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Paris

Fluctuations and generalized response theory in statistical physics: An application to granular materials

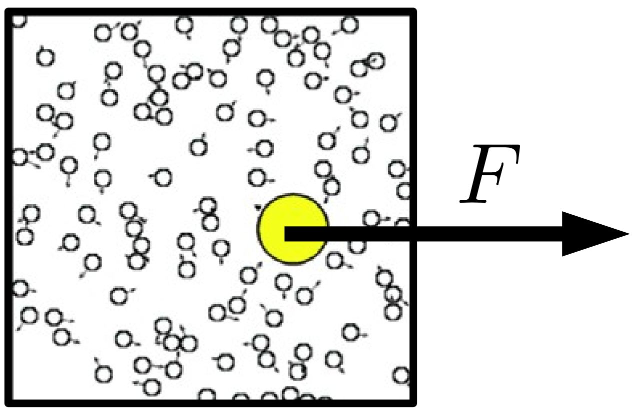
Strolling on Chaos, Turbulence and Statistical Mechanics

Roma, September 2014

Equilibrium fluctuation-dissipation relations (FDR)

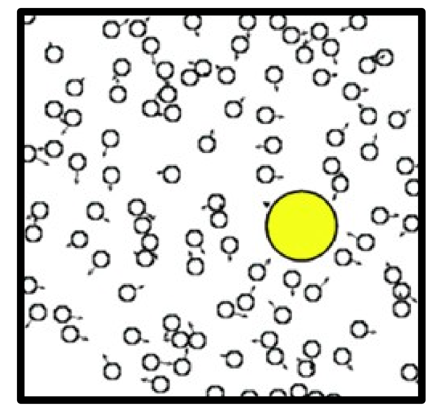
We shall assume that the average regression of fluctuations will obey the same laws as the corresponding macroscopic irreversible process

Onsager-Machlup



Perturbed system

$$\frac{\overline{\delta v(t)}}{\delta v(0)}$$



Unperturbed system

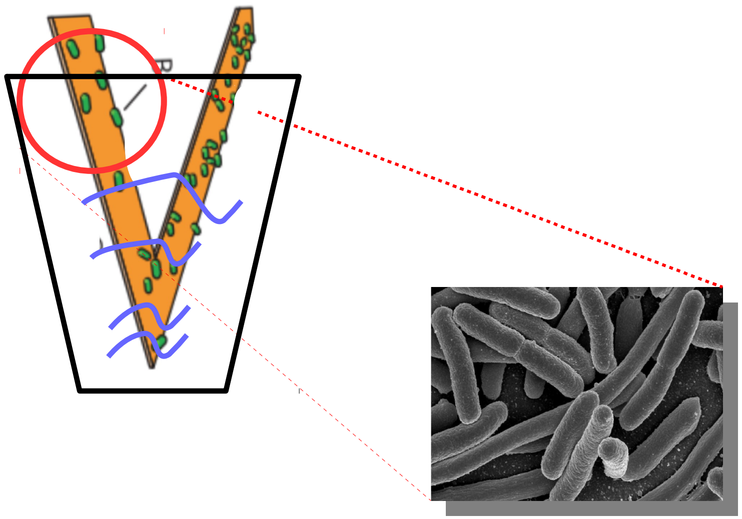
$$\langle v(t)v(0)\rangle$$

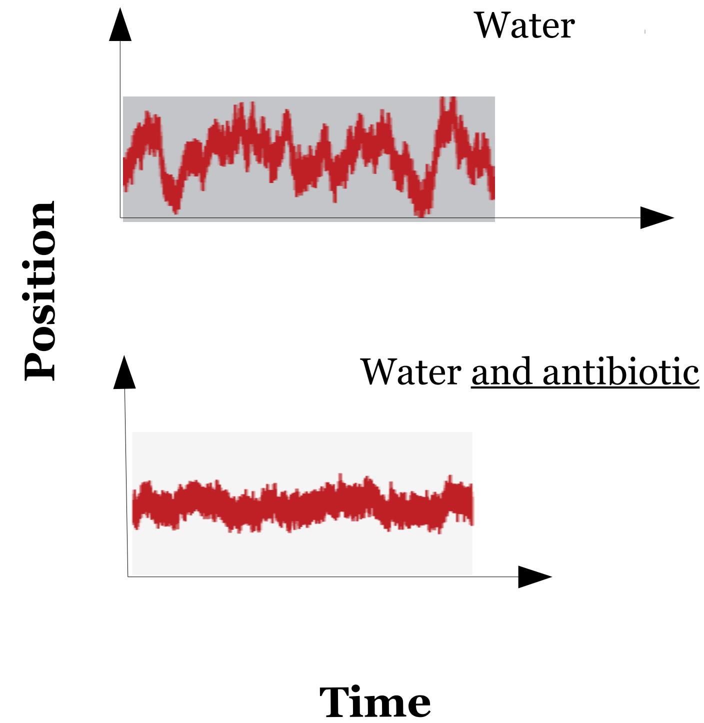
$$\mu = \frac{D}{T}$$

Einstein Relation

Fluctuations of a cantilever and living bacteria

G. Longo et al., Nature nanotechnology 8.7, 522-526 (2013)





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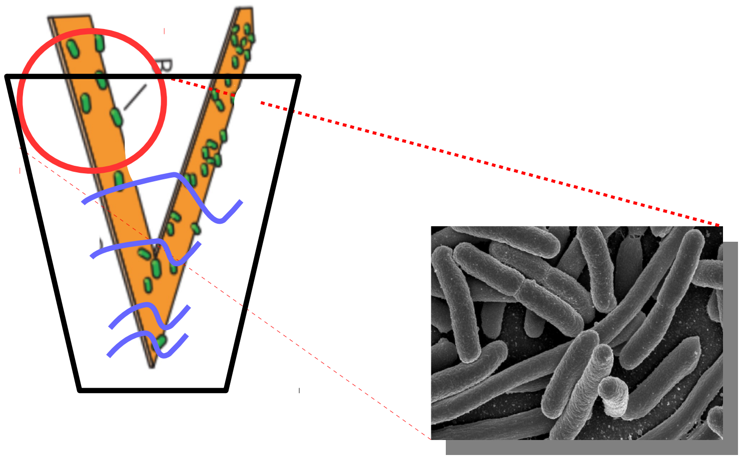
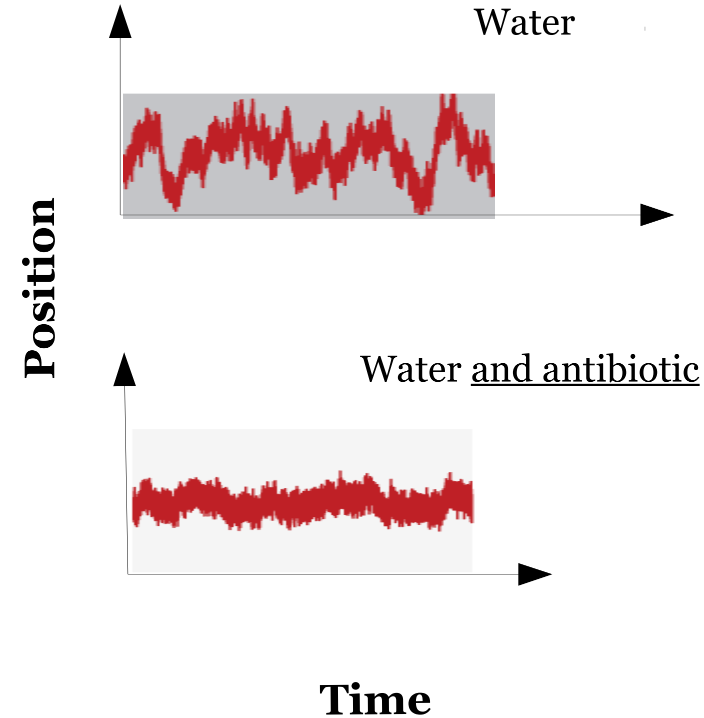
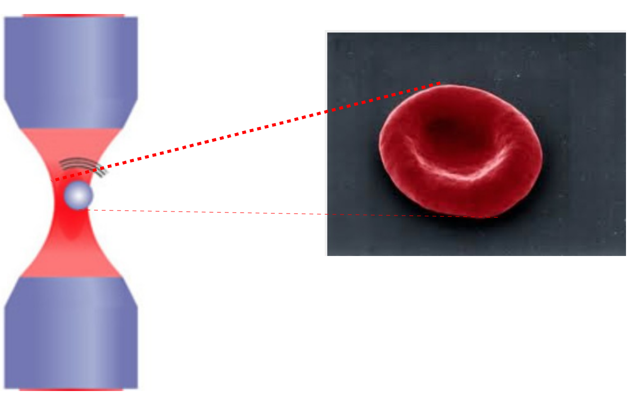
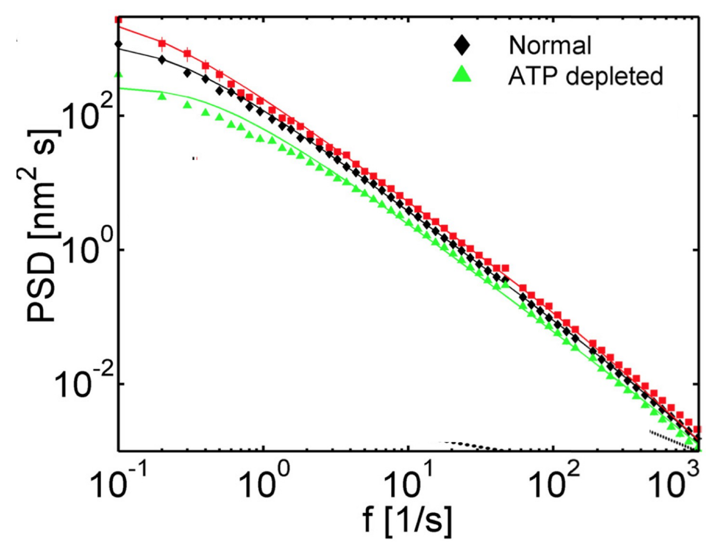


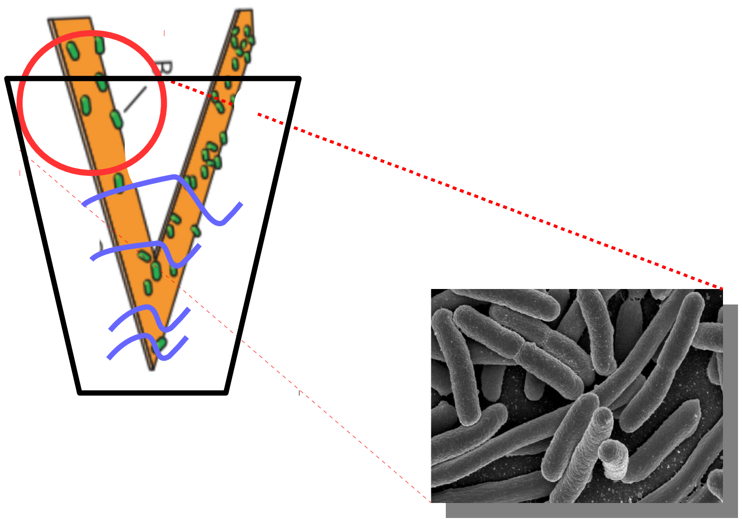


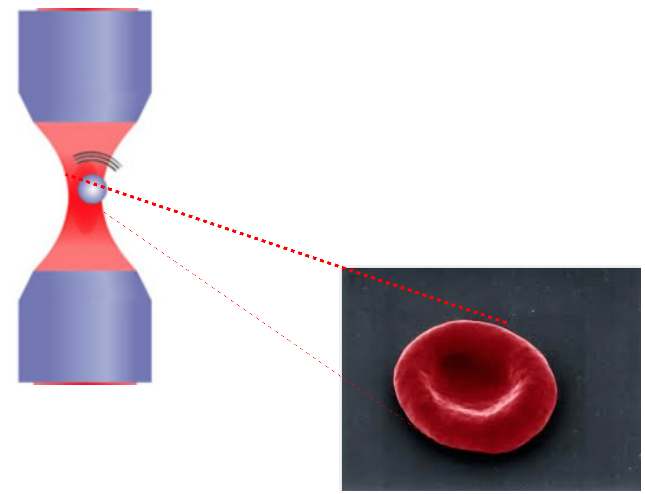
ATP-dependent fluctuations in a red-blood cell membrane

T. Betz et al. *PNAS* 106.36: 15320-15325 (2009)





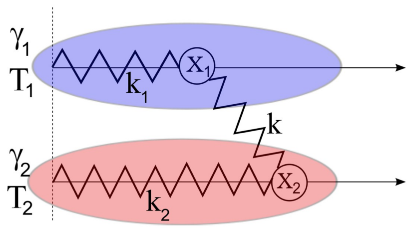




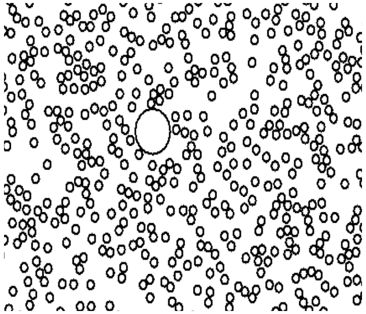
Active materials (Thermal *vs* Non-Thermal noise)

FDR “Violations”

Plan of the talk



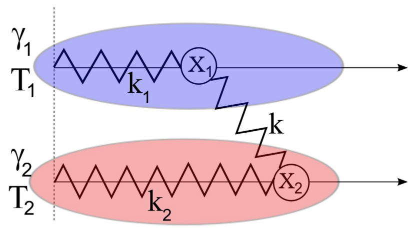
Generalized fluctuation response relations



Application to a granular gas

Experimental results

Plan of the talk



Generalized fluctuation response relations

Application to a granular gas

Experimental results

Generalized Fluctuation-dissipation relation (GFDR)

M Falcioni, S Isola, and A Vulpiani Physics Letters A , 144:341, 1990.

$$\mathbf{x}(t) = S^t \mathbf{x}(0) \xrightarrow{\text{Perturbed trajectory}} \mathbf{x}'(0) = \mathbf{x}(0) + \delta\mathbf{x}$$

$$\delta\mathbf{x} = (0, \ldots, \delta x_j \ldots, 0)$$

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$$\mathbf{x}(t) = S^t \mathbf{x}(0) \longrightarrow \mathbf{x}'(0) = \mathbf{x}(0) + \delta \mathbf{x}$$

Perturbed trajectory

$\delta \mathbf{x} = (0, \ldots, \delta x_j \ldots, 0)$

if $\rho(\mathbf{x})$ is non- vanishing and differentiable and the system is mixing

Relaxation of an external perturbation

Steady state correlation (unperturbed system)

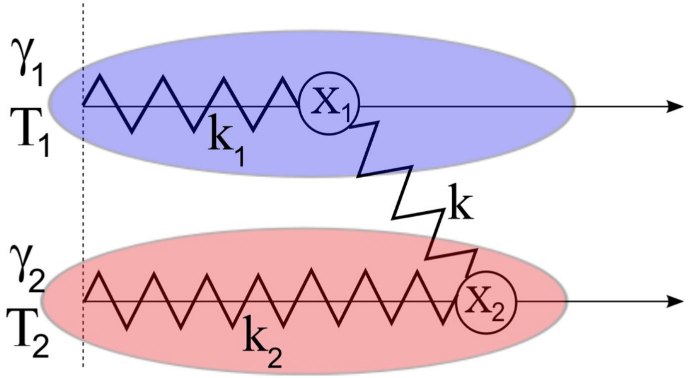
$$\overline{\frac{\delta x_i(t)}{\delta x_j(0)}} = - \left\langle x_i(t) \left. \frac{\partial \ln \rho(\mathbf{x})}{\partial x_j} \right|_{t=0} \right\rangle$$

Maxwell-Boltzmann distribution

$$P(v) \propto e^{-\frac{\beta}{2} v^2} \quad \Longrightarrow \quad \overline{\frac{\delta v(t)}{\delta v(0)}} = \beta \left\langle v(t) v(0) \right\rangle$$

Toy Model

D Villamaina, A Baldassarri, A Puglisi, A Vulpiani – J. Stat. Mech. P07024 (2009)



$$\gamma_1 \dot{x}_1 = k x_2 - (k_1 + k) x_1 + \sqrt{2\gamma_1 T_1}\phi_1$$

$$\gamma_2 \dot{x}_2 = k x_1 - (k + k_2) x_2 + \sqrt{2\gamma_2 T_2}\phi_2$$

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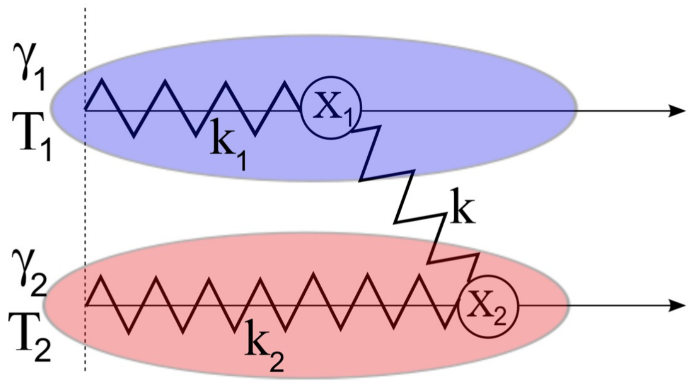

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Linear system: the response can be easily computed ($\rho(\mathbf{x})$ is multivariate Gaussian)

$$\overline{\frac{\delta x_1(t)}{\delta x_1(0)}} = A \frac{d}{dt} \langle x_1(t) x_1(0) \rangle + B \frac{d}{dt} \langle x_1(t) x_2(0) \rangle$$

Cross-correlation

Equilibrium case:

$$T_1 = T_2$$

$$\Longrightarrow \quad \overline{\frac{\delta x_1(t)}{\delta x_1(0)}} \propto \frac{d}{dt} \langle x_1(t) x_1(0) \rangle$$

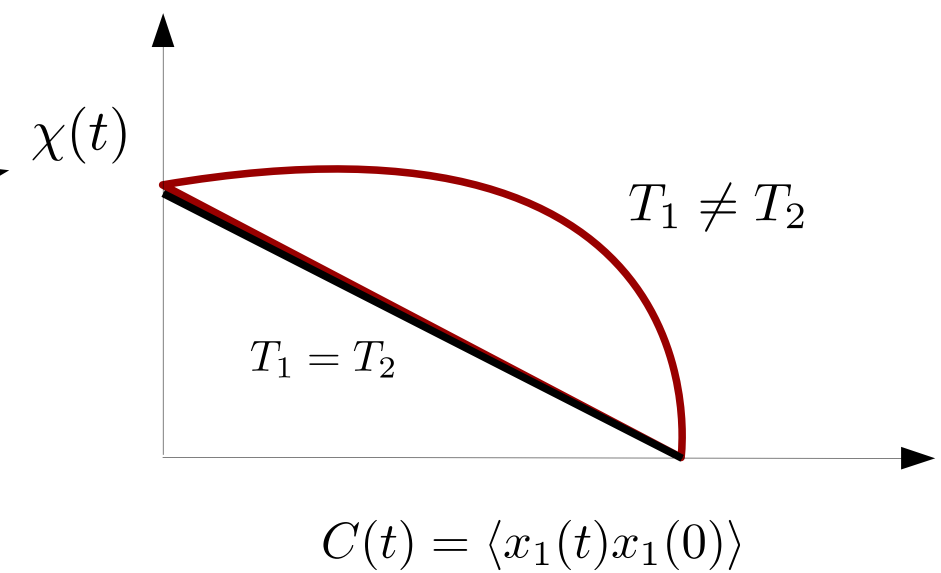
What happens if I have access only to one variable?

Toy Model: Response

Postulating what we want has many advantages: the same of theft over honest work” ***B. Russell***

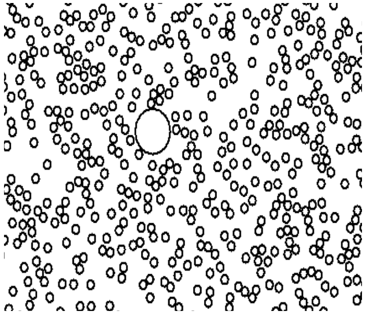
$$\dot{x_1}(t) = -kx_1(t) + \int_0^t \Gamma(t-t')x_1(t') + \mathcal{E}(t)$$

Integrated response



Plan of the talk

Generalized fluctuation response relations



Application to a granular gas

Experimental results

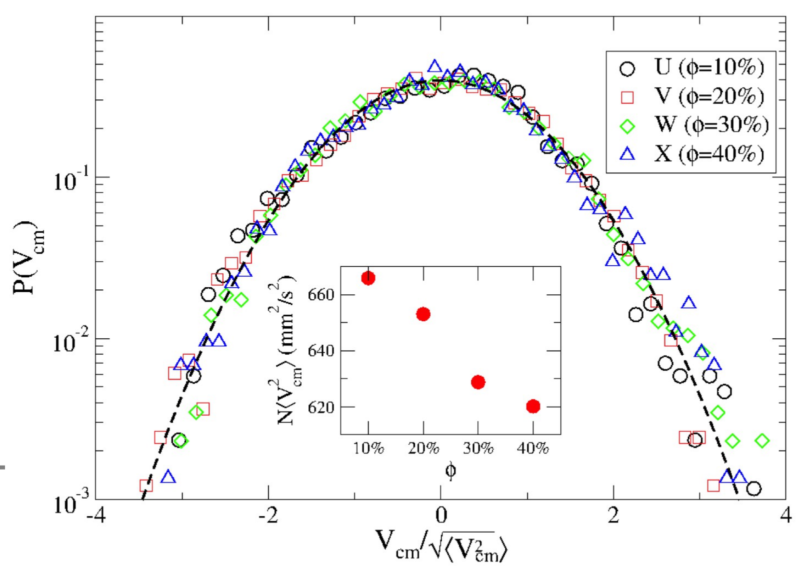
Application to a granular gas

Two time scales → Mean collision time τ_c ; Thermostat time τ_b

Equilibrium-like regime

$\tau_c \gg \tau_b$

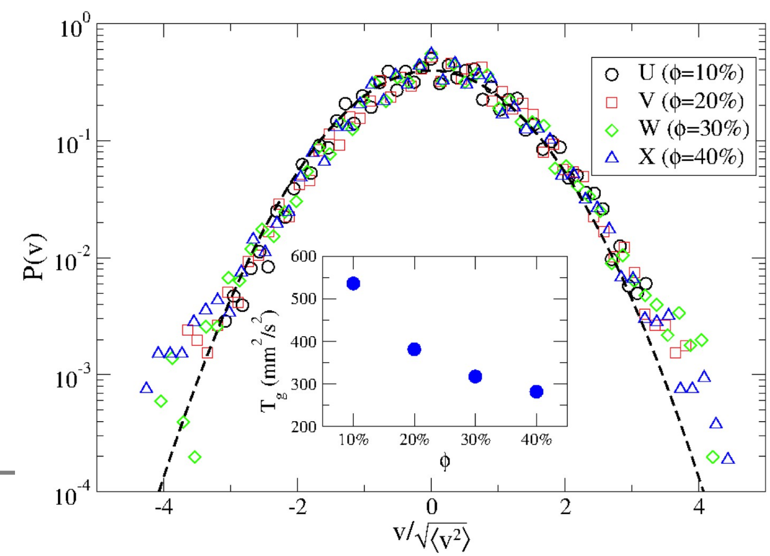
Homogeneous spatial distribution
Maxwell distribution of velocities



Colliding regime

$\tau_b \gg \tau_c$

Spatial inhomogeneity is present
Non-Maxwellian deviations can occur



The "Granular" Model

N Brownian particles of radius σ interacting via inelastic collisions

$$\frac{d}{dt}x_i(t) = v_i(t)$$
$$m\frac{d}{dt}v_i(t) = -\gamma_b v_i(t) + \eta_i(t)$$

$+$

$$\mathbf{v}_i' = \mathbf{v}_i - \frac{1+r}{2}\left[(\mathbf{v}_i' - \mathbf{v}_j') \cdot \hat{\mathbf{n}}\right]\hat{\mathbf{n}}$$
$$\mathbf{v}_i' + \mathbf{v}_j' = \mathbf{v}_i + \mathbf{v}_j$$

Elastic limit $r \to 1$

White noise: $\langle \eta(t)\eta(t') \rangle = 2\gamma_b T_b \delta(t - t')$

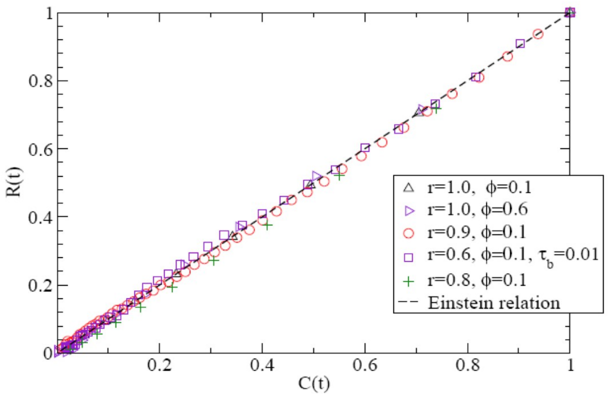
Thermostat "temperature"

The granular temperature is $T_g = m\left\langle v^2 \right\rangle$

Non-equilibrium effects can arise when $\tau_c \ll \tau_b \equiv \frac{1}{\tau_b}$ $\Rightarrow$ $T_g < T_b$

The “violations of the Fluctuation-dissipation relations”

Dilute regime



$$\rho(\{\mathbf{x},\mathbf{v}\}) = \rho_x(\mathbf{x}) \prod \rho_v(v_i)$$

$$\left\langle v_i(t) \frac{\ln \rho_v(v_i)}{\partial v_i}\right|_{t=0} \right\rangle \propto \langle v(t) v(0) \rangle$$

Einstein relation is restored, also if velocities are non Gaussian

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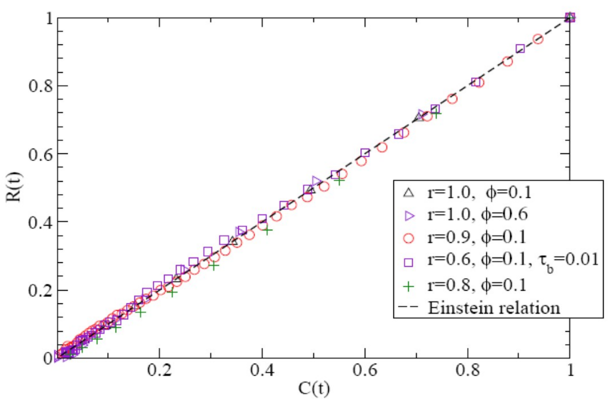

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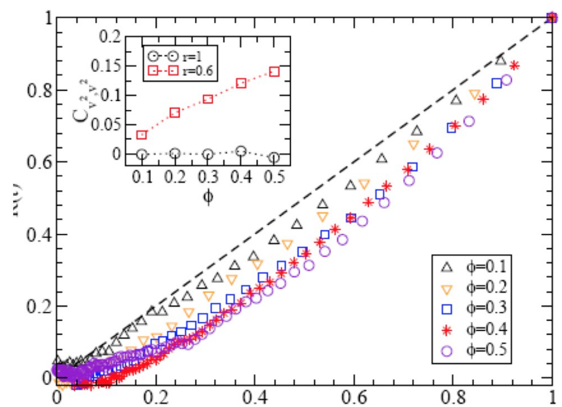
Dense regime

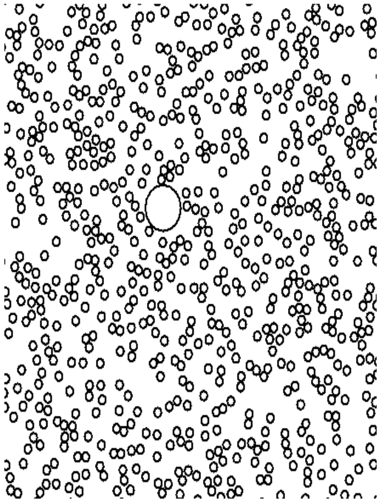


Single particle description fails

Non-linear relation between response and autocorrelation

Dilute case

$$\begin{cases} m\dot{v}_i = -\gamma_b v_i + \eta_b \\ M\dot{\mathbf{V}} = -\gamma_b \mathbf{V} + \eta_b \end{cases} + \text{collisions}$$

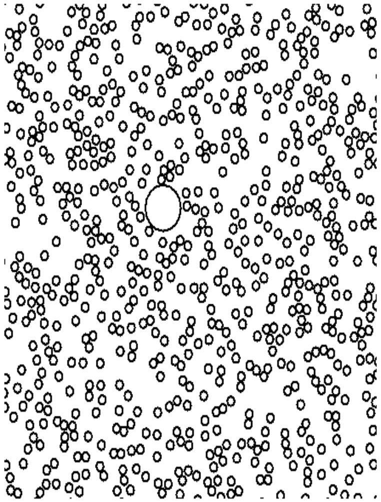


Boltzmann equation
Kramers Moyal expansion

$$\frac{m}{M} \gg 1$$

Dilute case

$$\begin{cases} m\dot{v}_i = -\gamma_b v_i + \eta_b \\ M\dot{\mathbf{V}} = -\gamma_b \mathbf{V} + \eta_b \end{cases} + \text{collisions}$$



Boltzmann equation
Kramers Moyal expansion

$$\frac{m}{M} \gg 1$$

$$M\dot{\mathbf{V}} = -\Gamma\mathbf{V} + \mathcal{E}$$

Effective equation for the tracer
(Linear only in the large mass limit)

$$\Gamma = \gamma_b + \gamma_g$$

$$\langle \mathcal{E}(t)\mathcal{E}(t')\rangle = 2(\gamma_b T_b + \gamma_g T_g)\delta(t-t')$$

This equation is reversible. The effect of the inelasticity has been **projected out.**

Dense case: the effect of recollisions

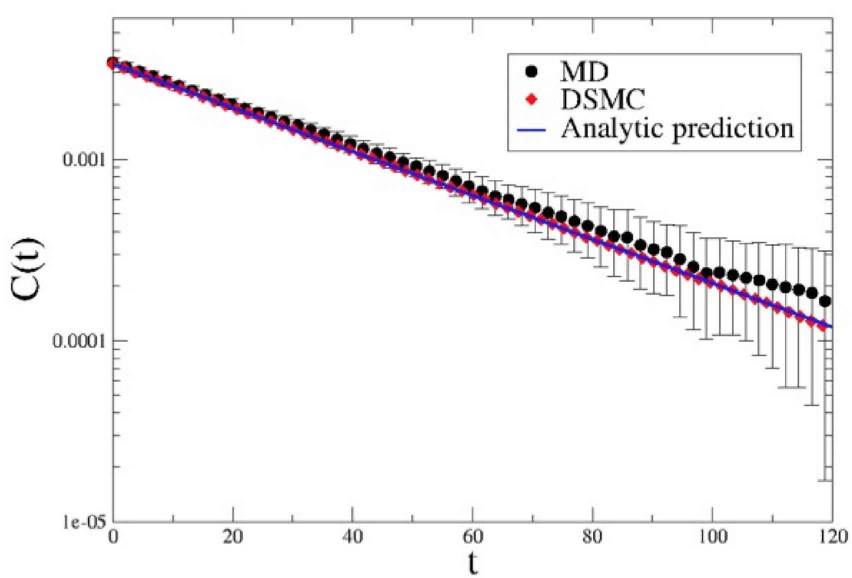
The dynamics of the intruder can be simply explained by its “effective” temperature

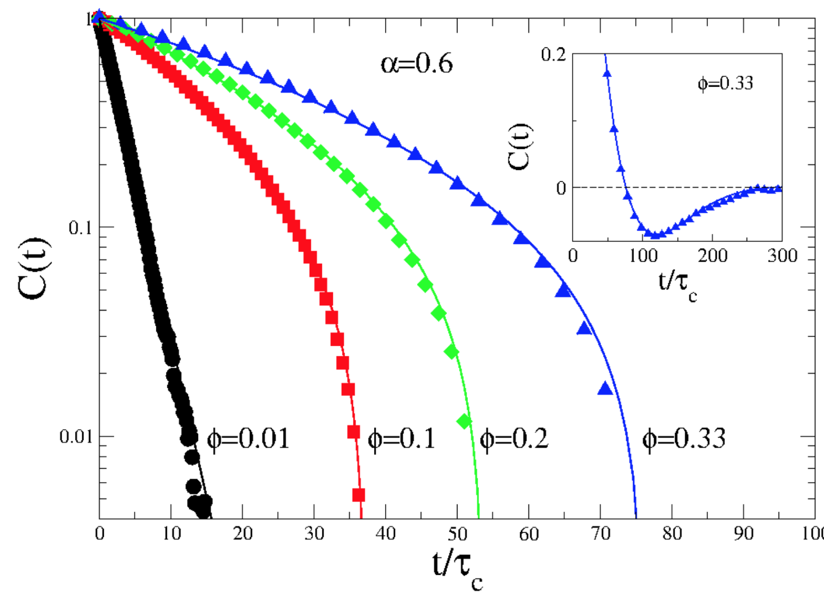
$$T_{eff} = \frac{\gamma_b T_b + \gamma_g T_g}{\gamma_b + \gamma_g}$$

The two noises act on the same time scale

$$\langle \mathcal{E}(t)\mathcal{E}(t')\rangle \propto \delta(t-t')$$

In the dense case, molecular chaos is broken and **a new time scale** appears in the velocity autocorrelation





Dense case: the effect of recollisions

$$M\dot{V}(t) = -\int_{-\infty}^{t} \Gamma(t-t')V(t') + \mathcal{E}(t)$$

$$\Gamma(t) = 2\gamma_1\delta(t) + \gamma_2/\tau_2 e^{-t/\tau_2}$$

Diluite limit $\tau_2 \to 0$

$$\langle \mathcal{E}(t)\mathcal{E}(t') \rangle = 2\gamma_1 T_1 \delta(t-t') + \gamma_2/\tau_2 T_2 e^{-t/\tau_2}$$

The two sources of energy act on different timescales

Memory as a hidden variable

How do you know you have taken enough variables, for it to be Markovian?

Lars Onsager

$$M\dot{V}(t) = -\int_{-\infty}^{t} \Gamma(t-t')V(t') + \mathcal{E}(t)$$

Mapped on a
two variables system
(in the case of an exponential kernel)

$$\begin{pmatrix} \dot{V} \\ \dot{u} \end{pmatrix} = A \begin{pmatrix} V \\ u \end{pmatrix} + \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

White noise
and
no Memory effects!

What is the physical interpretation of the hidden variable?

$$M\dot{V}(t) = -\Gamma V(t) - \Gamma' u(t) + \eta_1(t)$$

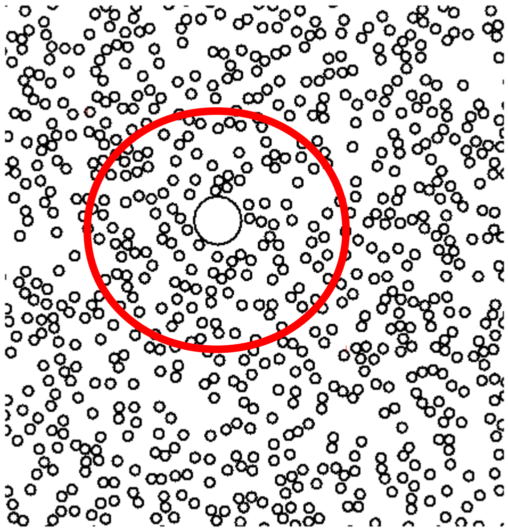
Stochastic force field

Restoring equilibrium-like relations

The mapping suggest that P(V,u) is a bivariate Gaussian

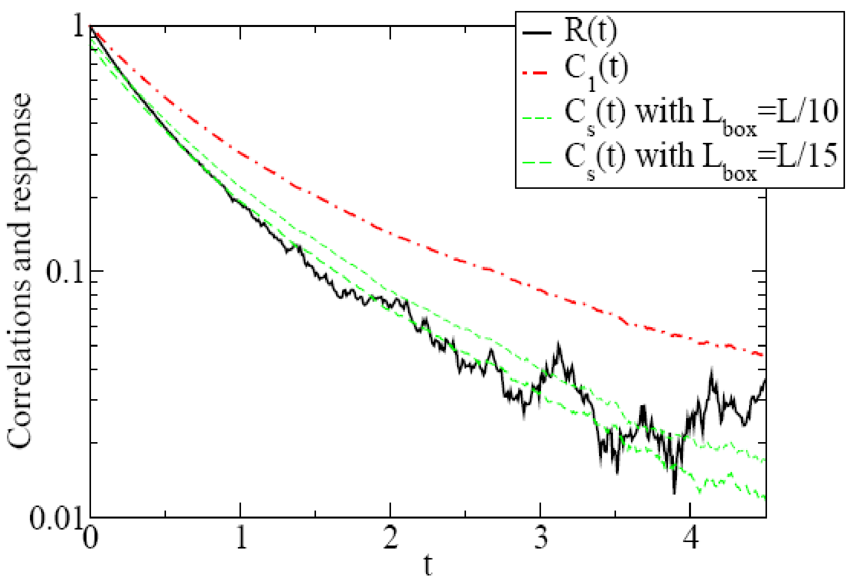
u(x) is assumed to be a **local force field**

u(x) is defined on a small cell **centered in the tracer**



$$\ln P(V,u) = \frac{a}{2}V^2 + buV + \frac{c}{2}u^2$$

$$\overline{\frac{\delta V(t)}{\delta V(0)}} = a\,\langle V(t)V(0)\rangle + b\,\langle V(t)U(0)\rangle$$



Elastic limit: $b \to 0 \qquad a \to \frac{1}{T_b}$

Some remarks

The projected dynamics does not have a sufficient information content, since it has a vanishing entropy production

A Crisanti, A Puglisi, D Villamaina – Phys. Rev. E 85, 061127 (2012)

$$\frac{1}{t}\log\frac{(P(\{\mathbf{V}\})}{P\left(\{\mathcal{I}\mathbf{V}\}\right)} \to 0$$

The memory alone (and the local velocity field) is present also in the **elastic dense case but is not coupled** to the tracer.

$$\langle V(t)u(t)\rangle_{eq} = 0$$

The **coupling with the local field** explains both the violation of the FDT and the entropy production of the system.

A Sarracino, D Villamaina, G Gradenigo, A Puglisi: EPL 92 34001 (2010)

Plan of the talk

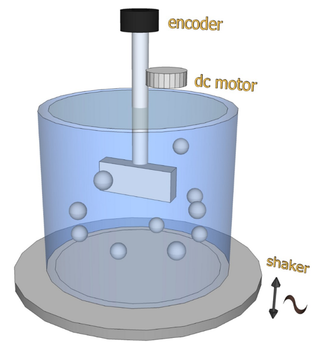
Generalized fluctuation response relations

Application to a granular gas

Experimental results

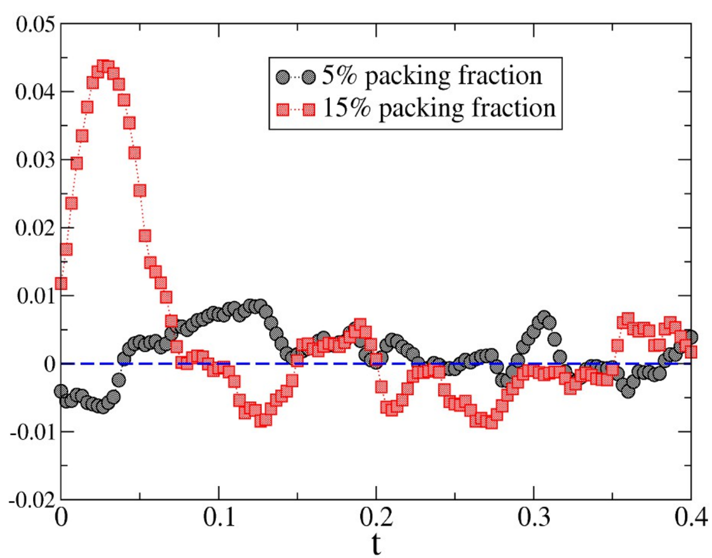
Experience 1: The intruder in a granular gas

A. Gnoli, A. Puglisi, A. Sarracino, A. Vulpiani, *PloS one*, *9*(4), e93720 (2014)



Strong coupling between the intruder and the local field

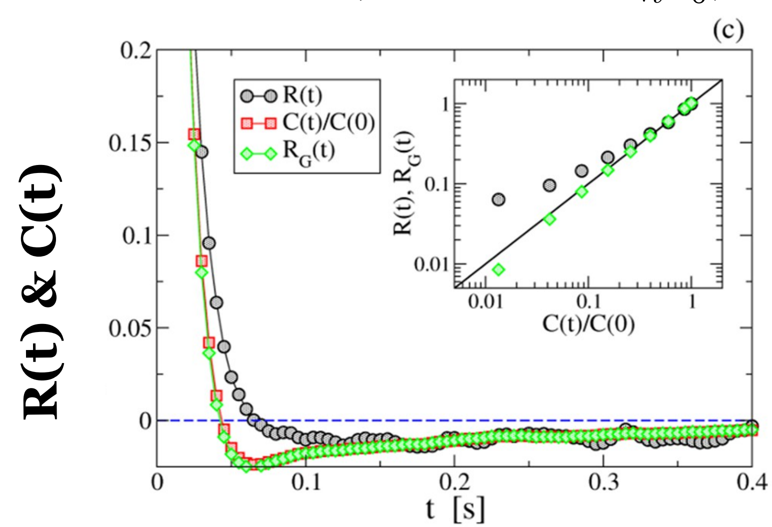
$$\langle V(t)U(0)\rangle$$



time

Neglegible role of non-Gaussianity

$$R(t) \neq -\left\langle V(t)\frac{\partial \ln P_v(V)}{\partial V}\bigg|_{t=0}\right\rangle$$



time

Experience 2: Hydrodynamical fluctuations

A. Puglisi, A. Gnoli, G. Gradenigo, A. Sarracino, D. Villamaina, *J. Chem. Phys.* *136*, 014704 (2012)
G. Gradenigo, A. Sarracino, D. Villamaina, A. Puglisi, *JSTAT*, P08017 (2011)

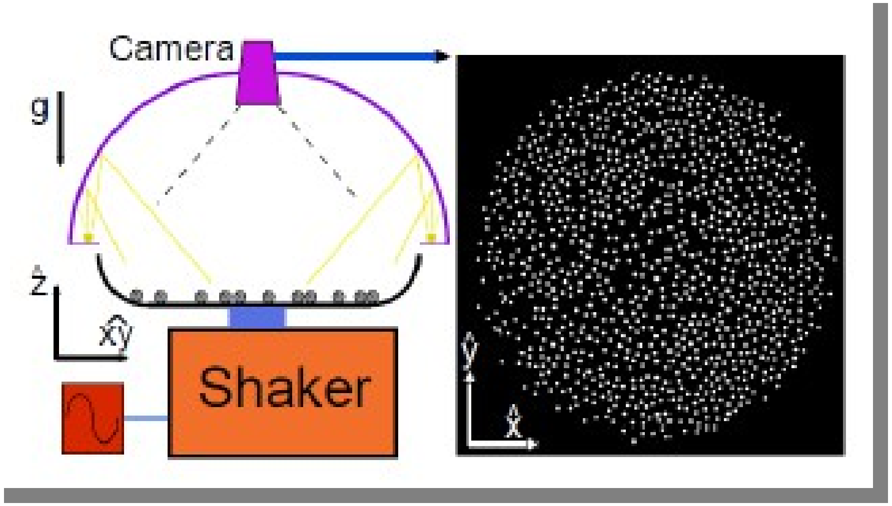
Statistics of the noise

$$\eta(t)$$

$$\Updownarrow$$

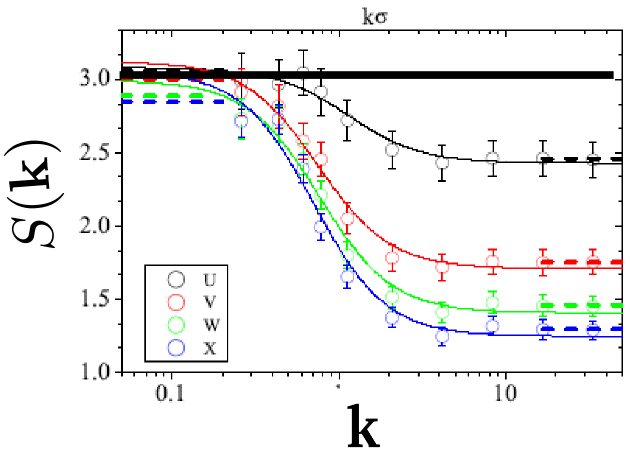
$$\mathbf{u}(\mathbf{r})$$

Statistics of velocity



Hydrodynamical fluctuations:

$$S(\mathbf{k}) = \frac{1}{N} \langle \mathbf{u}(\mathbf{k})\mathbf{u}(-\mathbf{k}) \rangle$$



T Equilibrium case

Granular case

Conclusions