# Kuramoto model of synchronization:equilibrium and non equilibrium aspects

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## Approach to equilibrium in a chain of nonlinear oscillators

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## Equipartition threshold in nonlinear large Hamiltonian systems: The Fermi-Pasta-Ulam model

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# Transition to equipartition in FPU

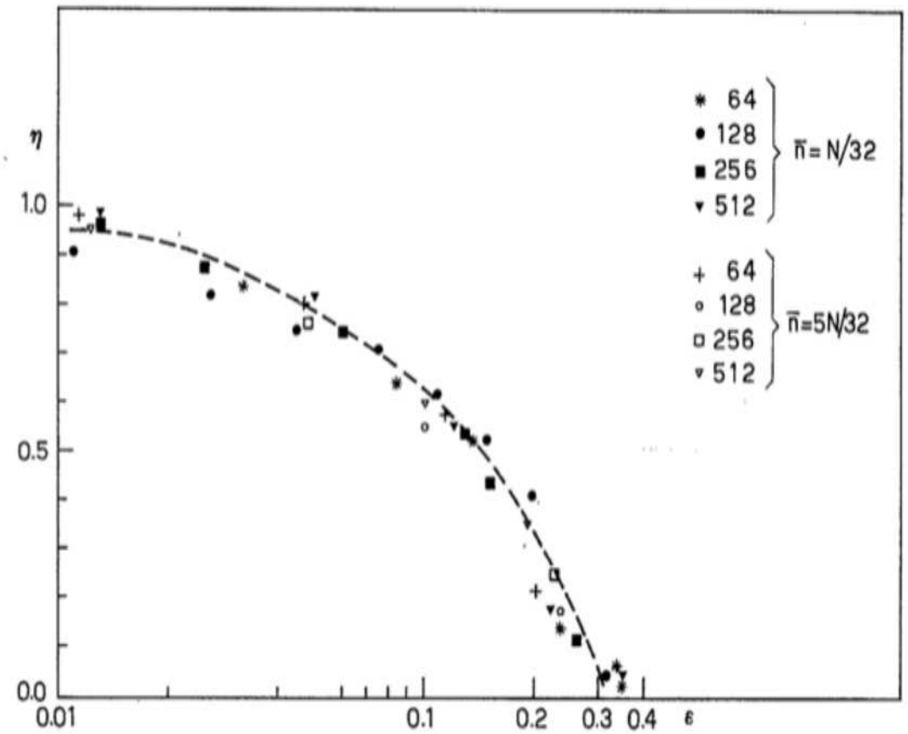


FIG. 1. $\eta$ vs $\epsilon$; $\bar{k}=2\pi\bar{n}/N$, $\overline{\Delta k}=2\pi\overline{\Delta n}/N$, $\overline{\Delta n}=N/16$, $\bar{n}=N/32$; $\frac{5}{32}N$; with $N=64,128,256,512$. Dashed line is a free-hand smoothing of the experimental results.

# Computers

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# References


- S. Gupta, M. Potters and S. Ruffo, *One-dimensional lattice of oscillators coupled through power-law interactions: Continuum limit and dynamics of spatial Fourier modes*, Phys. Rev. E, **85**, 066201 (2012).
- S. Gupta, A. Campa and S. Ruffo, *Overdamped dynamics of long-range systems on a one-dimensional lattice: Dominance of the mean-field mode and phase transition*, Phys. Rev. E, **86**, 061130 (2012).
- S. Gupta, A. Campa and S. Ruffo, *Nonequilibrium first-order transition in coupled oscillator systems with inertia and noise*, Phys. Rev. E, **89**, 022123 (2014)
- S. Gupta, A. Campa and S. Ruffo, *Kuramoto model of synchronization: equilibrium and nonequilibrium aspects*, J. Stat. Mech.: Theory and Exp., **R08001** (2014)

# Plan

- Kuramoto and Sakaguchi models
- Equilibrium vs. non equilibrium
- First-order phase transition
- Complete phase diagram
- Linear stability analysis
- $\alpha$-Kuramoto
- Zero-mode dominance

# Synchronization

Spontaneous synchronization: Coordination of events to operate a system in unison, in the absence of any ordering field

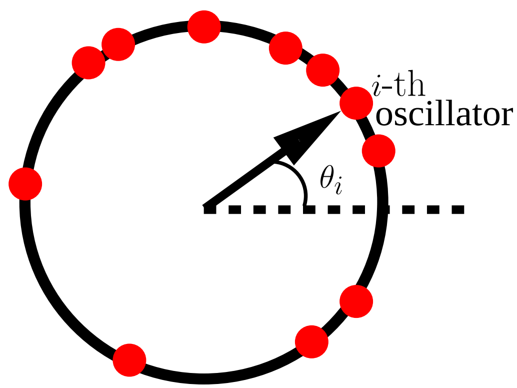
- Flashing fireflies
- Synchronized firing of cardiac pacemaker cells
- Phase synchronization in electrical power distribution networks

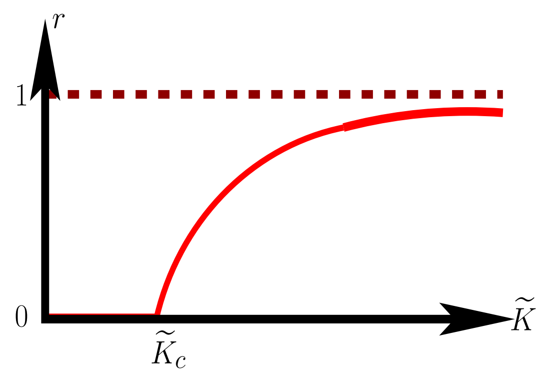
# The Kuramoto model

A framework to study spontaneous synchronization:
$N$ globally coupled oscillators with distributed natural frequencies

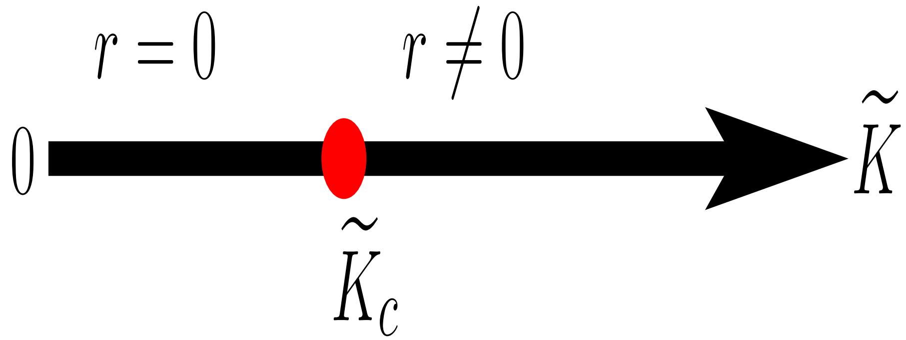
$$\frac{d\theta_i}{dt} = \omega_i + \frac{\tilde{K}}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i)$$

$\tilde{K}$ coupling constant, $\omega_i$ quenched random variables with distribution $g(\omega)$
Order parameter, fraction of phase-locked oscillators: $r = |1/N \sum_j \exp(i\theta_j)|$
High $\tilde{K}$: Synchronized phase , $r > 0$
Low $\tilde{K}$: Incoherent phase, $r = 0$.
For unimodal $g(\omega)$ continuous transition on tuning $\tilde{K}$.



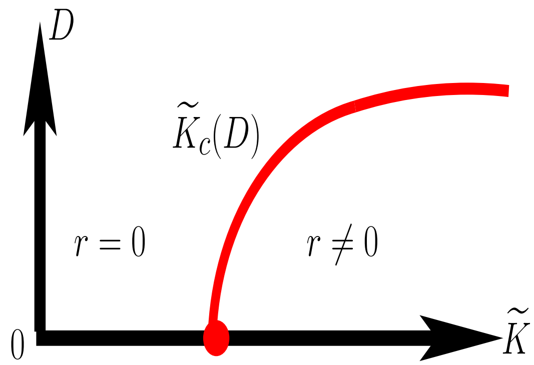
# The Kuramoto-Sakaguchi model




Stochastic fluctuations of the $\omega_i$ in time

$$\frac{d\theta_i}{dt} = \omega_i + \frac{\tilde{K}}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) + \eta_i(t)$$

$$< \eta_i(t) >= 0 \, , < \eta_i(t)\eta_j(t') >= 2D\delta_{ij}\delta(t - t')$$



# 2nd order dynamics

Two dynamical variables: $\theta_i$ (Phase); $v_i$ (Angular velocity)

$$\frac{d\theta_i}{dt} = v_i$$

$$m\frac{dv_i}{dt} = -\gamma v_i + \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) + \eta_i(t)$$

where $m$ is the inertia and $\gamma$ the friction constant.
Motivation:

- An adaptive frequency can explain the slower approach to synchronization observed in a particular firefly (the Pteropyx mallacae) `Ermentrout (1991)`
- Phase dynamics in electric power distribution networks in the mean-field limit `Filatrella, Nielsen and Pedersen (2008), Rohden, Sorge, Timme and Witthaut (2012), Olmi and Torcini (2014)`

# Previous studies

- No noise: Simulations for a Lorentzian $g(\omega)$ show a first-order synchronization transition `Tanaka, Lichtenberg and Oishi (1997)`
- Analysis in the continuum limit, based on a suitable Fokker-Planck equation analysis in the limit $N \to \infty$ for a Lorentzian $g(\omega)$: either larger inertia or larger $\omega$ spread makes the system harder to synchronize `Acebron and Spigler (1998); Acebron, Bonilla and Spigler (2000)`
- HOWEVER, THE COMPLETE PHASE DIAGRAM HAS NOT BEEN ADDRESSED

# Rescaling

One can analyze the model in the reduced parameter space $(T, \sigma, m)$

$$\frac{d\theta_i}{dt} = v_i$$
$$\frac{dv_i}{dt} = -\frac{1}{\sqrt{m}} v_i + \sigma\omega_i + \frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) + \eta_i(t)$$

where now:

- $g(\omega)$ has zero average and unit width
- $< \eta_i(t)\eta_j(t') >= \frac{2T}{\sqrt{m}}\delta_{ij}\delta(t - t')$

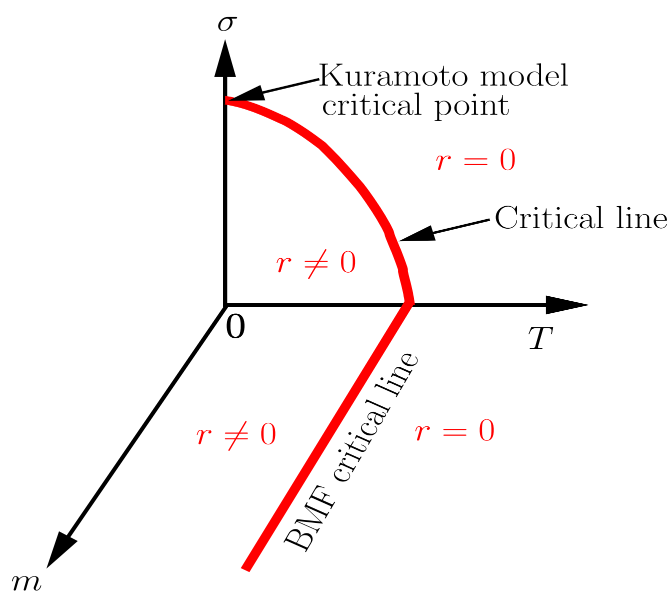
# Equilibrium and nonequilibrium

The role of $\sigma$

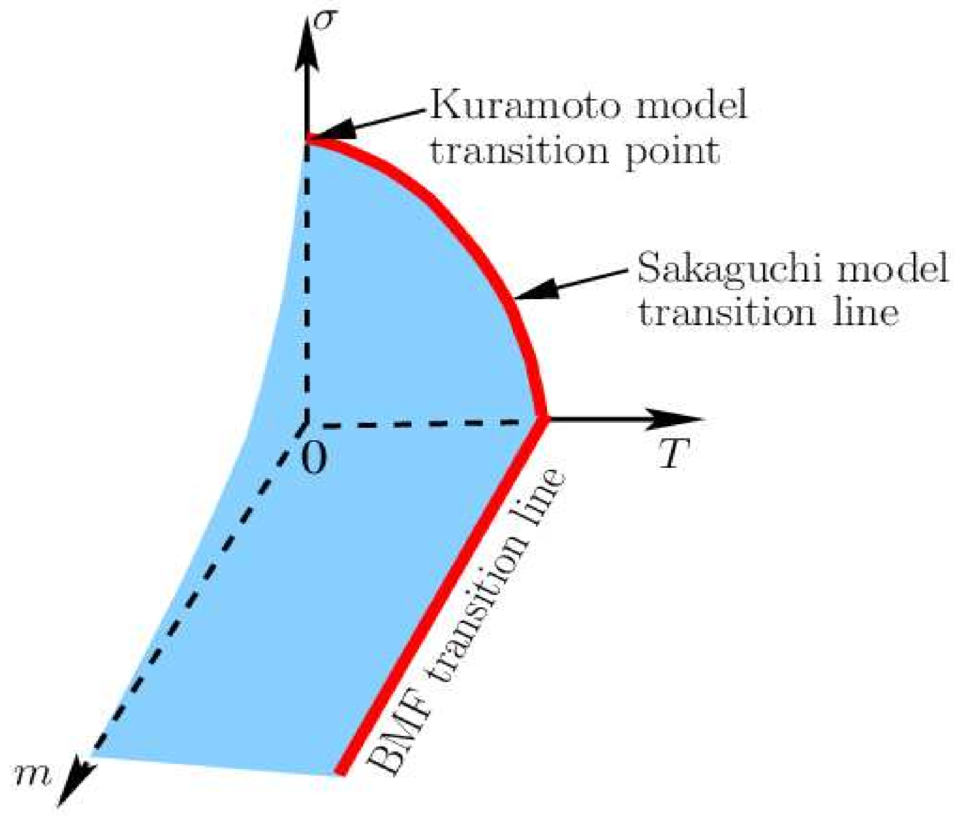
- $\sigma = 0$, $\Rightarrow$ no external drive $\Rightarrow$ detailed balance $\Rightarrow$ equilibrium stationary state
- $\sigma > 0 \Rightarrow$ non equilibrium stationary state

Continuous transition lines

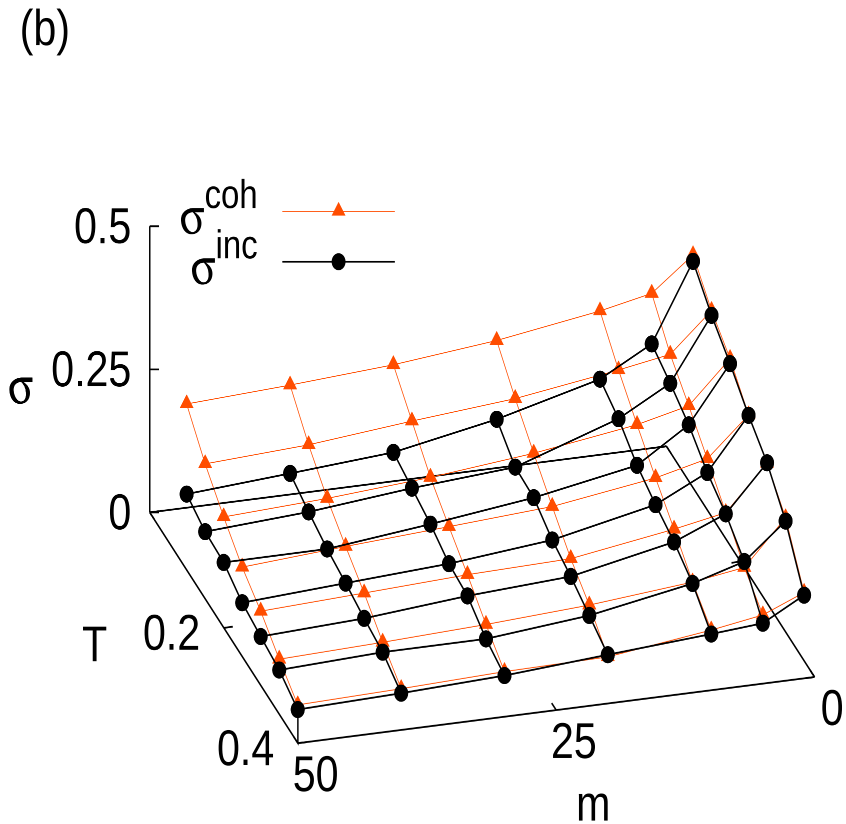
- $m = T = 0$, $\sigma > 0$, Kuramoto
- $m = 0$, $T > 0$, $\sigma > 0$, Sakaguchi, continuous transition, critical line
- $\sigma = 0$ Hamiltonian system + heat-bath (Brownian Mean Field model), continuous transition, critical line `Chavanis (2013)`



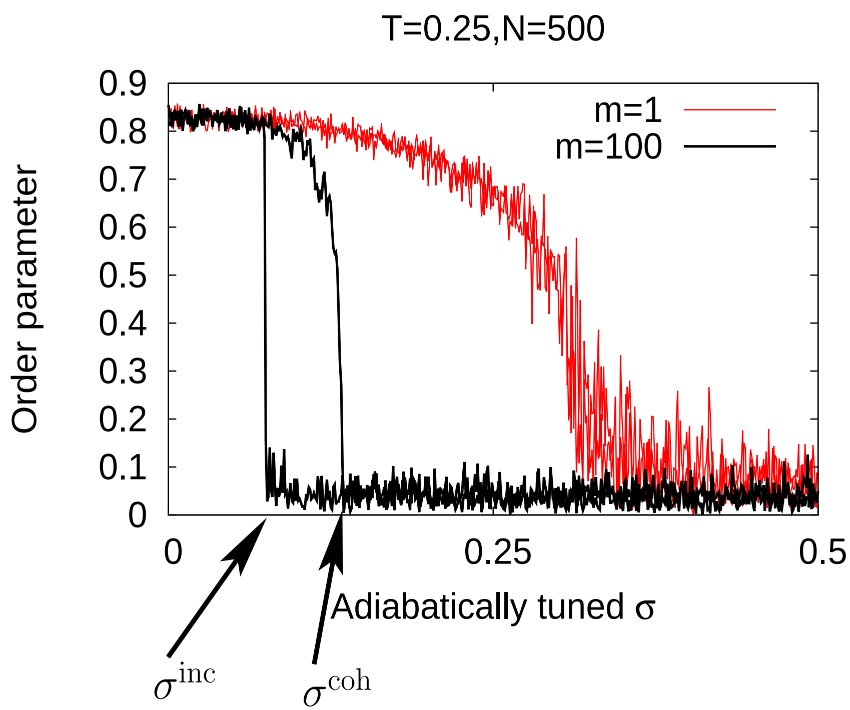
# Phase diagram-I

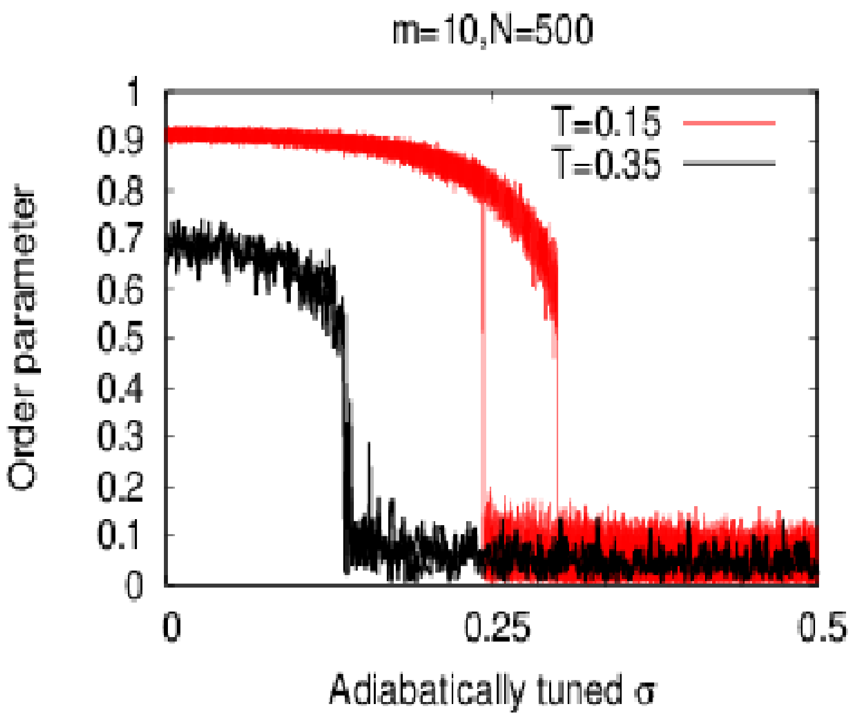


# Phase diagram-II

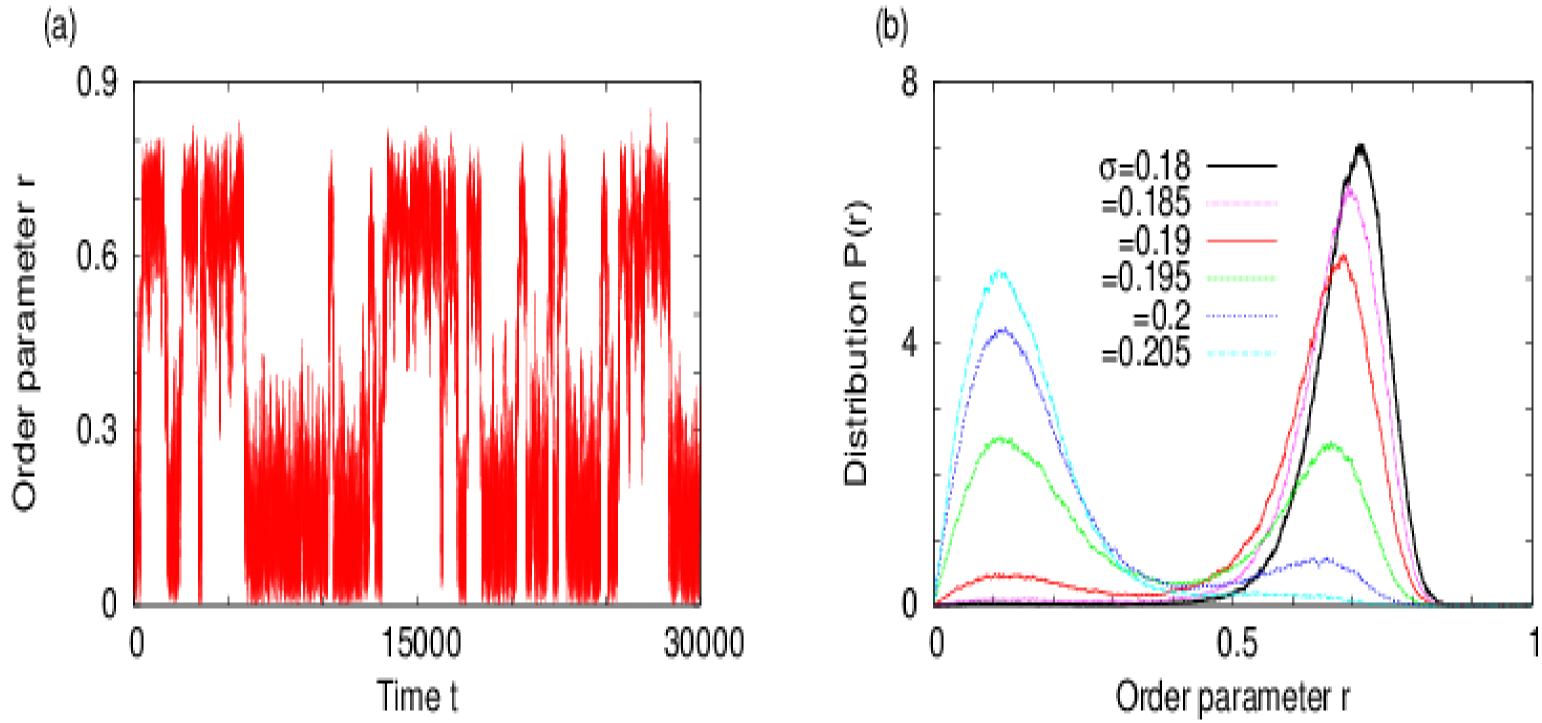


# Hysteresis


T=0.25,N=500
m=1
m=100
Order parameter
Adiabatically tuned σ
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.25
0.5
$\sigma^{\rm inc}$
$\sigma^{\rm coh}$


m=10,N=500
T=0.15
T=0.35
Order parameter
Adiabatically tuned σ
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.25
0.5

# Bistability




For $m = 20, T = 0.25, N = 100$, and a Gaussian $g(\omega)$ with zero mean and unit width, (left) shows, at $\sigma = 0.195$, $r$ vs. time in the stationary state, while (right) shows the distribution $P(r)$ at several $\sigma$'s around $0.195$.

# $N \to \infty$ continuum limit

Single-particle distibution $f(\theta, v, \omega, t)$: Fraction of oscillators at time t and for each $\omega$ which have phase $\theta$ and angular velocity $v$ (Periodic in $\theta$ and normalized).
Evolution by Kramers equation

$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial \theta} + \frac{\partial}{\partial v}\Big(\frac{v}{\sqrt{m}} - \sigma\omega - r\sin(\psi - \theta)\Big)f + \frac{T}{\sqrt{m}}\frac{\partial^2 f}{\partial v^2},$$

with self-consistent order parameter

$$r\exp(i\psi) = \iiint d\theta dv d\omega g(\omega)\exp(i\theta)f(\theta, v, \omega, t)$$

Homogeneous ($r = 0$) solution

$$f^{\mathrm{inc}} = \frac{1}{2\pi}\frac{1}{\sqrt{2\pi T}}\exp\left(-\frac{(v - \sigma\omega\sqrt{m})^2}{2T}\right)$$

# Linear stability results

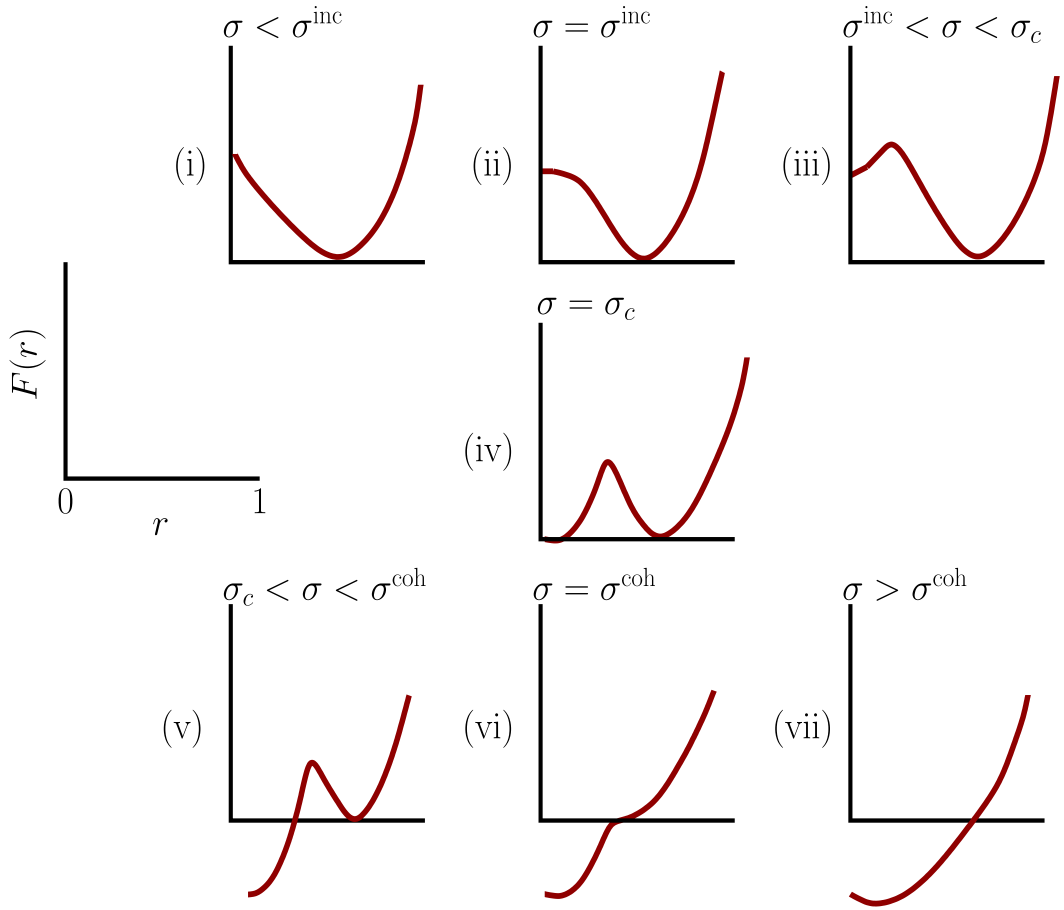
Stability analysis gives $\sigma^{inc}$: $f(\theta,v,\omega,t) = f^{\rm inc}(\theta,v,\omega) + e^{\lambda t}\delta f(\theta,v,\omega)$

$$\frac{2T}{e^{mT}} = \sum_{p=0}^{\infty} \frac{(-mT)^p(1+\frac{p}{mT})}{p!} \int_{-\infty}^{+\infty} \frac{g(\omega)d\omega}{1+\frac{p}{mT}+i\frac{\sigma\omega}{T}+\frac{\lambda}{T\sqrt{m}}}.$$

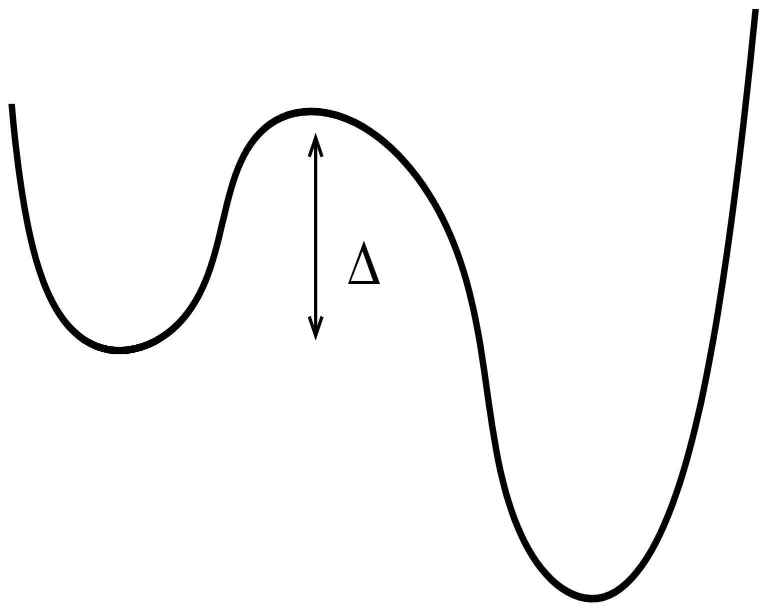
`Acebron, Bonilla and Spigler (2000)`

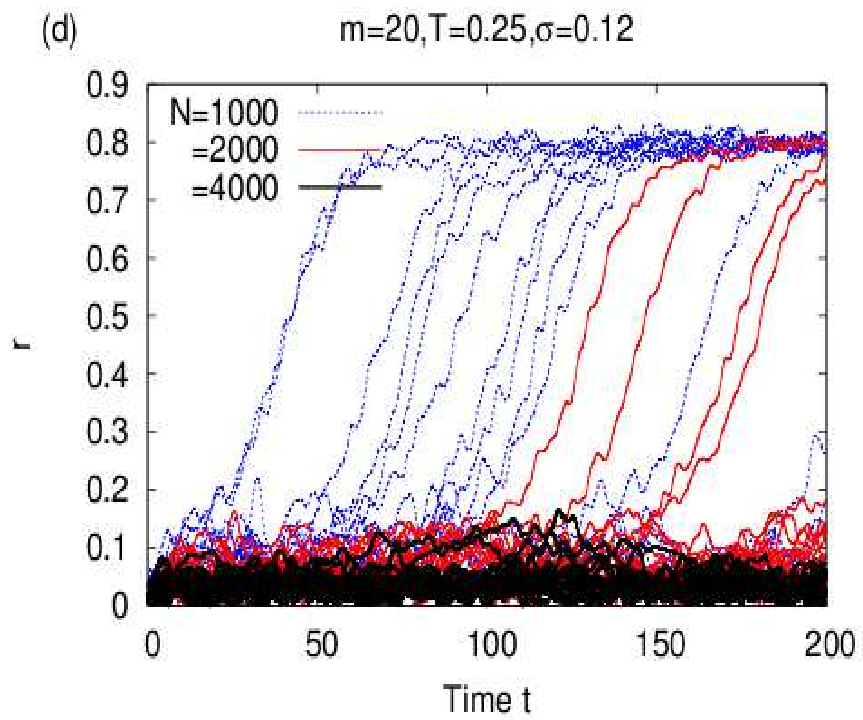
- The equation for $\lambda$ has at most one solution with a positive real part and, when the solution exists, it is necessarily real.
- Neutral stability $\Rightarrow \lambda = 0$ gives the stability surface $\sigma^{\rm inc}(m,T)$.
- Similarly, one can define $\sigma^{\rm coh}(m,T)$.
- The two surfaces enclose the first-order transition surface $\sigma_c(m,T)$.
- Taking proper limits, the surface $\sigma^{\rm inc}(m,T)$ meets the critical lines on the $(T,\sigma)$ and $(m,T)$ planes.
- The intersection of the surface with the $(m,\sigma)$ plane gives an implicit formula for $\sigma^{\rm inc}_{\rm noiseless}(m,\sigma)$.

# Landau free energy



# Mean-field metastability



# A non equilibrium perspective

- Kuramoto model as an overdamped limit of a long-range interacting system.
- General dynamics: (1) External drive, (2) Quenched vs. annealed randomness.
- No quenched randomness $\Rightarrow$ Equilibrium
  - Prob. distr. $\sim \exp(-\beta(K.E. + P.E.))$, product measure
  - Phase transition given by P.E. same for underdamped and overdamped dynamics
- Quenched randomness $\Rightarrow$ Non equilibrium stationary state
  - Prob. distr. $\neq \exp(-\beta(K.E. + P.E.))$, not product measure
  - Dynamics matters: Phase transitions different for underdamped and overdamped dynamics

# $\alpha$-Kuramoto

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{\widetilde{N}} \sum_{j=1, j\neq i}^{N} \frac{\sin(\theta_j - \theta_i)}{|x_j - x_i|^\alpha}$$

$\omega_i$ is a quenched random variable with distribution $g(\omega)$
In the continuum limit, the local density $\rho(\theta; \omega, x, t)$ satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \theta}\left(\rho \frac{\partial \theta}{\partial t}\right)$$

$$\frac{\partial \theta(\omega, x, t)}{\partial t} = \omega + \kappa(\alpha) K \int d\theta' d\omega' dx' \frac{\sin(\theta' - \theta)}{|x' - x|^\alpha} \rho(\theta'; \omega', x', t) g(\omega')$$

Linear stability analysis of the homogeneous state

$$\rho(\theta; \omega, x, t) = \frac{1}{2\pi} + \delta\rho(\theta; \omega, x, t)$$

Dispersion relation

$$1 - \frac{c_k(\alpha) K}{2} \int_{-\infty}^{\infty} d\omega \frac{g(\omega)}{(\lambda_k \pm i\omega)} = 0$$

# Stability of the incoherent state

If $g(\omega)$ is symmetric around the mean and non increasing then $\lambda_k$ is either real positive or zero.
The dispersion relation rewrites

$$1 - c_k(\alpha)K \int_0^\infty d\omega \frac{\lambda_k}{\lambda_k^2 + \omega^2} g(\omega) = 0$$

The limit $\lambda_k \to 0$ gives the critical couplings

$$K_c^{(k)} = \frac{2}{c_k(\alpha)\pi g(0)}$$

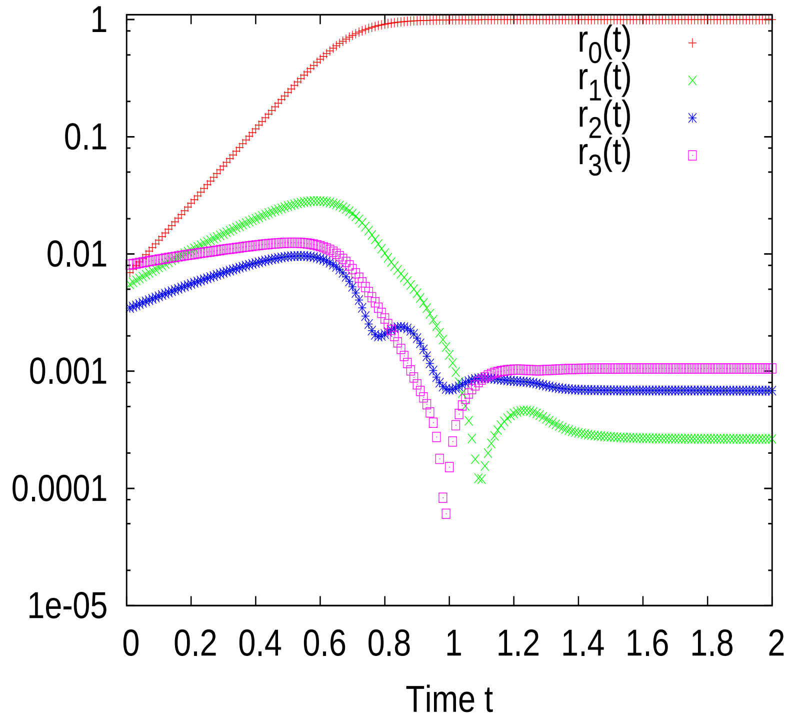
$$K_c^{(0)} < K_c^{(1)} < K_c^{(2)} < \ldots.$$

and the growth rates $\lambda_k$

$$\frac{2K}{\pi g(0) K_c^{(k)}} \int_0^\infty d\omega \; \frac{\lambda_k}{\lambda_k^2 + \omega^2} g(\omega) = \frac{K}{K_c^{(k)}} e^{\lambda_k^2/2} \mathrm{Erfc}\Big[\frac{\lambda_k}{\sqrt{2}}\Big] = 1.$$

for Gaussian $g(\omega)$.

# Zero-mode dominance




$$\alpha = 0.5\,, K = 15\,, K_c^{(0)} \approx 1.59577\,, K_c^{(1)} \approx 4.26696\,, K_c^{(2)} \approx 6.53664\,, K_c^{(3)} \approx 7.71516, \ldots,$$

so that the Fourier modes $0, 1, 2, 3$ are all linearly unstable.

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