Chaotic properties of oscillator networks

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Outline

- Hamiltonian mean field: microscopic chaos in a mean field model

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- From a general phenomenology to coupling pressure

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- From a general phenomenology to coupling pressure
- Collective dynamics in mean field models of identical oscillators
- Collective dynamics in the presence of disorder
- Collective dynamics in sparse networks

The model

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Hamiltonian

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} - \frac{1}{2N} \sum_{i,j=1}^{N} \cos(\theta_i - \theta_j)$$

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Equations of motion

$$\ddot{\theta}_i = M \sin(\phi - \theta_i)$$

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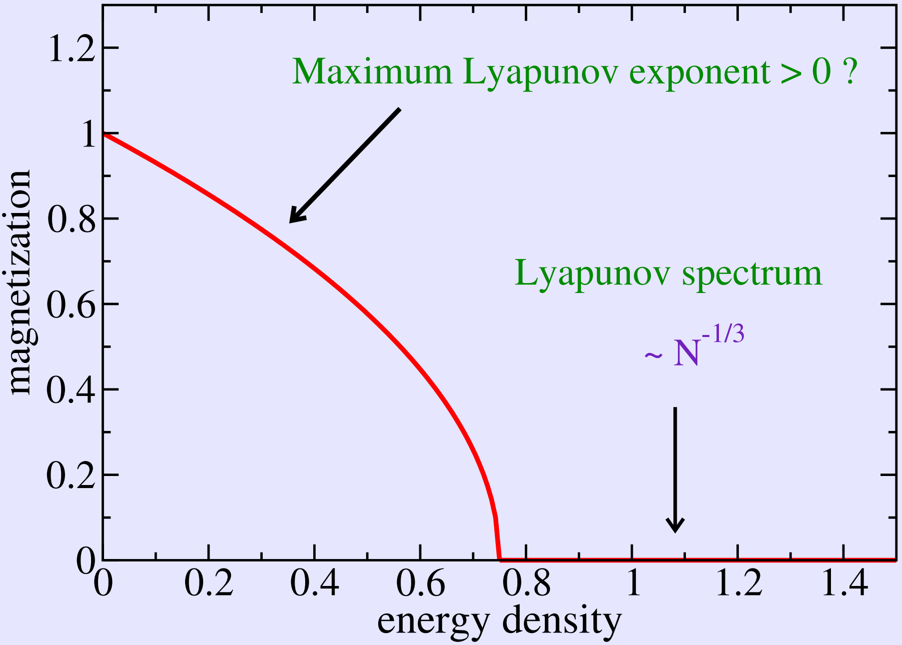
Order parameter

$$M\mathrm{e}^{i\phi} = \tfrac{1}{N} \textstyle\sum_j \mathrm{e}^{i\theta_j}$$



Maximum Lyapunov exponent > 0 ?
Lyapunov spectrum
~ N^-1/3
magnetization
energy density
0
0.2
0.4
0.6
0.8
1
1.2
1.4

Previous results

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Homogeneous phase

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Magnetized phase

- $\lambda_{max} > 0$ geometric appr. [Firpo, 1998]
- undecided field theoretic appr. [Tanase-Nicola Kurchan 2003]
- $\lambda_0 > 0$ numerical simulations (several authors)

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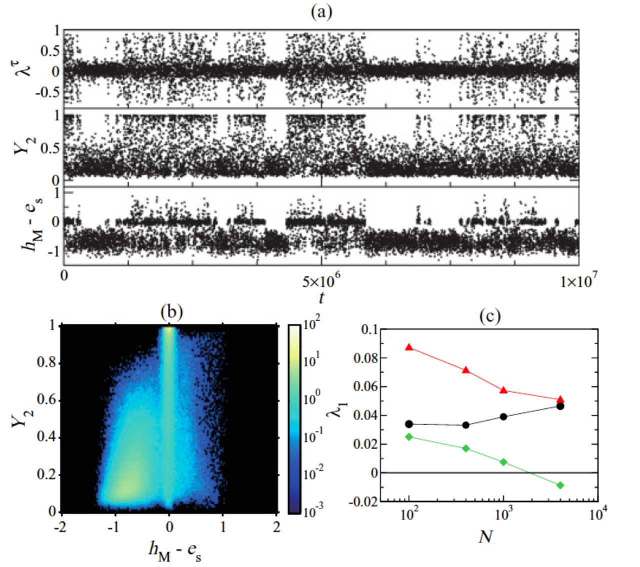
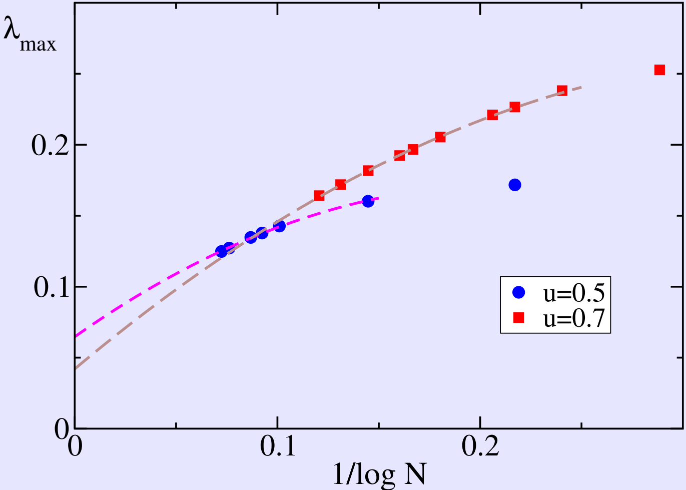
- $\lambda \to 0$ numerical simulations [Manos Ruffo 2010]

Some preliminaries: low energy behaviour

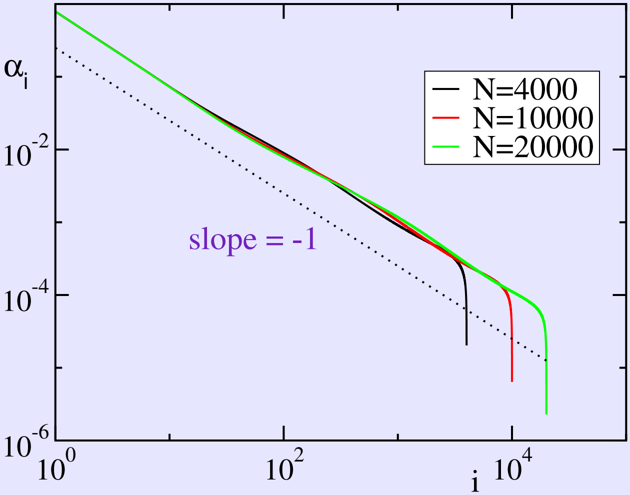


Tangent-space evolution

$$\delta\ddot{\theta}_i = -M\cos(\phi-\theta_i)\delta\theta_i + \frac{1}{N}\sum_j \cos(\theta_j-\theta_i)\delta\theta_j$$

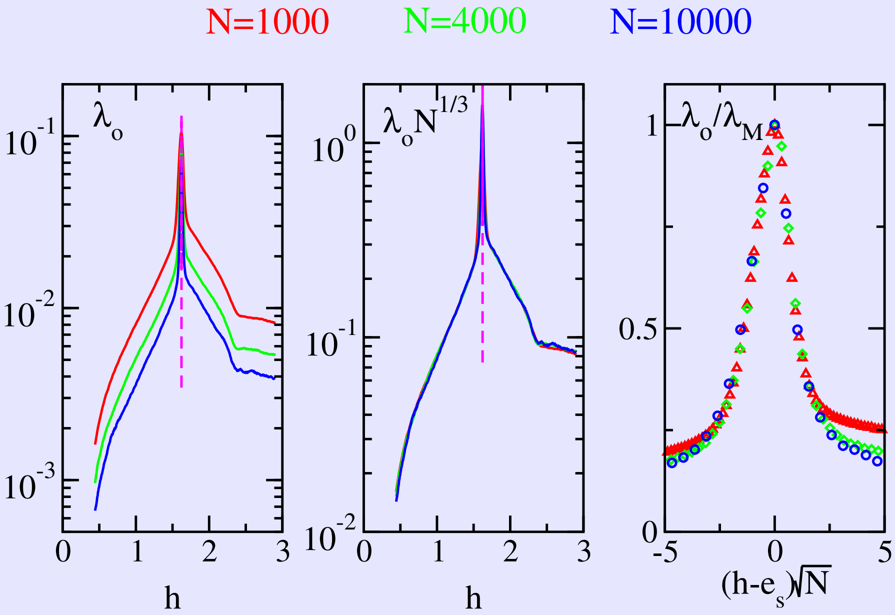


Localization of the perturbation

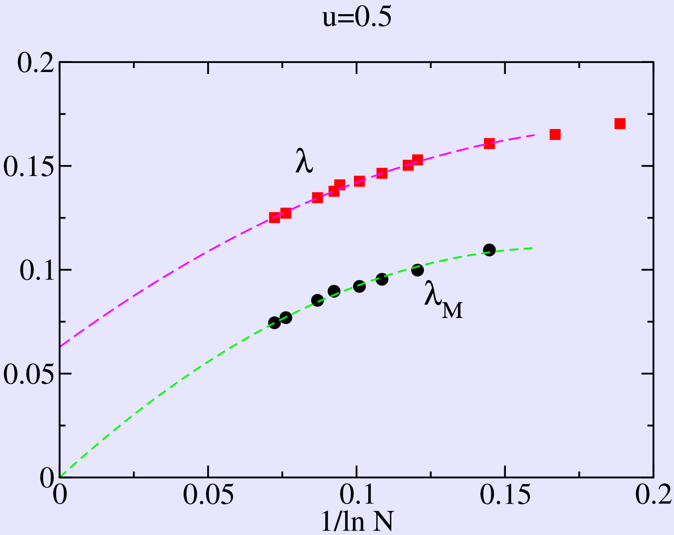


$$\alpha_i = \left\langle \sqrt{\delta\theta_i^2 + \delta\dot{\theta}_i^2} \right\rangle$$

Single-oscillator Lyapunov exponent



True vs. single-oscillator Lyapunov exponent



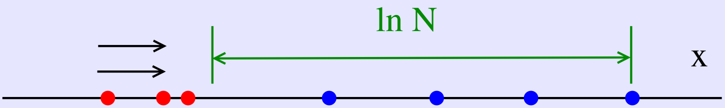
$\lambda_M \approx 1/lnN$

Including the coupling strength

$$\delta\ddot{\theta}_i = -M\cos(\phi - \theta_i)\delta\theta_i + \frac{1}{N}\sum_j \cos(\theta_j - \theta_i)\delta\theta_j$$

$$\delta\ddot{\theta}_i \approx -M\cos(\phi - \theta_i)\delta\theta_i + \frac{1}{N}\delta\theta_{max}$$
$$|\delta\theta_i| \rightarrow x = \log|\delta\theta_i|$$

- No coupling force
- Rapidly pushed ahead to a distance < ln N



Coupling pressure: chaotic units

extension of Daido (1984)

Fokker-Planck description

$\lambda_0 =$ Lyapunov exponent of the single unit

Choose a frame $u = x + \lambda t$ moving with the velocity λ of the coupled system

$$\frac{\partial}{\partial t} P(u,t) = -\frac{\partial}{\partial u}[(\lambda_0 - \lambda)P] + \frac{D}{2}\frac{\partial^2 P}{\partial u^2}$$

Stationary solution for a reflecting barrier set in $u = 0$

Impose $P(\ln N) \approx 1/N$

$$\Delta\lambda \equiv \lambda - \lambda_0 = \frac{D}{2}$$

K. Takeuchi et al. (2011)

Coupling pressure: HMF model

TWO populations

	p_1 (out-of-the saddle)		p_2 (on the saddle)
Lyapunov	$1/N^{1/3}$		$1/\ln N$
Diffusion	0		D_s
rate $1 \to 2$		α_1	
rate $2 \to 1$		α_2	

$$\alpha_2/\alpha_1 \approx 1/\sqrt{N}$$

Solution of the two coupled Fokker-Planck equations

$$e^{-\gamma x_{max}} = \tilde{d}_0 \left(1 - \frac{2\alpha_1}{D_\mathrm{s}\gamma^2}\right) \frac{1}{\sqrt{N}} + \mathcal{O}\left(\alpha_2^2\right)$$

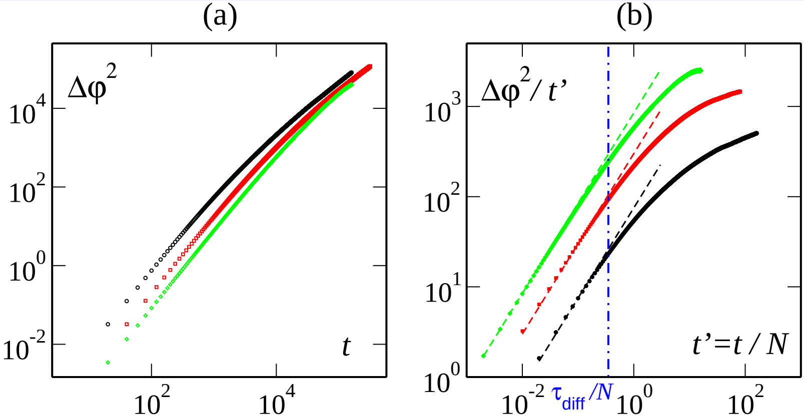
$\alpha_1 = 1/t_r$ where t_r is the residence time on the saddle
(to be determined)

Approximation of the Hamiltonian

$$h \simeq \tfrac{p^2}{2} + 2M_N - \tfrac{M_N}{2}\left(\theta - \phi_N - \pi\right)^2$$

$$\ddot{\theta} \sim M_N\left(\phi_N - \theta + \pi\right)$$

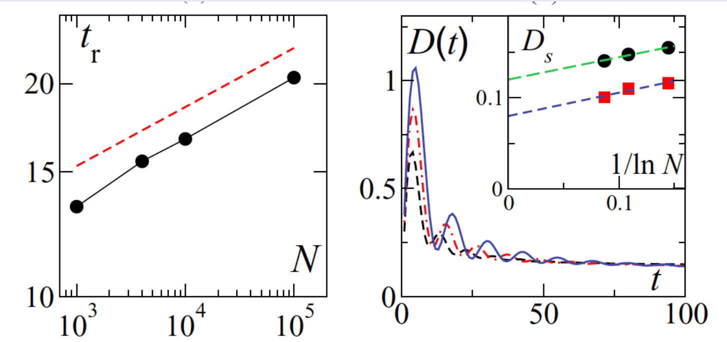
Dynamics of the phase of the magnetization



$$\Delta\phi \approx \omega t \quad \rightarrow \quad \Delta\theta \approx \omega t^3 \quad \rightarrow \quad \Delta h (= \Delta\theta^2) \approx \omega^2 t^6$$

$$\omega = 1/\sqrt{N} \quad \Delta h = 1\sqrt{N}$$

$$N^{-1/2} \approx N^{-1} t_r^6 \quad \rightarrow \quad \alpha_1 = t_r^{-1} \approx N^{-1/12}$$



Self-consistency and final expression

$$\lambda = D_s/4$$

A generic model of coupled oscillators

Generic setup

$$\dot{\mathbf{U}}_i = \mathbf{F}_i(\mathbf{U}_i) + \frac{g}{K} \sum_j \mathbf{J}_{ij} \mathbf{G}(\mathbf{U}_i, \mathbf{U}_j)$$

$$\dot{\mathbf{U}}_i = \mathbf{F}_i(\mathbf{U}_i) + \frac{g}{\sqrt{K}} \sum_j \mathbf{J}_{ij} \mathbf{G}(\mathbf{U}_i, \mathbf{U}_j)$$

g coupling gauge J_{ij} coupling constants

fully-coupled, massive and sparse networks

Presence vs. absence of disorder

Models with a balanced regime

$$h_i(t+1) = \gamma h_i(t) + \frac{g}{\sqrt{N}} \sum_j J_{ij} \left[a + \tanh h_j(t)\right]$$

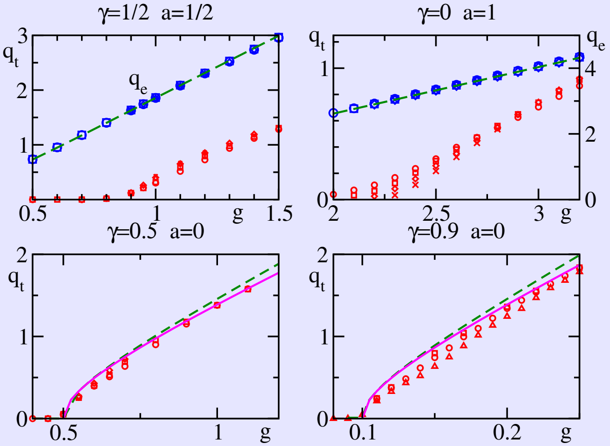
$\tanh h_i$ neural activity

γ local relaxation a external field

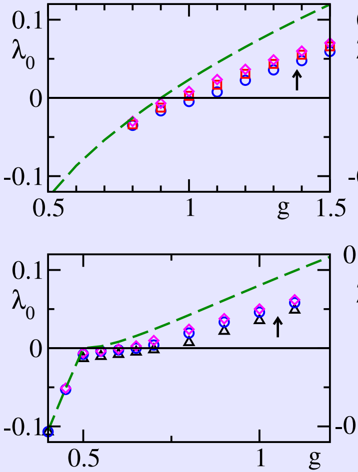
$\gamma \to 1$ ($g/(1-\gamma)$ constant); $a = 0$ Sompolinksy Crisanti Sommers (1988)

Order parameters

$$q_e^2(t) \equiv \overline{h_i^2}(t) \qquad q_t^2 \equiv \overline{\langle h_i^2\rangle_t - \langle h_i\rangle_t^2}$$



The maximum Lyapunov exponent



Identical units: unbalanced regime (logistic/tent maps)

$$x_{n+1}(i) = (1-g)f(x_n(i)) + g h_n(i)$$

$$f(x) = ax(1-x)$$

$$f(x) = a/2 - a|x - 1/2|$$

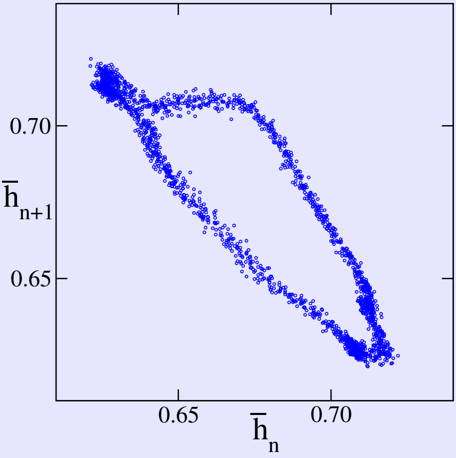
$\overline{h}_n$ = mean field

$$h_n(i) = \frac{1}{N}\sum_{j=1}^{N} f(x_n(j))$$

$$\sigma_h^2 = \langle \overline{h_n}^2 \rangle - \langle \overline{h_n} \rangle^2$$

Kaneko (1999)

Cencini Falcioni Vergni Vulpiani (1999)



Identical Stuart-Landau oscillators

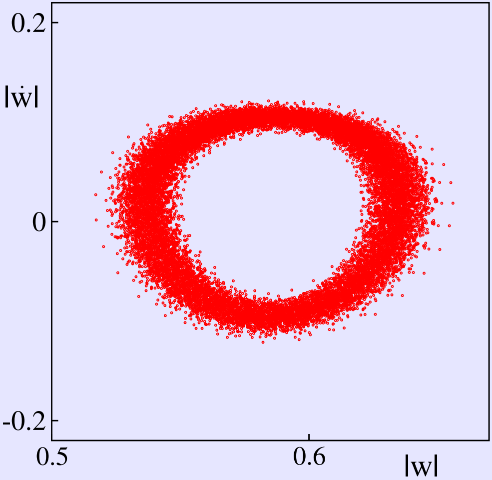
$$\dot{w}_i = w_i - (1 + ic_2)|w_i|^2 w_i + g(1 + ic_1)(W_i - w_i)$$

$W_i = \frac{1}{N}\sum_{j=1}^{N} w_j$

Nakagawa Kuramoto (1995)

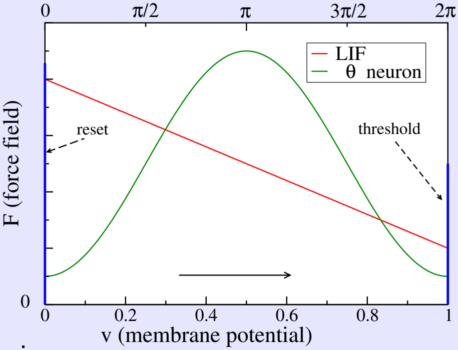
Chabanol Hakim Rappel (1997)

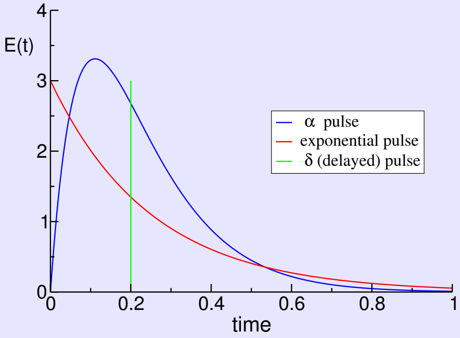
Takeuchi Ginelli Chaté (2009)



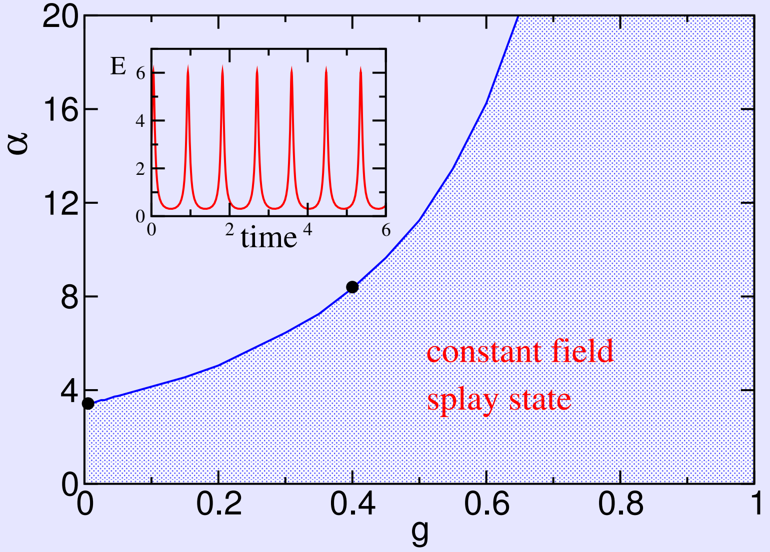
Identical pulse-coupled phase oscillators

$$\dot{v}_i = F(v_i) + \frac{g}{K} \sum_{n|t_n<t} P(t - t_n - t_d) = F(v_i) + gE_i$$





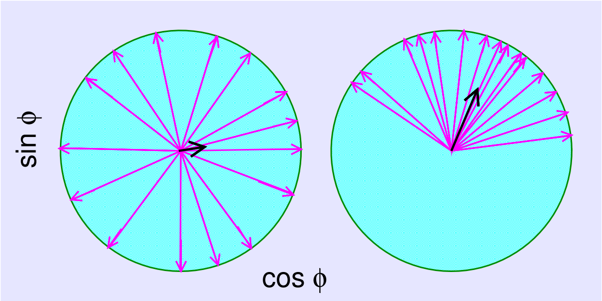
Self-consistent partial synchronization (LIF) (van Vreeswijk, 1996)



Adding disorder in the local dynamics: Kuramoto model

$$\dot{\phi}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\phi_j - \phi_i) + \xi_i(t)$$

order parameter $\quad Re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j}$



$$\dot{\phi}_i = \omega_i + KR\sin(\psi - \phi_i) + \xi_i(t)$$

Kuramoto-like setup: LIF with disorder and δ-pulses

$$\dot{v}_i = a_i - v_i + \frac{g}{N} \sum_{n|t_n<t} \delta(t - t_n - t_d)$$

global and homogeneous coupling

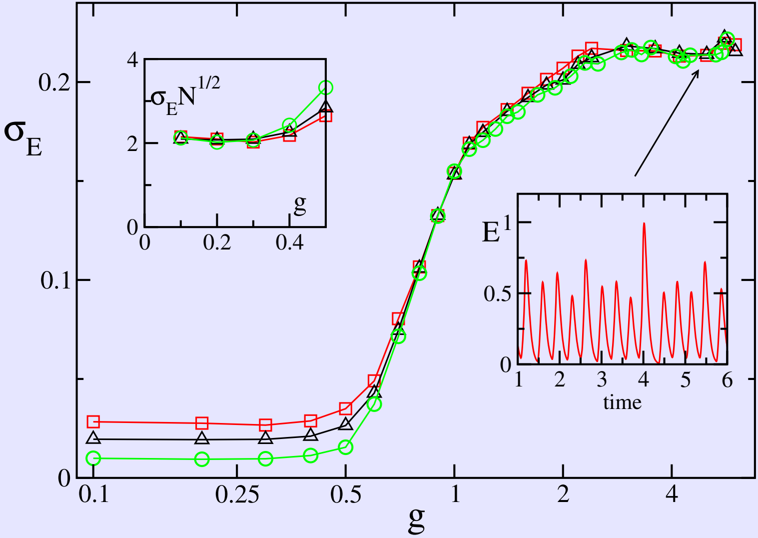
phase oscillators characterized by different frequencies

delayed coupling t_d

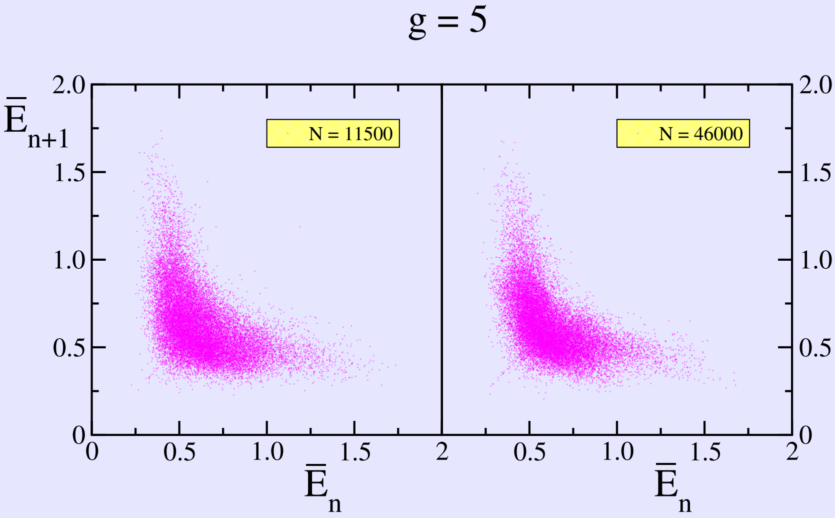
observable $\quad \ddot{E} + 2\alpha\dot{E} + \alpha^2 E = \frac{\alpha}{N} \sum \delta(t - t_n)$

Phase diagram

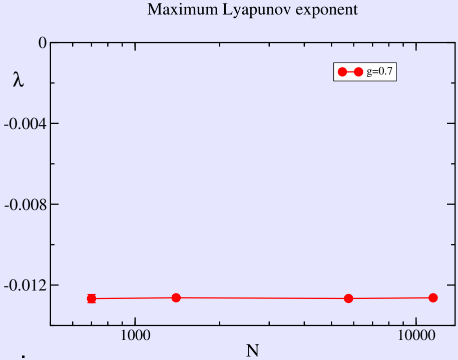
ORDER PARAMETER σ_e = standard deviation of E

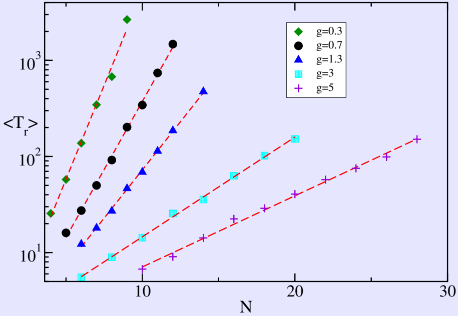


Return map of the field E(t) (on the sequence of maxima)



Stable chaos





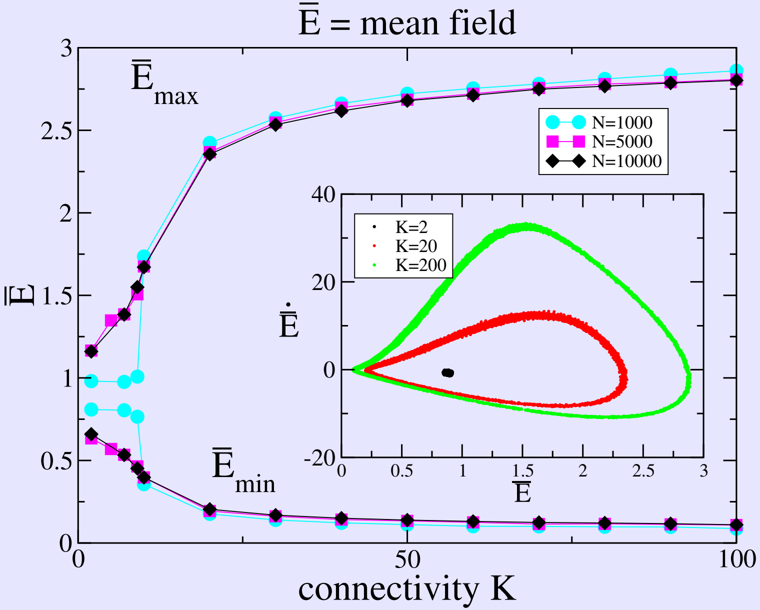
Disorder in the connections: from massive to sparse networks

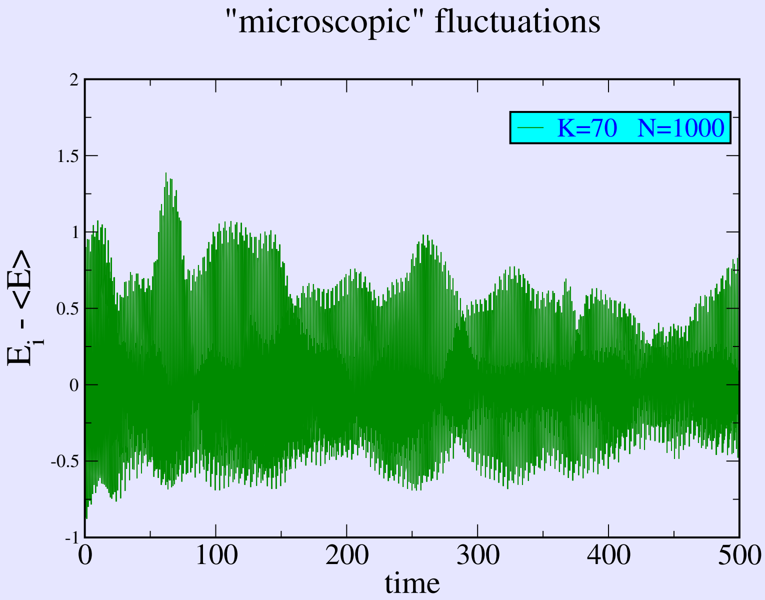
Massive (unbalanced) network: K/N finite

equivalent to a fully coupled network with a rescaled coupling strength.

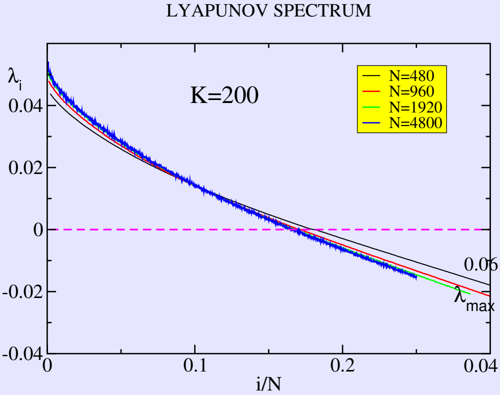
Sparse network: $K/N \to 0$ (K finite)

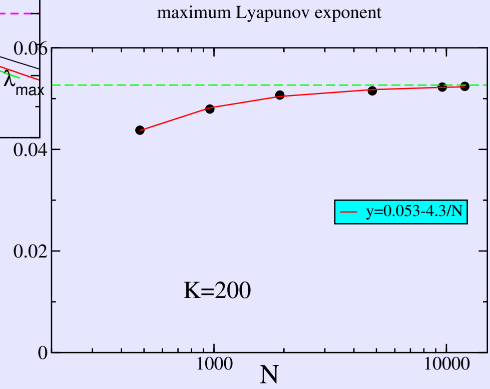
SPARSE NETWORKS: LIF NEURONS




"microscopic" fluctuations
K=70 N=1000
E_i - <E>
time
2
1.5
1
0.5
0
-0.5
-1
0
100
200
300
400
500

Characterization of microscopic chaos





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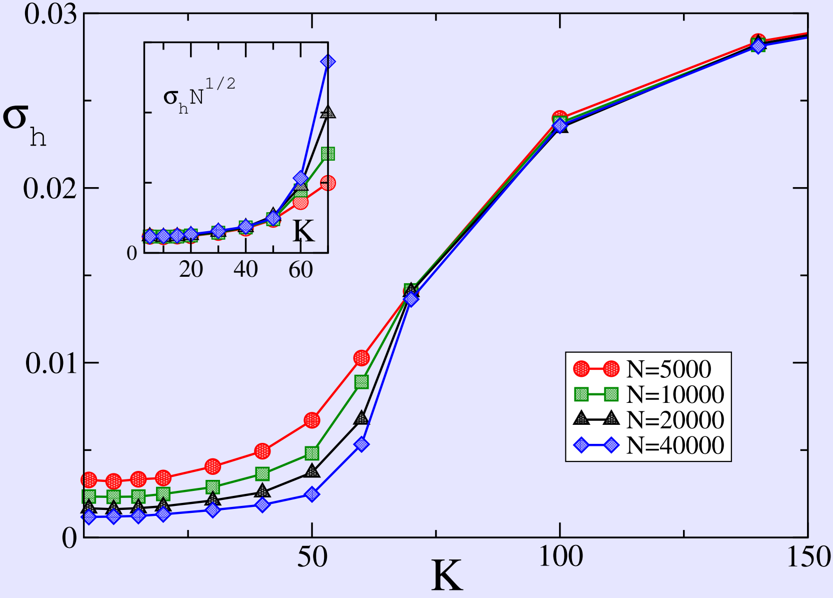
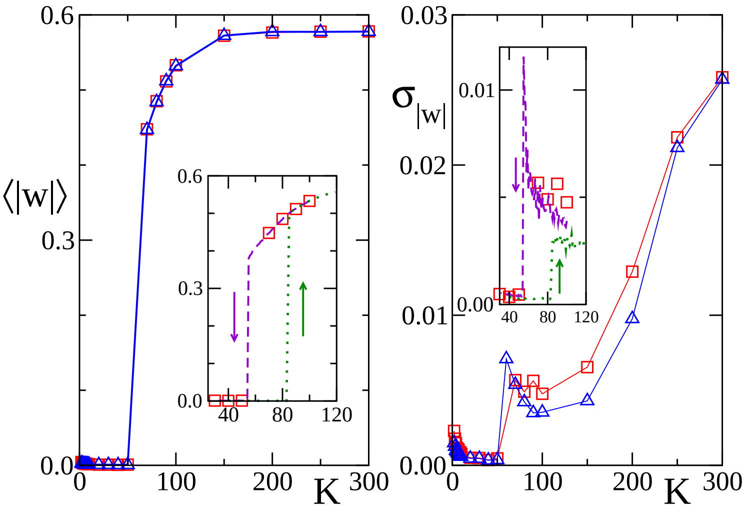
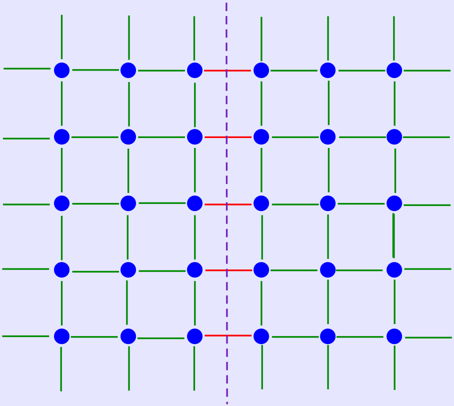
Collective dynamics in logistic maps



Collective dynamics in Stuart-Landau oscillators

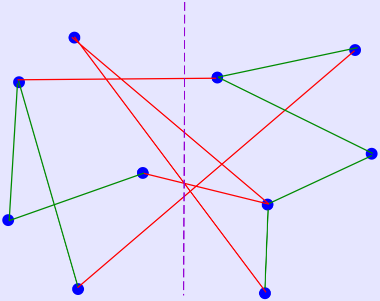


EXTENSIVITY AND ADDITIVITY



N nodes and $\sqrt{N}$ bound. links

N nodes and $\propto N$ bound. links



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