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Sporadicity and synchronization in one-dimensional asymmetrically coupled maps

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Abstract. A one-dimensional chain of sporadic maps with asymmetric nearest-neighbour couplings is studied numerically. It is shown that in the region of strong asymmetry the system becomes spatially fully synchronized, even in the thermodynamic limit, while the Lyapunov exponent is zero. For weak asymmetry the synchronization is no more complete, and the Lyapunov exponent becomes positive. At variance with the case of non-sporadic maps, where the coherence length does not change in time, here the coherence length exhibits strong fluctuations. In addition one has a clear relation between temporal and spatial chaos, i.e. a positive effective Lyapunov exponent corresponds to a lack of synchronization and vice versa.

An interesting property of extended systems is that they can exhibit complex behaviour both in space and time, that is the chaotic evolution of a spatial pattern. This kind of phenomena, called 'spatio-temporal' chaos, has been the focus of considerable interest in different fields, such as chemical reaction–diffusion systems, Bénard convection, turbulence, modelling of brain function, and so on [1,2]. Recent experiments have revealed the emergence of large-scale spatio-temporal patterns of activity in many brain areas, such as the olfactory system or the visual cortex [3]. Coherent spatial patterns also play an important role in the behaviour of turbulent fluids. Here numerical simulations and experiments show the emergence of strong coherent vortex structures which are responsible for intermittency and possible anomalous dimensions in the scaling law [4]. Moreover, coherent structures may play a non-trivial role in the predictability problem based on two-dimensional turbulence, an important subject from both a theoretical and meteorological point of view [5].

A direct study of the spatio-temporal behaviour from the full equations, such as the Navier–Stokes equations, in general, is quite complicated, even numerically. To overcome this problem, and gain more insight in the basic mechanisms, much work has been devoted to the study of models, such as coupled maps on a lattice. These are a crude, but non-trivial, approximation of extended systems with discrete space and time, but continuous states [2]. The simplest form is given by a set of N continuous variables x_i which evolve in (discrete) time as

$$x_i(n+1) = (1-\alpha_i-\beta_i)\, f(x_i(n)) + \alpha_i f(x_{i-1}(n)) + \beta_i f(x_{i+1}(n)) \,. \qquad (1)$$

The index i denotes the ith site on a one-dimensional lattice, and only nearest-neighbour interactions are taken into account. The strength of the couplings and their spatial

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homogeneity, as well as the type of boundary conditions used, may depend on the type of physical problem. For example, in the case of shear flow, boundary layers or convection, there is a privileged direction. This can be introduced in the model (1) by taking asymmetric couplings [6, 7]. We stress that in real systems, e.g., the boundary layer, one has laminar and intermittent phases both in space and in time. Therefore a 'good' model (1) should reproduce these features, at least on a qualitative level.

The most appropriate boundary conditions in these systems are open boundary conditions. A thermal reservoir can be included by giving a special evolution law to the sites on the boundary. We do not consider this generalization here.

In this paper we shall study a one-dimensional chain of coupled maps (1) with homogeneous asymmetric couplings and open boundary conditions. The equations of motion are given by (1), with couplings

$$\begin{aligned} &\alpha_i = \gamma_1 \qquad \beta_i = \gamma_2 \qquad \text{for} \quad i = 2, \ldots, N-1 \\ &\alpha_1 = 0 \qquad \beta_1 = \gamma_2 \qquad \alpha_N = \gamma_1 \qquad \beta_N = 0\,. \end{aligned} \tag{2}$$

Without loss of generality we assume $\gamma_1 > \gamma_2$. The system (1) with these couplings was studied in [7, 8] for the case of a chaotic single map $f(x)$. In these papers the interest was in the possible emergence of spatial coherent patterns due to the asymmetrical couplings.

A simple inspection reveals that the system (1), (2) possess the uniform solution

$$x_i(n) = \widetilde{x}(n) \qquad \widetilde{x}(n+1) = f(\widetilde{x}(n))\,. \tag{3}$$

To study the stability of the uniform state one can linearize (1) about the uniform solution (3) and study the spectrum of the fluctuations [7]. This consists of a uniform eigenmode with eigenvalue ρ_0 and $N-1$ non-uniform eigenmodes with spectrum

$$\rho(k) = \rho_0\left[1 - \gamma_1 - \gamma_2 + 2\sqrt{\gamma_1\gamma_2}\cos(k)\right] \tag{4}$$

where $k = \pi m/N$ with $m = 1, 2, \ldots, N-1$, and $\rho_0 = \exp(\lambda_0)$, where λ_0 is the largest Lyapunov exponent of the single map $f(x)$.

For $\gamma_1 \neq \gamma_2$ this spectrum possess a gap at $k = 0$ since $\rho(k \to 0)$ is less than the $k = 0$ eigenvalue ρ_0. Therefore, if γ_1 and γ_2 are such that $\rho_0 > 1$ and $|\rho(k)| < 1$ then all non-uniform fluctuations are stable. The only instability that is left is the instability to uniform fluctuations, which is inherent in the chaotic nature of the single map.

From this, one concludes that the uniform state (3) is stable. These predictions for the case of a single chaotic map $f(x)$ are essentially confirmed by numerical simulations [7, 8]. The finite coherence length l_c—that is, $x_i(n) \simeq \widetilde{x}(n)$ for $i < l_c$, while for $i > l_c$ the x_i are spatially irregular—is due to numerical noise. Indeed l_c decreases logarithmically with the noise level in the numerical simulations.

The open boundary condition can be seen as a defect in the chain. This scenario is essentially unchanged if one uses periodic boundary conditions and 'softer' defects, such as, e.g., the interchange of γ_1 and γ_2 in a finite fraction of sites [8].

The models discussed in [7, 8] are, however, unable to reproduce the spatio-temporal intermittency observed in real systems, e.g. boundary layer. The coherence length, in fact, does not fluctuate in time. It is therefore interesting to try to improve the model to grasp more complex behaviours.

The argument of [7] neglects the fluctuations of the chaotic degree along the trajectory, since the Lyapunov exponent λ_0 gives only the typical chaotic degree. If one considers a temporal window $[t - \Delta t/2, t + \Delta t/2]$ it is possible to repeat the arguments of [7] by simply replacing λ_0 in (4) with the effective Lyapunov exponent [9] which measures the local exponential rate of growth for the tangent vector z around the time t:

$$\chi_{\Delta t}(t) = \frac{1}{\Delta t} \ln \frac{|z(t + \Delta t/2)|}{|z(t - \Delta t/2)|}. \tag{5}$$

We expect that the scenario discussed in [7] may fail if the fluctuations of $\chi_{\Delta t}$ are strong, opening the way to reacher behaviours.

A rather natural way to address this problem is to study the system (1), (2) with the single map

$$f(x) = x + cx^z \bmod 1 \qquad (z \geqslant 1). \tag{6}$$

For $z \geqslant 2$ the dynamical system $x(n+1) = f(x(n))$ shows a sporadic behaviour [10], i.e. an initial disturbance grows in time as a stretched exponential with an exponent less than 1. Hence, the Lypunov exponent λ_0 vanishes.

In the case of $1 \leqslant z < 2$, the map (6) behaves as an ordinary chaotic system with positive Lyapunov exponent λ_0. Here we obtain for the system (1) the scenario discussed in [7, 8] for the logistic map. We thus find that for symmetric couplings the chain does not synchronize, while in the asymmetric case a finite coherence length may appear depending on the value of $\gamma_1 - \gamma_2$ and λ_0. To be more specific we can distinguish two cases. In the first case $\rho_0 > 1$ and $|\rho(k)| < 1$, e.g. when $z = 1.5$, for which $\lambda_0 = 0.56$, $\gamma_1 = 0.7$ and $\gamma_2 = 0.01$, so that all the non-uniform eigenmodes are stable and the system synchronizes with a finite l_c [7, 8]. In the second case, e.g. when $z = 1.5$, $\gamma_1 = 0.7$ and $\gamma_2 = 0.1$, there exist some k for which $|\rho(k)| > 1$, and the system does not synchronize.

We now turn to the more interesting case of $z \geqslant 2$, when the single map is sporadic. The results here reported are obtained from numerical simulations, where the parameter c of the single map has been kept fixed to 1, and with $\gamma_1 = 0.7$ and γ_2 which varies in the range $(0, \gamma_1]$. The typical system size considered is $N = 200$, but we checked some results also for larger values of N.

If the couplings are symmetric the system is not spatially synchronized, $l_c \simeq 1$. Moreover, even if the Lyapunov exponent of the single map λ_0 is zero, the Lyapunov exponent λ of the global system is always positive, confirming the non-relevance of the uniform mode for the dynamics.

As for the case of a non-sporadic chaotic map $f(x)$, when the couplings are not equal two qualitative different behaviours appear depending on the value of $\gamma_1 - \gamma_2$. However, unlike the previous case, here the synchronization, when present, is complete, i.e. $l_c = N$. In figure 1 we show τ, the fraction of the time that the system is completely synchronized, and λ, the Lyapunov exponent, as functions of γ_2.

A first regime, with $\tau \simeq 1$, appears for $\gamma_1 - \gamma_2$ large, which with our parameters means $\gamma_2 \leqslant 0.2$. Here $\rho_0 = 1$ and $|\rho(k)|$ is much smaller than one for all k, e.g. $\max_k |\rho(k)| = 0.85$ for $\gamma_2 = 0.2$. According to the above argument the uniform solution is locally stable. Indeed we find that the system synchronizes with $l_c = N$. A detailed analysis of the evolution reveals that the system spends most of the time completely synchronized. The ordered state is interrupted by intervals where the coherence length becomes very small. This is a very different qualitative behaviour with respect to the case $z < 2$ and the case discussed in [7, 8]. In fact in these latter cases the system is always partially synchronized, with l_c undergoing small fluctuations about its time average. All the simulations in this range of parameters give a value of τ very close to one. For example, when $\gamma_2 = 0.1$, we find, on

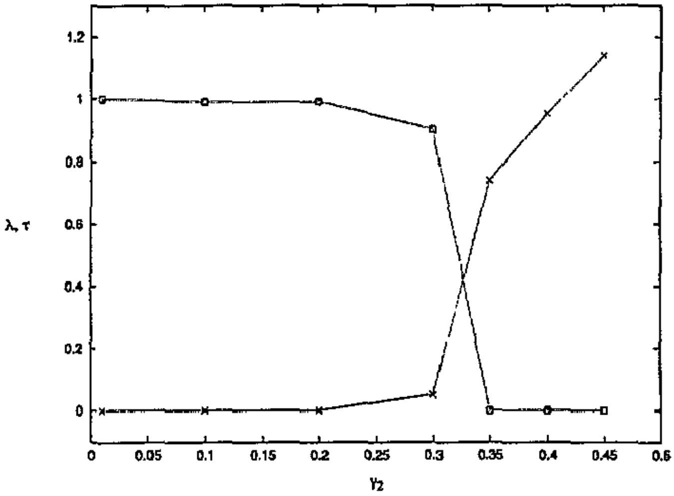

Figure 1. *Lyapunov exponent λ (crosses) and probability of complete synchronization τ* (squares) versus γ_2. $N = 200$ and $\gamma_1 = 0.7$. The integration steps are 3×10^8 for $\gamma_2 = 0.3$, 4.7×10^8 for $\gamma_2 = 0.2$ and a few millions of steps in the other cases.

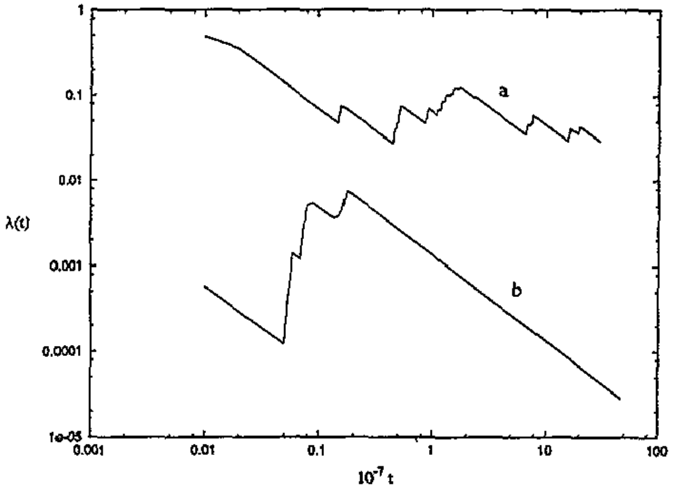

Figure 2. Time evolution of the mean value defining the Lyapunov exponent λ for $\gamma_2 = 0.2$ (lower curve) and $\gamma_2 = 0.3$.

a run of 5×10^6 steps, $\tau = 0.997$, while for $\gamma_2 = 0.2$, and on a run of 4.7×10^8 steps, we find $\tau = 0.999$. The Lyapunov exponent is always very small. It is difficult to decide if λ is really going to zero or stays very small. This difficulty is due to the fact that every time the system desynchronizes, the local Lyapunov exponent increases. As a consequence the numerical value of λ does not saturate in time but has kicks which increase its value, see figure 2. The quantity $(1/t)\ \ln(|z(t)|/|z(0)|)$ decreases as $1/t$ during the synchronized phase, and increases very fast when l_c is small and the system is not synchronized. To compute λ with a good numerical precision one needs many non-synchronized phases.

A different behaviour appears for $\gamma_1 - \gamma_2$ small which, with our parameters, means

$\gamma_2 \geqslant 0.35$. In this regime $\rho_0 = 1$ but $|\rho(k)|$ is larger than one for some values of k, e.g. when $\gamma_2 = 0.35$ we have $\max_k |\rho(k)| = 1.04$. Thus the uniform state is locally unstable. Indeed we find that the coherence length is almost always $l_c \simeq 1$ and the system is never completely synchronized—at least on a run of 3×10^7 steps. Moreover, τ is zero and λ is positive. For example, $\lambda = 0.74$ when $\gamma_2 = 0.35$.

These two behaviours appear to be related in a continuous way through the crossover region where $\max_k |\rho(k)|$ is very close to one, e.g., $\max_k |\rho(k)| = 0.92$ for $\gamma_2 = 0.3$. In this region λ begins to be definitely non-zero and τ becomes smaller than one. For example, $\lambda \simeq 0.05$ and $\tau \simeq 0.9$ again for $\gamma_2 = 0.3$ on a run of 3×10^8 time steps.

In order to stress the correlation between desynchronization and temporal chaoticity, we show in figure 3 the effective Lyapunov exponent $\chi_{\Delta t}(t)$ and $l_c(t)$ as functions of t. It is evident that a large $\chi_{\Delta t}(t)$ corresponds to $l_c \ll N$ and vice versa. This happens both for small and large values of $\gamma_1 - \gamma_2$.

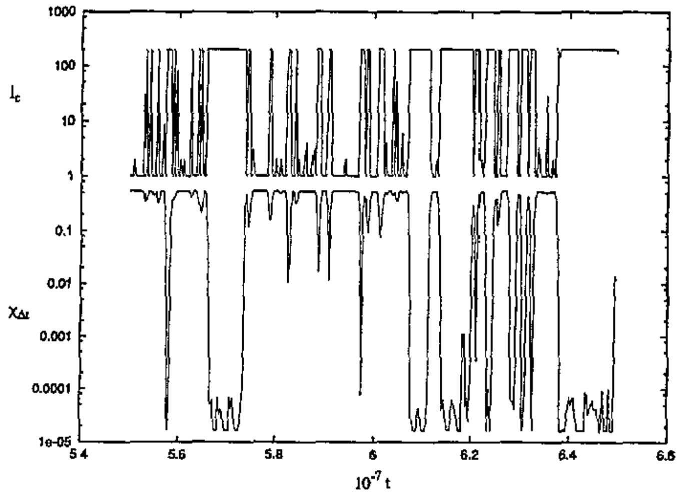

Figure 3. Effective Lyapunov exponent $\chi_{\Delta t}$ and coherence length l_c versus t. $N = 200$, $\gamma_1 = 0.7$, $\gamma_2 = 0.3$ and $\Delta t = 3 \times 10^4$.

The map (3) in the sporadic regime, i.e. when $f(x)$ is given by (6) with $z > 2$, does not possess an asymptotic measure, hence the initial condition may have a non-trivial role. This fact may be relevant for the coupled map system, at least for some values of γ_2, so that the numerical value of λ and τ computed for a long finite time can depend on the chosen initial condition. This dependence is more dramatic in the crossover region, and makes it difficult to define clear boundaries.

The strong correlation between the spatial behaviour—synchronization and desynchronization—and temporal features—laminar phase with small effective Lyapunov exponent and intermitted bursts with large $\chi_{\Delta t}(t)$ and small $l_c(t)$—reproduces, at least on a qualitative level, some properties observed in thermal convection and the boundary layer. We stress that this correlation is present in the system (1) only for the map $f(x)$ discussed here, and not in the systems studied in [7, 8].

We conclude with some remarks. The behaviour observed in the sporadic case is rather different from that previously studied [7, 8]:

(i) one can have for long time intervals a complete synchronization, i.e. $l_c = N$;

(ii) this complete synchronization does not change by the addition of a small noise: the effect of the noise is a small decrease in of the fraction of time in which $l_c = N$;

(iii) there is a strong correlation between the spatial and temporal behaviour, in particular one observes that the effective Lyapunov exponent $\chi_{\Delta t}(t)$ is large in the intervals of time in which $l_c(t) \ll N$, while $\chi_{\Delta t}(t) \simeq 0$ if $l_c(t) = N$.

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