

## Short-period attractors and non-ergodic behavior in the deterministic fixed-energy sandpile model

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**Abstract.** – We study the asymptotic behaviour of the Bak, Tang, Wiesenfeld sandpile automata as a closed system with fixed energy. We explore the full range of energies characterizing the active phase. The model exhibits strong non-ergodic features by settling into limit-cycles whose period depends on the energy and initial conditions. The asymptotic activity  $\rho_a$  (topplings density) shows, as a function of energy density  $\zeta$ , a devil’s staircase behaviour defining a symmetric energy interval-set over which also the period lengths remain constant. The properties of the  $\zeta$ - $\rho_a$  phase diagram can be traced back to the basic symmetries underlying the model’s dynamics.

Sandpile automata and the associated self-organized critical scenario have been widely used to study the occurrence of avalanche behavior in nature [1]. These automata are, in general, driven-dissipative systems in which matter (sand) or energy is externally added to the system and dissipated by the dynamics itself. Eventually, the input and output balance produces a stationary state with highly fluctuating bursts of activity, that in the limit of an infinitely slow input (time scale separation) is self-similar [2–4]. This condition is customarily embedded in the original sandpile model introduced by Bak, Tang and Wiesenfeld (BTW) [5], and the Manna model [6], which define a deterministic and stochastic relaxation dynamics, respectively.

Recently, fixed-energy sandpiles (FES) automata, which share the same microscopic dynamics of the corresponding BTW sandpiles without external driving and dissipation, have been studied [7–10]. The energy density or sand is therefore a conserved quantity that acts as a tuning parameter. FES present an absorbing state phase transition (APT) [11] at a value of the energy density which is identical to the stationary energy density of the driven-dissipative version of the model [8]. FES with stochastic dynamics (Manna dynamics) belong

to the universality class of APT with a conserved field [12, 13] and the FES with deterministic BTW dynamics defines a different critical behavior with anomalies associated to non-ergodic effects [8], similarly detected in the original driven model [14, 15].

Interestingly, FES automata allows the study of the full phase diagram of sandpile automata. In particular, the overcritical phase is relevant with respect to experimental situations since many avalanche phenomena are naturally poised in this regime. Indeed, relations among sandpile automata, charge density waves and linear interface models have been established and prompt to interesting behavior for the higher total energy properties [16–19].

In this letter we study the FES in order to provide a more detailed characterization of the model over the whole energy range. The density of active sites shows a step-like behavior for increasing energies, with constant plateaus in correspondence of energy intervals which form a hierarchical and symmetric interval set. The model shows strong non-ergodic features, and after a transient relaxes onto periodic orbits which depend upon energy and the initial conditions. Both the period lengths and transient times to reach the periodic orbits remain constant onto the same energy intervals characterizing the plateaus of the activity behavior. We tested the robustness of the observed behavior by looking at the scaling of activity and period lengths with the system size. As a preliminary understanding of some of the features observed in numerical simulations, we discuss analytically some basic symmetry properties of the BTW toppling dynamics that account for the symmetry observed in the activity behavior and the corresponding energy intervals. The present results might also be relevant to intermittency and predictability issues in slowly driven BTW automata [20, 21] and could provide new insights for the study of toppling invariants and recurrent states [22].

Here we consider the original BTW model in the square lattice with periodic boundary conditions. The configuration is specified by giving the *energy*,  $z_i$ , at each site. The energy may take integer values, and is non-negative in all cases. Each active site, *i.e.*, with (integer) energy greater than or equal to the *activity threshold*  $z_{\text{th}}$  ( $z_i \geq z_{\text{th}} = 4$ ), topples and redistributes its energy following the updating rules  $z_i \rightarrow z_i - z_{\text{th}}$ , and  $z_j \rightarrow z_j + 1$  at each of the 4 nearest neighbors of  $i$ . The BTW dynamics with *parallel* updating (all active sites topple at each update) is completely deterministic and Abelian; *i.e.* the order in which active sites are updated is irrelevant in the generation of the final (inactive) configuration [22].

In the FES, the energy density  $\zeta = L^{-2} \sum_i z_i$  is a conserved quantity since no external driving is present and periodic conditions are assumed at the boundaries of the lattice (no dissipation). The value of  $\zeta$  is fixed by the initial condition which is generated by distributing  $\zeta L^2$  particles randomly among the  $L^2$  sites. Once particles have been placed, the dynamics begins. If after some time the system falls into a configuration with no active sites, the dynamics is permanently frozen, *i.e.*, the system has reached an absorbing configuration. By varying  $\zeta$ , FES show a phase transition separating an absorbing phase (in which the system always encounters an absorbing configuration), from an active phase with sustained activity. This has been assumed to be a continuous phase transition, at which the system shows critical behavior. The order parameter is the stationary average density of active sites  $\rho_a$ , which equals zero for  $\zeta \leq \zeta_c$ . Previous studies focused on the behavior close to the critical point in order to characterize the scaling behavior  $\rho_a \sim (\zeta - \zeta_c)^\beta$ , for  $\zeta > \zeta_c$  [8]. Numerical results, however, pointed out possible failures of simple scaling hypothesis and non-ergodic behavior as also observed in slowly driven simulations [14, 15]. These features contradict to the standard scaling observed in stochastic sandpile models [8].

In order to exploit the effect of the deterministic BTW dynamics, we consider the active sites density behavior for the whole range of  $\zeta > \zeta_c$ , *i.e.* the active phase. Since for any finite size the number of possible configurations is finite and the dynamics is deterministic, after a transient the system enters a periodic orbit, visiting recursively a finite set of configurations.

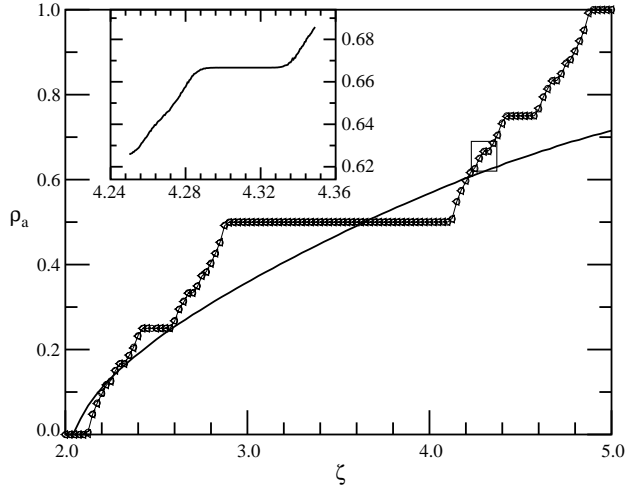


Fig. 1 – Stationary density  $\rho_a = n_T/T$  vs. energy density for lattice sizes:  $L = 50$  (square),  $L = 100$  (open circles),  $L = 200$  (triangles). Data are averaged over  $N = 200$  random initial conditions. The solid line curve represents the activity  $\rho_a$  for the stochastic Manna model ( $z_{th} = 4$ ) with size  $L = 100$ . Inset: enlargement of the region  $4.23 < \zeta < 4.26$  (box) showing that smaller plateaus are observed on finer scales.

In such a case,  $\rho_a$  can be computed as the ratio of the total number of topplings at a site in a period and the length of the period itself. In fig. 1 we plot the activity density  $\rho_a$  in the stationary state as a function of the energy  $\zeta$ .

The activity has a stairway structure with constant activity plateaus over energy intervals distributed symmetrically with respect to the energy value  $\zeta = 3.5$ . The step-like behavior is in contrast with the smooth and regular curve obtained in the case of a stochastic model in which, for each toppling event, energy grains are redistributed at random between neighbors [6]. The value of  $\rho_a$  corresponding to each value of  $\zeta$  is the average over at least  $N = 200$  initial conditions (IC). In the activity plateaus we do not observe a dependence of the stationary activity with respect to IC. For energies in the central and larger plateau, for instance, all initial conditions evolve to an orbit with period 2 and a single toppling per site per period for large lattice size. Correspondingly, the observed value of activity is delta-like distributed at  $\rho_a = 1/2$ . On the contrary, for energies outside of the plateaus, we find that different ICs may generate different values of the stationary  $\rho_a$  which exhibits a moderate dispersion around the mean reported in the plot of fig. 1. The mean value is not recovered in individual runs, signalling the non-ergodic properties already noticed in ref. [8]. It is worth remarking that this scenario presents striking analogies with complex phase-locking plateaus characterized by devil's staircases found in numerous non-linear driven systems [23].

A confirmation of the non-ergodic nature of the BTW model is provided by measuring, as a function of energy, the length of the periodic cycle where the same sequence of topplings is executed over and over (periodic orbit). The period lengths depend on the initial condition and on the system's energy. In fig. 2 we plot the average period length  $\langle T \rangle$  as a function of  $\zeta$  for a system of size  $L = 100$ . In the same figure  $\rho_a$  (suitably rescaled) is shown to better appreciate the very same correspondence of the plateau structures. On the plateaus the cycles period assumes extremely low values, such as  $T = 2, 4, 6$ . As for the activity  $\rho_a$ , the period lengths are delta-distributed in correspondence of the plateaus, *i.e.* all ICs define the same

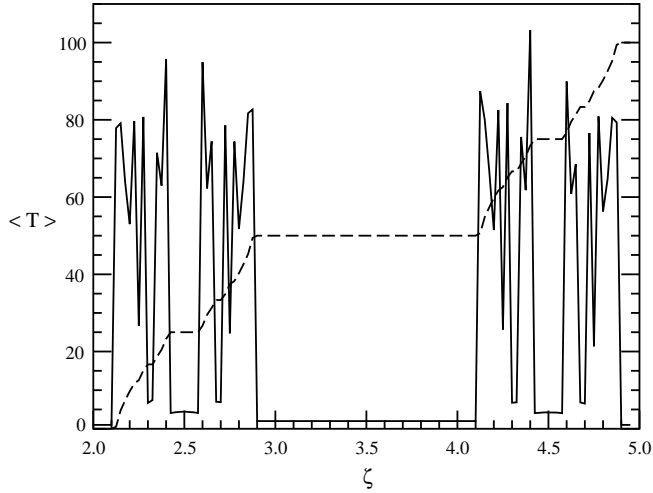


Fig. 2 – Behavior of average period  $\langle T \rangle$  as a function of energy density for system size  $L = 100$ . The average is performed over at least  $10^3$  randomly chosen initial conditions. The dashed line indicates the corresponding stationary  $\rho_a$  after a suitable rescaling for a direct comparison.

$T$ . Outside the energy plateaus, different IC generate different orbits defining a non-trivial period distribution. It is interesting to note that a similar irregular behavior of periods is found in other deterministic dissipative systems with continuous state variables which exhibit unpredictable transient dynamics but admit asymptotic periodic states as attractors [24]. The robustness of the present picture has been tested through simulations of systems of increasing size. In fig. 1 we report data for system sizes  $L = 50, 100$  and  $200$ . The discrete nature of the system allows to discriminate energies on a finer scale ( $1/L^2$ ), and the structure of the energy intervals defined by activity plateaus progressively reveals a richer structure [25]. With data at hand we cannot say whether, in the infinite-size limit, the system is characterized by an energy range made exclusively by infinitesimal intervals and, correspondingly, the curve  $\rho_a(\zeta)$  is discontinuous. In this case also the transition inactive-active would be discontinuous, ruling out the possibility even to define the critical exponent  $\beta$  characterizing the transition.

It is also worth discussing the non-ergodic effect with respect to IC. This feature does not disappear for increasing sizes, even though a single specific value for the period and the number of topplings per period (and therefore for  $\rho_a$ ) emerges with overwhelming frequency. As an illustration of the basic phenomenology, in fig. 3 we report the normalized histograms of periods  $T$  collected from a sample of  $10^3$  random IC, at different sizes and  $\zeta = 4.25$ . The results indicate that, as the size  $L$  grows, each distribution seems to become more and more peaked around a given period value  $T_M$  ( $T_M = 8$  for  $\zeta = 4.25$ ) and the inset of fig. 3 shows the increase of the histogram maximum  $P_M$  (peak) with  $L$ . This could be consistent with a period selection scenario in which, in the thermodynamic limit, all IC select a single period length. Finite-size effects, however, are still too large to conclude definitely that such a mechanism occurs for  $L \rightarrow \infty$ .

A first understanding of the behavior of the fixed-energy BTW model can be achieved analytically by exploiting the symmetry contained in the deterministic dynamical rules. The non-ergodicity with respect to the IC can be traced back to conserved invariants of the dynamics. For instance, the initial configuration in which all grains are concentrated in the central site of the lattice will necessarily evolve, preserving its central symmetry, and asym-

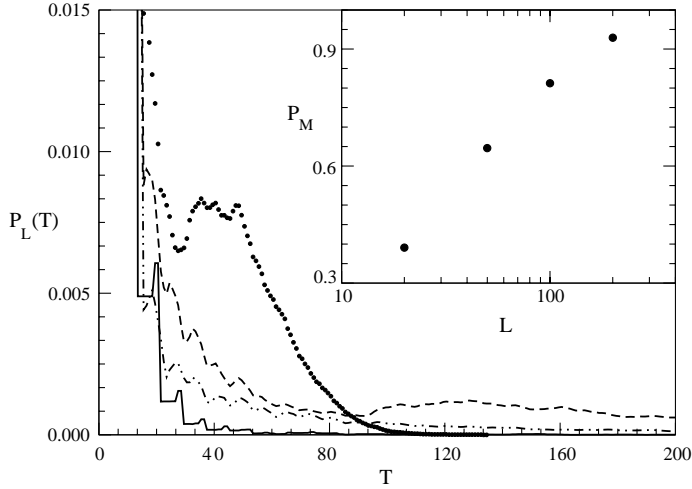


Fig. 3 – Histograms of periods collected in the stationary states at the energy density  $\zeta = 4.25$  and different system sizes:  $L = 20$  (circles),  $L = 50$  (dashed line),  $L = 100$  (dot-dashed line) and  $L = 200$  (solid line). The maxima  $P_M$  of the histogram, corresponding to  $T = 8$ , lie outside of the graph. The inset shows the increasing of  $P_M$  with the system size  $L$ ; this numerical evidence seems to support the view that period selection becomes stronger as the size increases.

metric configurations will never be visited. Other quantities conserved by the dynamics can be constructed by mimicking the center of mass and the angular momentum of an isolated Hamiltonian system. A straightforward calculation shows that

$$\begin{aligned}
 MX &= \sum_{x,y} xz_{x,y}, \\
 MY &= \sum_{x,y} yz_{x,y}, \\
 MXY &= \sum_{x,y} (x^2 - y^2)z_{x,y}
 \end{aligned} \tag{1}$$

are dynamical invariants if taken modulus the linear size of the lattice, with  $x$  and  $y$  indicating the integer coordinates of the lattice sites with respect to the origin ( $x = 0, y = 0$ ). The phase space relative to a given value of the energy is therefore partitioned according to the values assumed by conserved quantities [26]. A detailed and comprehensive discussion on toppling invariants can be found in ref. [27]. It is worth noticing that the previous considerations hold for any finite system, but it is not possible to generalize them to the thermodynamic limit.

A striking symmetry property of the model concerns the distribution of toppling per site within any given periodic orbit. In order to exploit this property, let  $n_i$  be the number of times that the site  $i$  topples during a period. Since after a period the initial configuration is restored, each site has given to its neighbors as many grains as it has received, which translates in

$$4n_i = \sum_{j \in \text{nn}} n_j, \tag{2}$$

where  $j \in \text{nn}$  denote the sum over the nearest neighbors of the site  $i$ . Let us suppose that not all  $n_i$  are the same and define  $i^*$  the site with the minimum number of topplings in a

period  $n_{i^*}$ . The quantity  $p_i = n_i - n_{i^*}$  satisfies the same equation as the  $n$ 's and  $p_{i^*} = 0$  is the minimum  $p$ . Since  $p_{i^*}$  is the sum of four  $p$ 's, that, by definition, are non-negative, we have that all  $p$ 's in the neighbors of  $i^*$  are zero. Iterating the argument, the entire lattice can be covered, and one obtains that all  $p$ 's are zero and therefore all  $n$ 's are the same. This results immediately implies that any site topples exactly the same number  $n_T$  of times over a given period of length  $T$  and eventually leads to the interesting result

$$\rho_a = \frac{n_T}{T}. \tag{3}$$

Finally, we can focus on the evident symmetries of the activity behavior  $\rho_a(\zeta)$  and the corresponding energy intervals. The overall symmetric structure of the energy intervals centered at  $\zeta = 3.5$  can be accounted for by the following argument. Let us consider two energy fields  $z_i(t)$  and  $z'_i(t)$ , that at time  $t$  are related as  $z_i(t) = 7 - z'_i(t)$ , but otherwise arbitrary. It can be easily verified that the above relation holds at all subsequent times. In fact, the evolution equation for  $z_i$ 's can be written as

$$z_i(t + 1) = z_i(t) + \sum_{j \in \text{nn}} \Theta[z_j(t) \geq 4] - 4\Theta[z_i(t) \geq 4], \tag{4}$$

and by substituting  $z_i(t) \rightarrow 7 - z'_i(t)$  we obtain

$$z_i(t + 1) = 7 - z'_i(t) + \sum_{j \in \text{nn}} \Theta[z'_j(t) \leq 3] - 4\Theta[z'_i(t) \leq 3]. \tag{5}$$

By using the relation  $\Theta[g \leq 3] = 1 - \Theta[g \geq 4]$  we readily obtain

$$z_i(t + 1) = 7 - z'_i(t + 1). \tag{6}$$

This show recursively that, if at a given time  $t$  the two fields are related by  $z_i(t) = 7 - z'_i(t)$ , they are similarly related at all subsequent times. In addition if a site topples in one of the two lattices, the corresponding site in the other will not, and vice versa, leading to  $\rho_a(z_i) = 1 - \rho_a(z'_i)$ . The full account of the observed symmetry under transformation  $\zeta \rightarrow 7 - \zeta$  and  $\rho_a(\zeta) \rightarrow 1 - \rho_a(7 - \zeta)$  would require that an equal proportion of IC settle into configurations related by  $z'_i = 7 - z_i$ . While it is not possible to show the latter statement analytically, the symmetry is recovered exactly in the diagram of fig. 1.

Despite these simple symmetry arguments account for several features of the BTW model, many other issues remain unsettled. In particular, the activity and period length plateaus extending over the energy intervals do not find a simple analytical explanation. As well, the continuous or discontinuous nature of the  $\rho_a(\zeta)$  behavior cannot be discriminated by the simple analytical arguments provided here.

In summary, we presented a numerical and analytical study of the BTW automaton with fixed energy. We find strong non-ergodic features related to the deterministic dynamics of the model in the whole energy range. Some features of the model can be understood in terms of the basic symmetries of the dynamics. A full rationalization of the present results could help to understand the scaling anomalies and the universality class of deterministic self-organized critical sandpile models.

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