes when $\mathbf{u}(\mathbf{x})$ is iterated to build a stochastic velocity field. First we show that spatially uncorrelated fields $\mathbf{u}(\mathbf{x})$ lead to both standard and anomalous diffusion equations. When the field $\mathbf{u}(\mathbf{x})$ is spatially correlated each particle performs a simple free Brownian motion, but the trajectories of different particles result to be mutually correlated. The two-point statistical properties of the field $\mathbf{u}(\mathbf{x})$ induce two-point spatial correlations in the particle distribution satisfying a simple but non-trivial diffusion-like equation. These displacement–displacement correlations lead the system to three possible regimes: coalescence, simple clustering and a combination of the two. The existence of these different regimes is shown, in the one-dimensional system, through computer simulations and a simple theoretical argument.

Keywords: irreversible aggregation phenomena (theory), stochastic particle dynamics (theory)

Contents

1. Introduction

The study of diffusion phenomena has a long history dating back to decades since the origin of statistical mechanics and includes many different subjects and applications ranging from irregular motion of particles in homogeneous or disordered media and osmosis [1, 2], to standard and anomalous transport [3, 4] of heat, mass and charges in materials, from coalescence of passive scalars [5] in highly turbulent fluxes to biological and ecological investigations on animal dispersal [6, 7].

The basis of the theory of diffusion is the random walk (RW) or random flight where a particle undergoes successive random displacements from its initial positions performing an irregular motion. The resulting character of the particle trajectory emerges from the statistical properties of these displacements. The wide applicability of the RW to natural phenomena relies just on the possibility to introduce appropriate generalizations on the probabilistic nature of the displacements. One of the straightforward generalizations is realized by introducing correlations in the displacements so to obtain the so-called correlated random walks (CRWs) [8]. This possibility extends also to a set of particles distributed in space leading to the definition of spatially correlated random walks. In this case one can wonder what kind of particle distribution emerges from reiterated displacements of the particles and how its properties can be directly inferred from the knowledge of the statistical correlation of the displacements [9, 10]. This approach is usually adopted to model the motion of a system of particles in a disordered environment where the randomness changes in time. The study of this kind of models is strictly related to the mathematical theory of the so-called point processes [11], i.e., stochastic point-wise particle distributions. The theory which investigates the geometrical and statistical properties of the stochastic processes generating particle distributions constitutes a very important field of research with important physical and cross-disciplinary applications [12].

In this paper we derive different results on the physics of diffusion from the exact transformation equation [13] for the two-point correlations of a particle distribution subjected to a single step stochastic deformation. More precisely, we deal with

the transformation that a stochastic displacement field induces on a given particle distribution once it is applied to the particles. Statistically independence between the particle distribution and the displacement field is assumed, but arbitrary displacement–displacement correlations and initial density–density correlations can be present. The main purpose of this study was originally to understand, through the exact formula presented for one- and two-point particle correlations in [13], the discretization and finite size effects in the preparation of the initial conditions in cosmological n-body [14, 15] simulations which are usually built by applying a suitable stochastic deformation field to a regular lattice or other very uniform configurations of particles with equal mass. In the present context, instead, we use the same approach to derive by reiteration all the famous equations of ordinary diffusion, super-diffusion and sub-diffusion. We finally consider the most complex case of clustering and coalescence of particles observed in fully developed turbulence due to spatially correlated diffusion induced by turbulent flows on passive pollutants.

2. From one-step displacement to spatially correlated random walks

In this section we recall first some notations for the one- and two-point correlation function of a spatially homogeneous point process, then we briefly summarize the main results of [13].

Let us consider a d-dimensional spatial distribution of N particles with equal unitary mass (i.e., a so-called 'point process') in a volume V which we assume to coincide asymptotically with $\mathbb{R}^d$. The microscopic density is by definition

$$n(\mathbf{x}) = \sum_{i=1}^{N} \delta(\mathbf{x} - \mathbf{x}_i),$$

where the limit is taken keeping fixed $N/V = n_0$, and $\mathbf{x}_i$ indicates the position of the ith particle. The system is supposed to be statistically homogeneous and uniform on large scales, therefore the average density is well defined and positive: $\langle n(\mathbf{x}) \rangle = n_0 > 0$, where the average $\langle .. \rangle$ runs over the ensemble of realizations of the point process. Moreover, we define the normalized connected two-point correlation function as

$$\xi(\mathbf{x}) = \frac{\langle n(\mathbf{x}_0) n(\mathbf{x}_0 + \mathbf{x}) \rangle}{n_0^2} - 1 = \frac{\delta(\mathbf{x})}{n_0} + h(\mathbf{x}). \tag{1}$$

The *covariance* function $h(\mathbf{x})$ is the off-diagonal two-point correlation function and it gives the true correlation between different particles. The power spectrum of the particle distribution is defined as the limit

$$S(\mathbf{k}) = \lim_{V \to \infty} \frac{\langle |\hat{n}(\mathbf{k}; V)|^2 \rangle}{N} - 2\pi n_0 \delta(\mathbf{k}),$$

where $\hat{n}(\mathbf{k}; V) = \int_V \mathrm{d}^3 x \, \exp(-\mathrm{i}\mathbf{k} \cdot \mathbf{x}) n(\mathbf{x})$ is the Fourier transform (FT) of the density on the finite volume V. Because of the hypothesis on statistical homogeneity, $S(\mathbf{k})$ is simply given by the FT of $\xi(\mathbf{x})$ multiplied by n_0:

$$S(\mathbf{k}) = n_0 \mathcal{F}[\xi(\mathbf{x})] = 1 + n_0 \hat{h}(\mathbf{k}), \tag{2}$$

with $\hat{h}(\mathbf{k}) = \mathcal{F}[h(\mathbf{x})]$ and $\mathcal{F}[..] = \int \mathrm{d}^d x \, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}(..)$ being the usual d-dimensional infinite volume Fourier transform. It is clear that for a statistically homogeneous particle distribution, $S(\mathbf{k})$ and $\xi(\mathbf{x})$ contain the same information.

When a statistically homogeneous and arbitrarily correlated stochastic displacement field $\mathbf{u}(\mathbf{x})$ is applied to the particle distribution, each particle moves from its old position $\mathbf{x}_i$ to the new one $\mathbf{x}_i + \mathbf{u}(\mathbf{x}_i)$. We assume that the displacement field and the particle positions are statistically independent. We are interested in the change of the two-point correlation properties of the particle distribution under the effect of the displacements. The complete statistics of $\mathbf{u}(\mathbf{x})$ is given by a probability density functional $\mathcal{P}[\mathbf{u}(\mathbf{x})]$ giving the statistical weight of each realization of the stochastic field. However, in our hypotheses, the behavior of the two-point correlation function or the power spectrum under the system displacements [13] is only determined through the knowledge of the probability density function (PDF)

$$\phi(\mathbf{w}; \mathbf{x}) = \iint \mathrm{d}^d u \, \mathrm{d}^d v \, f(\mathbf{u}, \mathbf{v}; \mathbf{x}) \delta(\mathbf{w} - \mathbf{u} + \mathbf{v}) \tag{3}$$

that two particles, separated by the vector distance $\mathbf{x}$, undergo a relative displacement $\mathbf{w}$. In equation (3) we used the property that $\phi(\mathbf{w}; \mathbf{x})$ is, in turn, related to the joint PDF $f(\mathbf{u}, \mathbf{v}; \mathbf{x})$ that two particles separated by the vector $\mathbf{x}$ perform the displacements $\mathbf{u}$ and $\mathbf{v}$ respectively. If we denote by $S_{\mathrm{in}}(\mathbf{k})$ and $\xi_{\mathrm{in}}(\mathbf{x}) = [\delta(\mathbf{x})/n_0 + h_{\mathrm{in}}(\mathbf{x})]$ respectively the power spectrum and the two-point correlation function of the particle distribution before the application of the displacements and by $S_f(\mathbf{k})$ and $\xi_f(\mathbf{x}) = [\delta(\mathbf{x})/n_0 + h_f(\mathbf{x})]$ the corresponding quantities after the displacements, we can write [13] the equation

$$S_f(\mathbf{k}) = 1 - \int \mathrm{d}^d q \, \tilde{\phi}(\mathbf{k}, \mathbf{q}) + n_0 \int \mathrm{d}^d x \, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \hat{\phi}(\mathbf{k}; \mathbf{x})[1 + \xi_{\mathrm{in}}(\mathbf{x})] - (2\pi)^d n_0 \delta(\mathbf{k}), \tag{4}$$

where

$$\hat{\phi}(\mathbf{k}; \mathbf{x}) = \int \mathrm{d}^d w \, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{w}} \phi(\mathbf{w}; \mathbf{x})$$

is the characteristic function[1] of the random displacement $\mathbf{w}$, and $\tilde{\phi}(\mathbf{k}, \mathbf{q}) = \int \mathrm{d}^d x \, \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}} \hat{\phi}(\mathbf{k}; \mathbf{x})$. The only hypotheses for the validity of equation (4) are: (i) spatial homogeneity of both particle distribution and displacement field, (ii) statistical independence between the particle positions and the displacement field. Notice that no 'small displacements' approximation or special properties of the displacement correlations are required. Let us define

$$G_{\mu\nu}(\mathbf{x}) = \overline{u_\mu(\mathbf{x}_0) u_\nu(\mathbf{x}_0 + \mathbf{x})} \qquad \mu, \nu = 1, \ldots, d, \tag{5}$$

the displacement–displacement correlation function which is a symmetric tensor of rank 2, whose FT is a non-negative definite symmetric tensor for all the k-vectors, with $\overline{(..)}$ indicating the average over $\mathcal{P}[\mathbf{u}(\mathbf{x})]$. For symmetric distributions $\mathcal{P}[\mathbf{u}(\mathbf{x})] = \mathcal{P}[-\mathbf{u}(\mathbf{x})]$, the positive value $G_{\mu\mu}(\mathbf{0})$ represents the variance of the μth component of $\mathbf{u}(\mathbf{x})$ at any point. Moreover, in the case of a Gaussian field, the function $G_{\mu\nu}(\mathbf{x})$ determines completely the probabilistic properties of the field [12].

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[1] From equation (3), it is immediate to verify also that $\hat{\phi}(\mathbf{k}; \mathbf{x}) = \hat{f}(\mathbf{k}, -\mathbf{k}; \mathbf{x})$ and $\tilde{\phi}(\mathbf{k}, \mathbf{q}) = \tilde{f}(\mathbf{k}, -\mathbf{k}; \mathbf{q})$ where respectively $\hat{f}(\mathbf{k}, \mathbf{k}'; \mathbf{x}) = \iint \mathrm{d}^d u \, \mathrm{d}^d v \, f(\mathbf{u}, \mathbf{v}; \mathbf{x}) \mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{u} + \mathbf{k}'\cdot\mathbf{v})}$ and $\tilde{f}(\mathbf{k}, -\mathbf{k}; \mathbf{q}) = \int \mathrm{d}^d x \, \hat{f}(\mathbf{k}, -\mathbf{k}; \mathbf{x}) \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}}$.

The purpose of this paper is to deduce from equation (4) the time-evolution of the two-point correlation function and power spectrum of a particle distribution in the limit that each particle performs a Brownian trajectory, but the motions of different particles can be arbitrarily spatially correlated. In other words, we consider an assigned displacement field statistics, $f(\mathbf{u}, \mathbf{v}; \mathbf{x})$, at every time step Δt, but no time correlation between consecutive time-steps. Then we take $\Delta t \to 0$ in such a way to have a well defined diffusional limit. Two cases have to be basically distinguished:

(i) The field $\mathbf{u}(\mathbf{x})$ is a spatially uncorrelated stochastic process at each time step, i.e., $\mathbf{u}(\mathbf{x})$ and $\mathbf{u}(\mathbf{y})$ are completely independent if $\mathbf{x} \neq \mathbf{y}$. In this case,

$$f(\mathbf{u}, \mathbf{v}; \mathbf{x}) = \begin{cases} p(\mathbf{u})p(\mathbf{v}), & \text{if } \mathbf{x} \neq 0, \\ \delta(\mathbf{u} - \mathbf{v})p(\mathbf{u}), & \text{if } \mathbf{x} = 0, \end{cases}$$

where $p(\mathbf{u})$ indicates the single displacement PDF [13]. Basically, the displacement field is simply a white noise both in space and time[2]. Moreover $f(\mathbf{u}, \mathbf{v}; \mathbf{x})$ is a discontinuous function at $\mathbf{x} = 0$, that is $G_{\mu\nu}(\mathbf{x}) = 0$ for $\mathbf{x} \neq 0$, whereas $G_{\mu\nu}(0) \neq 0$ and in particular $G_{\mu\nu}(0) = \delta_{\mu\nu}\overline{u^2}/d > 0$ when assuming that $p(\mathbf{u}) = p(u)$. Then, we can simply show that $\hat{\phi}(\mathbf{k}; \mathbf{x}) = |\hat{p}(\mathbf{k})|^2$ where $\hat{p}(\mathbf{k}) = \mathcal{F}[p(\mathbf{u})]$, and, because of the discontinuity in $\mathbf{x} = 0$, $\int d^d q \, \tilde{\phi}(\mathbf{k}; \mathbf{q}) = |\hat{p}(\mathbf{k})|^2 \neq \hat{\phi}(\mathbf{k}; 0) = 1$. This implies that equation (4) reads

$$S_f(\mathbf{k}) = 1 + |\hat{p}(\mathbf{k})|^2 [S_{\text{in}}(\mathbf{k}) - 1]. \tag{6}$$

Note that this relation is local in $\mathbf{k}$, i.e., each $\mathbf{k}$-mode of the particle density evolves independently one of each other.

(ii) $\mathbf{u}(\mathbf{x})$ is a real correlated and continuous stochastic process. In this case [16], it is well known that $G_{\mu\nu}(\mathbf{x})$ is a continuous function of $\mathbf{x}$, and in the limit when $\mathbf{x} \to 0$, it approaches with continuity its value $G_{\mu\nu}(0)$ [16]. In other words, $f(\mathbf{u}, \mathbf{v}; \mathbf{x})$ is continuous in $\mathbf{x}$ and

$$\lim_{\mathbf{x} \to 0} f(\mathbf{u}, \mathbf{v}; \mathbf{x}) = \delta(\mathbf{u} - \mathbf{v})p(\mathbf{u});$$

this implies that $\int d^d q \, \tilde{\phi}(\mathbf{k}; \mathbf{q}) = \hat{\phi}(\mathbf{k}; 0) = 1$ [13], and therefore equation (4) becomes

$$S_f(\mathbf{k}) = n_0 \int d^d x \, e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{\phi}(\mathbf{k}; \mathbf{x})[1 + \xi_{\text{in}}(\mathbf{x})] - (2\pi)^d n_0 \delta(\mathbf{k}). \tag{7}$$

Unlike equation (6), this equation is nonlocal in $\mathbf{k}$ due to the presence of displacement–displacement spatial correlations which couple different modes of the particle density before and after the application of the displacement field.

We can now study the spatial diffusion of a particle distribution in which at each time step the particles move under one of the stochastic displacement fields just described above. We will assume for simplicity that the time can be discretized in time-steps of size Δt at which the particle distribution is displaced by a realization of the field $\mathbf{u}(\mathbf{x}, t)$; moreover, different time-steps are supposed to be statistically independent.

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[2] Apart from the possible correlations in $d > 1$ between the different components of the displacement $\mathbf{u}(\mathbf{x})$ at a single point $\mathbf{x}$. However, if we consider the case that $p(\mathbf{u})$ depends only on $u = |\mathbf{u}|$ (i.e., the displacement field is isotropic), perpendicular displacements are uncorrelated.

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3. Spatially uncorrelated displacements

In order to illustrate the general formalism to derive a continuous time equation from the discrete one (single step), it is instructive to consider the simplest case of spatially and temporally uncorrelated displacement fields. As we show below, it corresponds to the homogeneous diffusion equations, either standard or fractional depending whether the variance of displacements is finite or infinite. This discussion is useful in view of the more interesting case of random walks generated by spatially correlated displacements. As clarified above, all the statistics of the field is contained in the one-displacement PDF $p(\mathbf{u})$. Let us firstly consider the statistically isotropic case for $\mathbf{u}$: $p(\mathbf{u}) = p(u)$, implying that $\hat{p}(\mathbf{k}) = \hat{p}(k)$. We have to distinguish the two cases of finite and infinite variance $\overline{u^2}$. In the former, each particle performs an ordinary d-dimensional random walk and the paths of different particles are independent one of each other, while in the latter, each particle undergoes a d-dimensional Levy walk independently of the others.

3.1. The continuous time limit

For statistical isotropic displacements, we can expand at small k as

$$\hat{p}(k) = 1 - Bk^\alpha + o(k^\alpha), \tag{8}$$

where $\alpha = 2$ and $B = \overline{u^2}/2d$ when $\overline{u^2}$ is finite, while $0 < \alpha < 2$ when $\overline{u^2}$ diverges and $p(u) \simeq Au^{-(\alpha+d)}$ at large u, with B proportional to the amplitude of the tails $A > 0$. In the first case each particle undergoes an independent standard random walk while, in the second, an independent Levy flight [17]. A well-defined diffusional continuous time limit is attained by requiring that $2B = D\Delta t$ where $D > 0$ is a constant independent of the time step Δt. Therefore in the limit $\Delta t \to 0$, the substitution of equation (8) into equation (6) leads, for $k^\alpha \ll 2/(D\Delta t)$, to

$$\partial_t S(\mathbf{k}, t) = Dk^\alpha \left[1 - S(\mathbf{k}, t)\right], \tag{9}$$

whose solution is

$$S(\mathbf{k}, t) = [S_{\text{in}}(\mathbf{k}) - 1] \exp(-Dk^\alpha t) + 1. \tag{10}$$

The approach of $S(\mathbf{k}, t)$ to a homogeneous Poisson power spectrum $S(\mathbf{k}) = 1$ is exponentially fast, and each $\mathbf{k}$-mode of the two-point correlation relaxes with a rate Dk^α $(0 < \alpha \le 2)$. We see immediately that the smaller α the faster the approach to the completely uncorrelated stationary state. In other words inhomogeneities diffuse $(\alpha = 2)$ or super-diffuse $(0 < \alpha < 2)$ until reaching the uniform stationary state $\Gamma_s(\mathbf{x}) = n_0$. Equation (9) can be recast in a more familiar form by rewriting equation (10) for $\hat{h}(\mathbf{k}, t)$:

$$\partial_t \hat{h}(\mathbf{k}, t) = -Dk^\alpha \hat{h}(\mathbf{k}, t) \tag{11}$$

and then taking the inverse FT. For simplicity, let us consider a one-dimensional system. By taking the FT of equation (11) we can write

$$\partial_t h(x, t) = D\partial_x^\alpha h(x, t), \tag{12}$$

where ∂_x^α is for $\alpha = 2$ the usual Laplacian in $d = 1$, giving the celebrated standard diffusion equation, while ∂_x^α for $0 < \alpha < 2$ is the fractional derivative [18] of order α in x. Equation (12) is for $0 < \alpha < 2$ the well-known equation of spatially fractional diffusion [18, 19] and it describes how the connected two-point correlations super-diffuse in space toward an uncorrelated Poisson steady state. Equation (12) takes a more familiar form when written for the average conditional density $\Gamma(x, t)$:

$$\partial_t \Gamma(x, t) = D \partial_x^\alpha \Gamma(x, t), \tag{13}$$

which is the fractional diffusion equation for the density of particles seen in average by a generic particle of the system. It belongs to a larger family of fractional diffusion equations whose general form is

$$\partial_t^\beta \Gamma(x, t) = D \partial_x^\alpha \Gamma(x, t), \tag{14}$$

with $0 < \beta \leq 1$ and $0 < \alpha \leq 2$. This kind of equation is encountered in the description of key aspects of anomalous transport as for instance that occurring on disordered peculiar structures (fractal supports) or due to spatial non-locality [18, 20]. More precisely the fractional diffusion equation (14) can be obtained in the context of continuous time random walks (CTRWs) [21] under the assumption that the joint PDF $\psi(\mathbf{u}, t)$ to make a step of size $\mathbf{u}$ in the time interval $[t, t + \mathrm{d}t)$ factorizes as $\phi(t)p(\mathbf{u})$. In order to obtain $\beta = 1$ it is necessary that the mean value of $\phi(t)$ is finite, and to obtain $\alpha = 2$ we need a finite variance for $p(\mathbf{u})$. Otherwise we have respectively $\beta < 1$ and $\alpha < 2$ depending on the power law tails of the two functions $\phi(t)$ and $p(\mathbf{u})$. Indeed we have obtained equation (13) via the choice $\phi(t) = \delta(t - \Delta t)$ with $(\Delta t \to 0^+)$ which has finite mean value equal to Δt itself.

4. Spatially correlated displacements

We now turn to the case where the stochastic displacements, acting on the particles at each time-step, are generated through the realizations of a continuous and spatially stationary correlated stochastic field $\mathbf{u}(\mathbf{x})$ defined by a probability density functional $\mathcal{P}[\mathbf{u}(\mathbf{x})]$. For instance, in $d = 1$ we can consider a Gaussian stochastic field defined by the probability density functional

$$\mathcal{P}[u(x)] \sim \exp\left[-\int\int \mathrm{d}x\, \mathrm{d}y\, u(x) K(x - y) u(y)\right],$$

with $K(s)$ being the positive definite and continuous correlation kernel. Since we assume again no time correlation between successive realizations of the field $\mathbf{u}(\mathbf{x})$, at each time-step the power spectrum of the particle distribution evolves according to equation (7). In principle, a time-dependent PDF $\mathcal{P}[\mathbf{u}(\mathbf{x}); t]$ can be also considered, referring to a displacement field whose correlation properties depend on time. Here we limit the discussion to time-independent functionals $\mathcal{P}$, focusing on the case of a d-dimensional Gaussian displacement field with finite variance $\overline{u^2} = \sum_{\mu=1}^{d} G_{\mu\mu}(\mathbf{0}) < +\infty$, where the displacement–displacement correlation matrix $G_{\mu\nu}(\mathbf{x})$ has been defined in section 2. However, as shown explicitly below, the validity of the evolution equations we derive is

not restricted to this case. If the displacement field is Gaussian its characteristic function reads [13]

$$\hat{\phi}(\mathbf{k}; \mathbf{x}) = \exp\left\{-\sum_{\mu,\nu}^{1,d} k_\mu k_\nu [G_{\mu\nu}(\mathbf{0}) - G_{\mu\nu}(\mathbf{x})]\right\}, \tag{15}$$

where k_μ indicates the μth component of $\mathbf{k}$.

In analogy with the simpler case of uncorrelated displacements, we have to assume

$$G_{\mu\nu}(\mathbf{x}) = \Delta t c_{\mu\nu}(\mathbf{x})$$

for the existence of a smooth time limit, with $c_{\mu\nu}(\mathbf{x})$ independent of Δt in order to obtain the correct diffusional processes in the limit $\Delta t \to 0$. In fact the above condition implies that the variance of each component μ of the field satisfies $G_{\mu\mu}(0) = \overline{|u_\mu(\mathbf{x},t)|^2} \sim \Delta t$ at any time t. Therefore, if $\mathbf{u}(\mathbf{x}, n\Delta t)$ is the displacement in the point $\mathbf{x}$ at time $t = n\Delta t$, we can write

$$\overline{\mathbf{u}(\mathbf{x}, n\Delta t)} = 0, \tag{16}$$

$$\overline{u_\mu(\mathbf{x}, n\Delta t) u_\nu(\mathbf{x}', n'\Delta t)} = \Delta t c_{\mu\nu}(\mathbf{x} - \mathbf{x}')\delta_{nn'}; \tag{17}$$

thus, in the limit $\Delta t \to 0$ the field $\eta(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t)/\sqrt{\Delta t}$ becomes a spatially correlated and temporally delta-correlated noise for the motion of particles. In a hydrodynamic analogy, in which the displacement field is interpreted as a turbulent main flow advecting the particles of a passive pollutant, the quantities $d_{\mu\nu}(\mathbf{x}) = [c_{\mu\nu}(0) - c_{\mu\nu}(\mathbf{x})]$ are usually called [22] the *structure functions* of the flow.

In the limit of small Δt we can expand equation (15):

$$\hat{\phi}(\mathbf{k}; \mathbf{x}) = 1 - \Delta t \sum_{\mu,\nu}^{1,d} k_\mu k_\nu d_{\mu\nu}(\mathbf{x}) + o(\Delta t). \tag{18}$$

When plugged into equation (7) this gives

$$S(\mathbf{k}, t + \Delta t) = S(\mathbf{k}, t) + \Delta t \sum_{\mu,\nu}^{1,d} k_\mu k_\nu \left[n_0 \hat{c}_{\mu\nu}(\mathbf{k}) - c_{\mu\nu}(0)S(\mathbf{k}, t)\right]$$

$$+ \Delta t \sum_{\mu,\nu}^{1,d} k_\mu k_\nu \int \frac{\mathrm{d}^d q}{(2\pi)^d} S(\mathbf{k} - \mathbf{q}, t)\hat{c}_{\mu\nu}(\mathbf{q}) + o(\Delta t), \tag{19}$$

where $\hat{c}_{\mu\nu}(\mathbf{k}) = \mathcal{F}[c_{\mu\nu}(\mathbf{x})]$ is the renormalized power spectrum tensor of the instantaneous displacement field. The limit $\Delta t \to 0$ on equation (19) yields

$$\partial_t S(\mathbf{k}, t) = \sum_{\mu,\nu}^{1,d} k_\mu k_\nu \left[n_0 \hat{c}_{\mu\nu}(\mathbf{k}) - c_{\mu\nu}(0)S(\mathbf{k}, t) + \int \frac{\mathrm{d}^d q}{(2\pi)^d} \hat{S}(\mathbf{k} - \mathbf{q}, t)\hat{c}_{\mu\nu}(\mathbf{q})\right], \tag{20}$$

which in terms of FT of the covariance function takes the same forms

$$\partial_t \hat{h}(\mathbf{k}, t) = \sum_{\mu,\nu}^{1,d} k_\mu k_\nu \left[\hat{c}_{\mu\nu}(\mathbf{k}) - c_{\mu\nu}(0)\hat{h}(\mathbf{k}, t) + \int \frac{\mathrm{d}^d q}{(2\pi)^d} \hat{h}(\mathbf{k} - \mathbf{q}, t)\hat{c}_{\mu\nu}(\mathbf{q})\right] \tag{21}$$

and under the inverse FT becomes

$$\partial_t h(\mathbf{x}, t) = \sum_{\mu,\nu}^{1,d} \partial_{\mu\nu}^2 \left[d_{\mu\nu}(\mathbf{x}) h(\mathbf{x}, t) - c_{\mu\nu}(\mathbf{x}) \right]. \tag{22}$$

Consequently, for $\Gamma(\mathbf{x}) = n_0[1 + h(\mathbf{x})]$ we obtain

$$\partial_t \Gamma(\mathbf{x}, t) = \sum_{\mu,\nu}^{1,d} \partial_{\mu\nu}^2 \left[d_{\mu\nu}(\mathbf{x}) \Gamma(\mathbf{x}, t) \right]. \tag{23}$$

It is noteworthy that if $P(\mathbf{x}, t | \mathbf{x}_0, t_0)$ is the PDF of the separation $\mathbf{x}$ at time t between an arbitrary pair of particles of the distribution, given their initial distance $\mathbf{x}_0$ at time t_0, we can write

$$\Gamma(\mathbf{x}, t) = \int \mathrm{d}^d x_0 \, P(\mathbf{x}, t | \mathbf{x}_0, t_0) \Gamma(\mathbf{x}_0, t_0).$$

It is straightforward to verify that the transition probability $P(\mathbf{x}, t | \mathbf{x}_0, t_0)$ also satisfies equation (23), and for this reason $P(\mathbf{x}, t | \mathbf{x}_0, t_0)$ is called the *propagator* of the diffusion operator defined by equation (23). On the other hand equation (23) is the Fokker–Planck (FP) equation associated to the stochastic Langevin equation (LE) in the Îto representation [23],

$$\dot{\mathbf{x}}(t) = \mathbf{w}(t), \tag{24}$$

describing the time evolution of the two-particle separation $\mathbf{x}(t)$, where $\mathbf{w}(t)$ is a Gaussian noise with the following one- and two-time correlation properties:

$$\begin{cases} \overline{w_\mu(t)} = 0 \\ \\ \overline{w_\mu(t) w_\nu(t')} = 2 d_{\mu\nu}(\mathbf{x}) \delta(t - t') \end{cases} \quad \mu, \nu = 1, \dots, d. \tag{25}$$

Therefore equations (23) and (24) are equivalent.

In the hyper-isotropic condition, corresponding to the choice $d_{\mu\nu}(\mathbf{x}) = \delta_{\mu\nu} d(x)$ (i.e., $c_{\mu\nu}(\mathbf{x}) = \delta_{\mu\nu} c(x)$), equation (23) can be rewritten as

$$\partial_t \Gamma(\mathbf{x}, t) = \nabla^2 [d(x) \Gamma(\mathbf{x}, t)]. \tag{26}$$

Note that all these results are not restricted to the case of Gaussian displacement fields. In fact, as shown in [13], in general if $G_{\mu\nu}(\mathbf{x}) = \overline{u_\mu(\mathbf{x}_0) u_\nu(\mathbf{x}_0 + \mathbf{x})}$ is finite for $x = 0$, the small k expansion

$$\hat{\phi}(\mathbf{k}; \mathbf{x}) = 1 - \sum_{\mu,\nu}^{1,d} k_\mu k_\nu [G_{\mu\nu}(\mathbf{0}) - G_{\mu\nu}(\mathbf{x})] + o(k^2)$$

is always valid.

It is easy to prove, but important to note, that when the displacement field $\mathbf{u}(\mathbf{x}, t)$ can be decomposed into two independent components, a spatially correlated field $\mathbf{u}_1(\mathbf{x}, t)$ characterized by equation (17) and a spatially uncorrelated and statistically isotropic field

$\mathbf{u}_2(\mathbf{x}, t)$ of variance $\overline{u_2^2} = d \times D\Delta t$, equation (23) takes the form

$$\partial_t \Gamma(\mathbf{x}, t) = D\nabla^2 \Gamma(\mathbf{x}, t) + \sum_{\mu,\nu}^{1,d} \partial_{\mu\nu}^2 \left[d_{\mu\nu}(\mathbf{x})\Gamma(\mathbf{x}, t) \right]. \tag{27}$$

That is, the motion of the set of particles results in the superposition of a standard diffusion, the first term in equation (27), with a spatially correlated diffusion, the second term. Equations of the type (27) are the generalized diffusion equations which are usually encountered in the context of turbulent transport of passive scalars (i.e., pollutant) [5], where $n(\mathbf{x}, t)$ is the density of the passive particles advected by the velocity field $\mathbf{v}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t)/\Delta t$ of the synthetic turbulent flow. It is important to note that for $x \to 0$, the term $D\nabla^2\Gamma(\mathbf{x}, t)$ of equation (27) dominates because $d_{\mu\nu}(0) = 0$, therefore the small scale motion occurs via standard diffusion. The interesting case is obtained when $D \to 0$, and consequently $\mathbf{x} = 0$ becomes a singularity of equation (27) [23]. In particular in turbulence a complete solution has been given for the scale-free case in which

$$d_{\mu\nu}(\mathbf{x}) = ax^\xi \delta_{\mu\nu} + bx^{\xi-2}x_\mu x_\nu, \tag{28}$$

with $0 < \xi \leq 2$, a and b constants such that the tensor $c_{\mu\nu}(\mathbf{x})$ has a positive definite FT. The one-dimensional case is recovered by putting $d = 1$ and $b = 0$. The class of models defined by the structure tensor (28) is referred to as a *generalized Kraichnan ensemble* [24, 5] and a complete classification of their solutions, in terms of the singular behavior around $\mathbf{x} = 0$, has been given in [25, 26]. For $0 < \xi < 2$, it consists in three possible behaviors:

(i) For $(b/a) > (d-2)$ and at the same time $(\xi-1)(b/a) \geq (d-\xi)$, the only possible solution of equation (23) is such that different particles coalesce in finite time and no stationary state exists. In practice a pair of particles collides in finite time with vanishing relative velocity and therefore remains attached for the rest of the dynamics. In this case $\Gamma(\mathbf{x}) \simeq \alpha(t)\delta(\mathbf{x}) + \beta(t)x^{2-\xi-d}$ with a time increasing coefficient $\alpha(t)$, which signals the coalescence phenomenon.

(ii) For $(\xi-1)(b/a) < (d-\xi)$ and at the same time $(b/a) \leq (d-2)$ the only possible solution is such that the probability of finding more than one particle in a single spatial point is zero at all times. In this phase particles form only clusters and the diffusion of particles converges to a stationary state in which $\Gamma(\mathbf{x}) \sim x^{-\gamma}$, where $\gamma = \xi + b(d-1)/(a+b)$.

(iii) Finally, for $(b/a) > (d-2)$ and $(\xi-1)(b/a) < (d-\xi)$ particles can collide at finite time, but with non-zero relative velocity. This obliges us to fix a boundary condition at $\mathbf{x} = 0$ for the diffusion equation. For an *absorbing* boundary condition one has an effective behavior as in case (i) above. Instead for a *reflecting* boundary condition the effective behavior is similar to case (ii) above. Adopting a mixed boundary condition, a composition of the two above behaviors appears (called a 'sticky' phase in [26]).

The method used to get this classification is quite complex and consists in the theory of boundary conditions of elliptic operators. In the rest of the paper we do not enter into the details of this rigorous analysis, but limit our study to the one-dimensional case for a generic choice of the displacement–displacement correlation function through computer simulations and simple theoretical arguments.

4.1. The one-dimensional spatially correlated diffusion

In $d = 1$ equation (23) becomes

$$\partial_t \Gamma(x,t) = \partial_x^2[d(x)\Gamma(x,t)], \tag{29}$$

where $d(x) = [c(0) - c(x)]$ is the structure function of the stochastic velocity field (i.e., displacements) with the two-point correlation function $c(x)$. It is important to note that the general properties of any correlation function $c(x)$ constrains $d(x)$ to have the small x behavior $d(x) \simeq ax^\xi$, with $a > 0$ and $0 < \xi \le 2$. In this paper we do not treat the 'smooth' case $\xi = 2$ as it has been solved rigorously elsewhere [10], and we focus our analysis on the 'rough' stochastic velocity fields for $0 < \xi < 2$. The aforementioned classification of the solutions of the Kraichnan ensemble immediately implies that for $b = 0$ and $d = 1$ in equation (28), only the first and the third cases are possible in one dimension around the singularity $x = 0$. In particular we have the first behavior for $1 \le \xi < 2$, and the third for $0 < \xi < 1$. For $\xi = 2$ (when the velocity field is smooth at small scales) one can see that particles coalesce but in an infinite mean time.

First of all we note that, if it exists, the only possible stationary solution of equation (29) is

$$\Gamma_s(x) = n_0 \frac{c(0)}{d(x)}. \tag{30}$$

$\Gamma_s(x)$, due to its definition of average conditional density, must converge to n_0 for $|x| \to \infty$ where density–density correlations have to disappear. In order to decide about the acceptability or not of this stationary solution, it is necessary to study its small scale behavior. We know that by definition $\Gamma_s(x)$ is acceptable only if it is integrable at small x. We see immediately that in $d = 1$, it happens only for $\xi < 1$. Thus for $\xi \ge 1$ equation (29) admits no stationary solution in agreement with the aforementioned classification of solutions of the generalized Kraichnan ensemble [25, 26]. In fact equation (30) corresponds to the stationary state which the family of solutions with simple clustering and no coalescence converge to, obtained by imposing a reflecting boundary condition at the singularity $x = 0$. This solution is analogous to the Poisson stationary correlation function $\Gamma_s(x) = n_0$ for the ordinary diffusion equation to which corresponds the well-known propagator

$$P(x,t|0,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right), \tag{31}$$

satisfying the reflecting condition at $x = 0$ and which shows how two particles at initial vanishing distance spread when a reflecting condition is imposed about their collisions. An analogous propagator can be found also in relation to the stationary state (30) of our more complex diffusion equation (29). It can be found by looking for a scaling solution of the form $P(x,t|0,t) = t^{-\beta} f(x/t^\beta)$. Plugging this scaling form into equation (29), one finds

$$P(x,t|0,t) = \frac{C}{t^{(1-\xi)/(2-\xi)}} x^{-\xi} \exp\left(-\frac{x^{2-\xi}}{a(2-\xi)^2 t}\right), \tag{32}$$

where $C > 0$ is the normalization constant. Also equation (32) describes the spreading of a pair of particles at initial vanishing distance with reflecting condition at $x = 0$. In fact

$\partial_x[x^\xi P(x,t|0,t)]|_{x=0} = 0$. Moreover it becomes the ordinary diffusing Gaussian for $\xi \to 0$. Let us now analyze in detail equation (30). First of all we note that at all x the covariance function $h(x) = [\Gamma(x) - n_0]/n_0$ has the same sign of the displacement correlation function $c(x)$, i.e., those scales at which displacements are positively (negatively) correlated asymptotically become also positively (negatively) correlated scales for the density of particles. Moreover $\Gamma_s(x)$ for large x approaches the average density n_0 in the following way:

$$\Gamma_s(x) \simeq n_0 \left[1 + \frac{c(x)}{c(0)}\right],$$

i.e., $h(x)$ is

$$h(x) \simeq \frac{c(x)}{c(0)}.$$

In other words the iterated displacement field injects exactly its large scale correlations in the particle system. This is interesting because, as one can check by expanding equation (7) at small k [13], in a single step the displacement field injects only a large scale contribution to the power spectrum $S(k)$ [or $\hat{h}(k)$] of order $k^2 \hat{c}(k)\Delta t$, which vanishes k^2 times faster than $\hat{c}(k)$. Finally, at small x we have $\Gamma_s(x) \sim x^{-\xi}$, meaning that particles form at small scales clusters with fractal dimension $D = (1 - \xi)$.

We now turn to the problem of what happens for $\xi \geq 1$ and how the other 'non-reflecting' solutions for $\xi < 1$ behave. This is a more difficult task as the singularity at $x = 0$ of equation (29) generates a coalescence dynamics which cannot be described through only smooth functions. In order to study this case, we adopt a sort of mean field approximation in an appropriately transformed LE (in the Îto representation) for the separation x between a pair of particles associated to the FP equation (29). We suppose that at initial time $t = 0$ we have $x_0 \equiv x(0) > 0$. At sufficiently small x equation (24), for our one-dimensional case, can be written as

$$\dot{x}(t) = A[x(t)]^{\xi/2}\eta(t), \tag{33}$$

where $A = \sqrt{2a} > 0$ and $\eta(t)$ is a white Gaussian noise such that $\overline{\eta(t)} = 0$ and $\overline{\eta(t)\eta(t')} = \delta(t - t')$. Let us now apply the following change of variables to equation (33):

$$y = \frac{2}{A}\frac{x^{1-\xi/2}}{2-\xi}, \tag{34}$$

i.e.,

$$x = \left[\frac{A}{2}(2-\xi)y\right]^{2/(2-\xi)}. \tag{35}$$

By using the rule of change of variables in the Îto representation [23], we get the LE for $y(t)$:

$$\dot{y}(t) = -\frac{C}{y(t)} + \eta(t),$$

where $C = \xi/(4 - 2\xi)$. If we suppose that the initial value $y(0) > 0$ is sufficiently small, the noise $\eta(t)$ can be neglected, obtaining

$$\dot{y}(t) = -\frac{C}{y(t)},$$

whose solution is

$$y(t) = \sqrt{y^2(0) - \frac{\xi}{2 - \xi}t}, \tag{36}$$

where $y(0) = (2/A)(x_0^{1-\xi/2}/(2 - \xi))$ from equation (34). This equation implies that at $t^* = (2 - \xi)y^2(0)/\xi$ we get $y(t^*) = 0$ and therefore $x(t^*) = 0$, i.e., the two particles collide. Moreover the velocity $\dot{x}(t)$, in the approximation where we neglect the noise, is

$$\dot{x}(t) = \frac{\mathrm{d}x}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} \sim -\left[y^2(0) - \frac{\xi}{2 - \xi}t\right]^{(\xi-1)/(2-\xi)}.$$

Therefore $\dot{x}(t^*) = 0$ for $1 \leq \xi < 2$, while $\dot{x}(t^*) \to -\infty$ for $\xi < 1$. Consequently, for $1 \leq \xi < 2$, independently of the boundary condition at $x = 0$, once $x(t)$ vanishes it keeps this value forever, i.e., for $1 \leq \xi < 2$ this coalescing solution is the only possible solution as predicted by the general classification given above for the Kraichnan ensemble. Instead for $\xi < 1$, since the velocity $\dot{x}(t)$, when $x(t) = 0$, is non-zero, it is necessary to choose by hand the boundary condition to fix the kind of solution. For reflecting boundary conditions we have the above presented solution converging to the stationary state $\Gamma_s(x)$ given in equation (30), while for absorbing boundary conditions no stationary solution is reached and we have a behavior similar to that for $\xi \geq 1$. We now derive the small x behavior of the propagator $P(x, t|x_0, 0)$ with sufficiently small $x_0 > 0$ from equation (36) valid for both cases $0 < \xi < 1$ with absorbing boundary condition and for $1 < \xi < 2$. Equation (36) says that at time t all pairs of particles with $y^2(0) \leq \xi/(2 - \xi)t$, i.e., with initial separation $x_0 \leq [(a\xi(2 - \xi)/2)t]^{1/(2-\xi)} \equiv x_{\max}(t)$, have already collided at time t. For $1 \leq \xi < 2$ such pairs of particles always coalesce as $\dot{x}(t) = 0$ when $x(t) = 0$. Instead for $0 < \xi < 1$ they coalesce only if we impose the absorbing condition (i.e., completely inelastic collision) at $x = 0$. In both cases we can say that $P(x, t|x_0, 0)$ has developed a singular contribution $m(t)\delta(x)$ at $x = 0$ where

$$m(t) = \int_0^{x_{\max}(t)} \mathrm{d}x_0\, p(x_0)$$

and $p(x_0)$ is the PDF of the initial pair distance x_0. If $\Gamma_s(x, 0) = n_0$ (i.e., the initial particle distribution is a homogeneous random Poisson one) for small x_0 (i.e., $x_0 \ll 1/n_0$) we have $p(x_0) \simeq n_0$, from which we obtain $m(t) \simeq n_0 x_{\max}(t) \sim t^{1/(2-\xi)}$. On the other hand for all pairs with $x_0 > x_{\max}(t)$ we can develop the following argument. Since in the present approximation the dynamics is deterministic one can derive $P(x, t|x_0, 0)$ at $x > 0$ directly by a simple change of variable in $p(x_0)$. More precisely let us call $x_t = x(t)$ and $P(x, t|x_0, 0) = p_t(x_t)$; we can write by conservation of probability

$$p_t(x_t) = p(x_0)\frac{\mathrm{d}x_0}{\mathrm{d}x_t}, \tag{37}$$

where the relation between x_t and x_0 is given by equations (36), (34) and (35). By considering again $p(x_0) \simeq n_0$, it is simple to derive for small $x > 0$ that

$$p_t(x) \sim t^{(\xi-1)/(2-\xi)} x^{1-\xi}.$$

Note that $\lim_{x \to 0} p_t(x) = 0$ for $\xi < 1$ as it has to be for an absorbing condition at $x = 0$. Finally at small x and sufficiently small t, if $\Gamma(x \to 0, 0) = n_0$, the expression

$$\Gamma(x, t) \sim P(x, t | x_0, t) = bt^{1/(2-\xi)} \delta(x) + ct^{(\xi-1)/(2-\xi)} x^{1-\xi}, \tag{38}$$

represents the solution for $1 < \xi < 2$, while it is a solution for the case $0 < \xi < 1$, only when considering absorbing conditions at $x = 0$. With b and c positive constants. This completely agrees with what found in [25, 26].

4.2. Numerical results

In the light of the previous one-dimensional analysis, we discuss an important example in $d = 1$ where $u(x)$ is a spatially correlated Gaussian displacement field with short range correlation function

$$c(x) = c(0) \exp\left\{ -\left| \frac{x}{x_0} \right|^\xi \right\}, \qquad 0 < \xi \leq 2, \tag{39}$$

where x_0 determines the correlation length. We recall that according to the Wiener–Khinchin theorem [16], a given $c(x)$, such that $c(0) > 0$, is a well-defined correlation function of a continuous stochastic field if and only if $c(x)$ is continuous for all x, $\hat{c}(k) \geq 0$ for any k and finally $\int_{-\infty}^{+\infty} dk \, \hat{c}(k) < +\infty$. All these conditions are satisfied by equation (39). Varying ξ between 0 and 2 allows us to study numerically all the phases above described theoretically:

(i) for $0 < \xi < 1$, in the case of reflecting boundary conditions on particle–particle collisions, the particle diffusion admits the stationary solution equation (30) implying a small x behavior

$$\Gamma_s(x) \simeq n_0 \left| \frac{x}{x_0} \right|^{-\xi}, \tag{40}$$

which describes a fractal distribution with dimension $D_f = 1 - \xi$ for $x < x_0$, and exhibits a crossover to a uniform particle distribution with average density n_0 for $x > x_0$. For $x \gg x_0$ instead correlations decay exponentially fast as $h_s(x) \simeq \exp(-|x/x_0|^\xi)$;

(ii) for $1 \leq \xi < 2$, the system admits no stationary behavior and the unique solution corresponds to particle coalescence as described by equation (38).

We run computer simulations to check these behaviors of particle distributions under the effects of a repeated application of the displacement field with correlation (39). The numerical implementation of the dynamics requires the generation of an array of correlated Gaussian random variables $\{u_i\}$ with the prescribed correlator $\langle u_i u_j \rangle \sim c(x_i - x_j)$, where $c(x)$ is given by equation (39) and x_i is the position of the ith particle ($i = 1, \ldots, N$). We used two methods:

(i) The first one considers the Cholesky decomposition method [27] for the covariance matrix. According to this algorithm, an array $\{u_1, \ldots, u_N\}$ of N correlated Gaussian variables with correlation matrix $\hat{C}$ is obtained from the set of $\{w_1, \ldots, w_N\}$ independent random Gaussian variables with zero mean and unitary variance by applying the linear transformation $u = Aw$, where A is the lower diagonal matrix A such that $C = AA^t$ (Cholesky decomposition of the matrix C).

(ii) The second method makes use of discrete fast Fourier transform (FFT) to generate the stationary Gaussian field $\{u(x_n)\}$ over a grid of sites x_n with $n = 1, \ldots, \mathcal{N}$. The values of the field on the site x_n is expressed as the Fourier sum

$$u(x_n) = \frac{1}{\mathcal{N}} \sum_k \mathrm{e}^{\mathrm{i}kx_n} z(k),$$

with $k = 2\pi m / \mathcal{N}$ and $m = 0, 1, \ldots, \mathcal{N} - 1$ to avoid aliasing. Note that $z(k) = z^*(-k)$ to ensure that $u(x_n)$ is real. The choice $z(k) = [\alpha(k) + \mathrm{i}\beta(k)]/2$, with $\alpha(k)$ and $\beta(k)$ independent real Gaussian variables of zero mean and variance $\mathcal{N}\hat{c}(k)/2$, guarantees the set of random variables $\{u(x_n)\}$ representing the discretized version of the field to have the correct power spectrum $\hat{c}(k)$.

Once the field is generated the position of a particle i is updated according to the Euler scheme

$$x_i(t + h) = x_i(t) + \sqrt{h}u(x_i), \tag{41}$$

with time step h. When the Gaussian displacement field is generated through FFT, there is clearly a problem associated to discretization due to the grid of step Δx where the FFT is computed. We assign a particle the displacement u_n if its position at time t falls in the nth bin $[n\Delta x, (n + 1)\Delta x]$ determined by the grid. The choice of the time step is such that $h\langle u(x_i)^2 \rangle \simeq \Delta x^2$ in order to sample the maximal resolution scale allowed by the discretization.

Spatio-temporal patterns obtained via simulations of a system of N particles subjected to displacements of correlation $c(x)$ with $x_0 = 3$ are shown in figure 1 for cases $\xi = 0.5$ and $\xi = 1.5$. The particle are initially distributed uniformly in a one-dimensional simulation box with density $\rho = 1$, and periodic boundary conditions are applied at the ends of the box.

For $\xi = 0.5$ (left panel) and assuming elastic collisions (a reflecting boundary condition when particle trajectories intersect one another), the system exhibits simple particle clustering toward a stationary state. Instead for $\xi = 1.5$ (right panel), particle trajectories coalesce more and more in time and no stationary regime is actually reached. We measured during each run the density correlation function $\Gamma(x)$, as the histogram of the relative particle distance $|x_i - x_j|$ and the results, averaged over several independent runs starting from the uniform particle distributions, are plotted in figures 2 and 3, for $\xi = 0.5, 0.95, 1.5$ respectively. The stationary state for the case $\xi = 0.5$ coincides with that described by theoretical equation (30) which is represented in figure 2 (together with the one of the case $\xi = 0.95$ which is in the same class of behavior). The discrepancy in the amplitudes between numerical and theoretical stationary $\Gamma_s(x)$ is due to the following finite size effect: in simulations of N particles in a volume V, the average conditional density $\Gamma(x)$ is subjected to the integral constraint $\int_V \Gamma(x)\mathrm{d}x = (N - 1)$. Consequently,
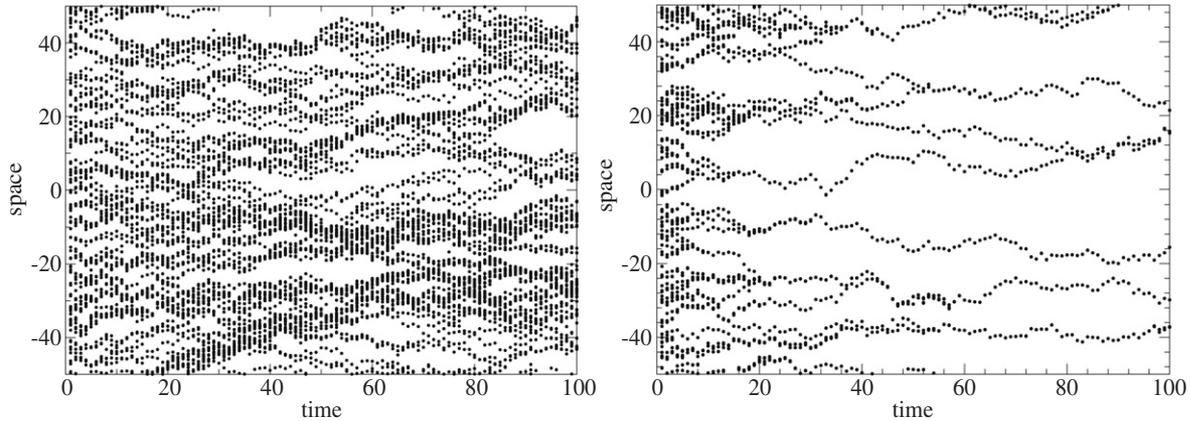
Figure 1. Detail of the space–time pattern generated by the trajectories of $N = 512$ random walkers undergoing the evolution equation (41) with time step $h = 10^{-4}$ and displacements that are not time correlated but have a spatial correlation (39), with $x_0 = 3$. The left panel corresponds to $\xi = 0.5$ showing simple clustering and the right refers to $\xi = 1.5$, for which a coalescence regime occurs.
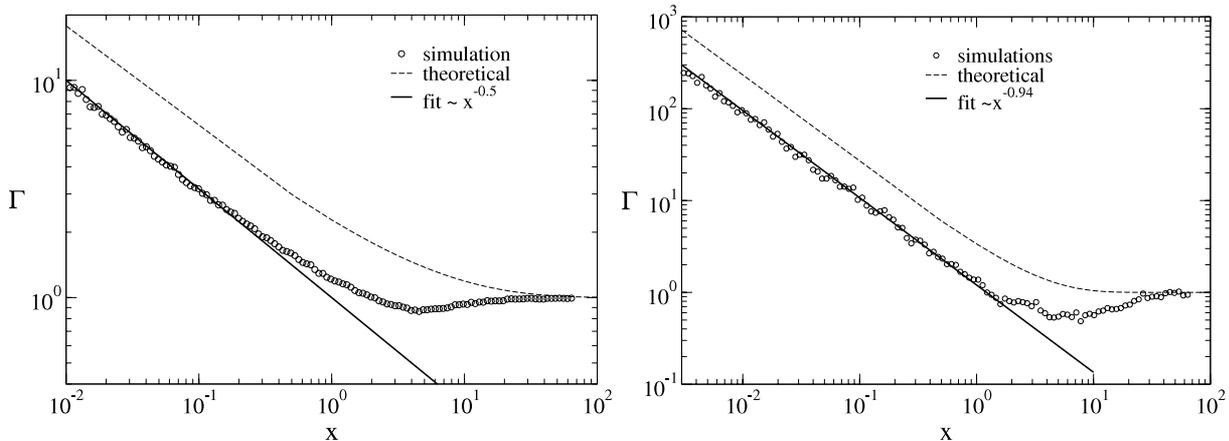

Figure 2. Log–log behavior of the average conditional particle density $\Gamma(x)$ as a function of the separation x generated by the iterations of equation (41) with time step $h = 10^{-4}$ and a displacement field with correlation (39) defined by parameters $x_0 = 3$, $\xi = 0.5$ (left) and $\xi = 0.95$ (right). $\Gamma(x)$ is computed as the histogram of the interparticle distance binned exponentially in 128 intervals. The number of particles is $N = 256$ and simulation data are the results of the average over 1500 independent runs. The small scale decay is expected to be a power law with exponent ξ. The solid line indicates the result of a power law fitting with exponents -0.5, -0.95 respectively and the dashed lines refer to equation (40).

if the dynamics develops a strong positive density correlation at small scale, an artificial negative correlation must appear at larger scales. The typical behavior of $\Gamma(x, t)$ for $\xi = 1.5$ is represented in figure 3. It shows correctly the small x behavior proportional to $x^{1-\xi}$ predicted by theoretical results.
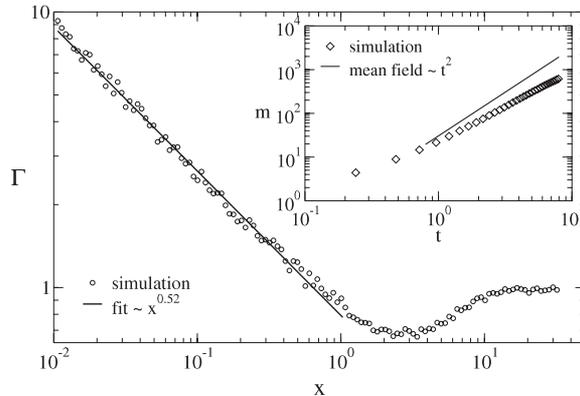
Figure 3. Log–log plot of the average conditional density $\Gamma(x,t)$ as a function of the separation x at time $t = 8$, for $\xi = 1.5$, $x_0 = 3$ and $N = 64$ particles, averaged over 2200 independent runs. the integration time step in equation (41) amounts to $h = 10^{-5}$. The small scale decay is fitted by a power law (solid line) with exponent 0.52, consistent with the theoretical value $(1 - \xi) = 0.5$. The inset shows the time increasing behavior of the coefficient of the δ function contribution to equation (38) due to particle coalescence. The solid line indicates the mean field scaling behavior $\sim t^{1/(2-\xi)}$.

5. Conclusions and discussion

The evolution of several many-particle systems can be described and also generated via the iterated application on the particles of a suitable displacement field which determines their trajectories. In this context, the question we addressed concerns the connection between the statistical properties of the displacement field at a single time and the spatial correlations that arise in the particle distribution during the evolution. We have shown how to derive the partial differential equations describing the continuous time evolution of the two-point correlation function of the particle density under the iterative application of a stochastic displacement field with no temporal memory from the corresponding exact single step evolution equation. This continuous time equations are of a diffusion type and describe simple, fractional, or spatially correlated diffusion of density fluctuations, depending only on the two-point statistical properties of the elementary displacement field. Simple and fractional diffusion occurs in absence of spatial correlations of displacements. Which of the two regimes prevails depends only on the finiteness of the elementary displacement variance. Spatially correlated displacements, instead, determine a Fokker–Planck (FP) equation for density correlations equivalent to a simple Langevin equation whose multiplicative noise is defined again by the two-point displacement correlation function. Such a FP equation predicts a rich phenomenology, ranging from a simple particle clustering to coalescence. These two regimes depend uniquely on the small scale behavior of displacement correlations.

We characterized and classified these regimes in terms of the properties of the solution $\Gamma(x,t)$ of the FP equation for the density correlations which depend on the small scale statistical properties of the displacements fields. This classification can be given in terms of the theory of boundary conditions of the FP equation [28]. Our equations present indeed in

general a singularity at the origin corresponding to the vanishing of the structure function of the velocity field and which has to be treated as a possible additional boundary [23]. By developing a simple theoretical approach in $d = 1$, we have compared the results with the solution classification for the Kraichnan ensemble for d-dimensional turbulence [25], finding a perfect agreement.

Computer simulations, implementing the evolution of a one-dimensional system of particles driven by a Gaussian correlated displacement field, confirm the presence of the clustering and coalescence regimes in agreement with our theoretical predictions.

It is noteworthy that the problem we have considered is different from the motion of particles in random potentials for which in general different FP equations hold [28]. This point is simply clarified by the observation that, in our model, for any spatial correlation of the displacement field, any single particle performs a simple Brownian motion, and the effect of the two-point displacement correlations amounts to correlating the Brownian trajectories of different particles. On the contrary, in the presence of an external random potential even the motion of a single particle is far from being simply Brownian.

It is also interesting to discuss the analogies with the random one-dimensional coagulation process $A + A \rightarrow A$ [29], where particles perform mutually independent random walks and when they collide, one of them is removed according to an assigned rule. The particles are considered to be 'sticky' as they undergo perfectly inelastic collision and coalesce in a single particle independently of the relative velocities. In our case, in contrast, for $1 < \xi < 2$, particles are not 'sticky' and their coalescence is only a consequence of the small scale correlation properties of the displacement field which imply particle–particle collisions occurring at vanishing relative velocity. Therefore, even though we would impose perfectly elastic collisions, particles would coalesce in the same way. The process $A + A \rightarrow A$ is recovered in the limit $\xi \rightarrow 0$ (implying collisions at non-zero relative velocity with probability one) and imposing inelastic collisions at interparticle contact.

In summary, via simple physical arguments, we have generalized the result of [10], concerning only the smooth case $\xi = 2$, to the case of $0 < \xi < 2$, where ξ is the exponent defining the behavior of the displacement correlation $c(x)$ at short scales: $d(x) = [c(0) - c(x)] \sim x^{\xi}$.

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