Discontinuous percolation transitions: How cooperativity can lead to disasters

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Angelo Vulpiani's 60th birthday
Rome, September 22, 2014

Obvious:

If it's the bad guys that cooperate!

So title & talk are trivial?

NO!!

Real title is:

HOW (!!!) can cooperativity lead to disasters?

Obvious:

By percolative phase transitions!

Are these in general first or second order ?

Neither!

If sufficiently cooperative, then transitions are first/second order mix

For each variant of the model, this mix is different

Short take-home message:

For cooperative percolation,
forget about standard phase transition classification!

Why is this a problem?

Percolation is unbiquitous;
It deals with establishing / breaking long range connectivity,
if sufficiently many short range "bridges" are established / broken.

Take e.g. your breakfast time:
If you have established long-range connectivity, then you:

- ... can eat your egg without a spoon

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- ... can eat your egg without a spoon
- ... can make a phone call
  or start an on-line game of "go"
- ... win this game of "go" or
- ... might have cought the flu.

In spite of this, the folklore is that percolation is simple & well understood:

- Second-order (continuous) phase transition
- Universal (same off / on lattice, same for different lattice types)
- Standard finite size scaling (i.e., observables are described by power laws $\times$ homogeneous functions of dimensionless ratios)
- Probability P that a cluster starting at random point becomes infinite
  and density S of the infinite cluster (= prob. that an infinite cluster reached a random point)
  are equally good order parameters.

Early warning that things might not be so simple:

“bootstrap percolation”

Seen by most as curiosity & not taken very serious.

Taken more serious:
”Explosive percolation” (Achlioptas *et al.*)
Claimed to be first order, but is actually continuous
with non-standard FSS

Other recent "non-standard" percolation models:

- Percolation in growing networks (Callaway *et al.*)
- Percolation in 1-d lattice models with long range links (Singh *et al.*, P.G.)
- Both are largely explained by:

  Non-amenable infinite networks
  (networks where arbitrarily large chunks can be cut off by arbitrarily few cuts)
  show critical *phase* (Lyons, Hasegawa, ...).

Bootstrap percolation relies on cooperativity
("a site only remains in the infinite cluster, if it has k neighbors in the infinite cluster"),

... and cooperation is also ubiquitous!

Cooperation between “infecting” (“wetting”, “convincing”, ...) neighbors:

- Spreading of some fad, opinion, measles, computer virus ...
  if “infection” needs more > 1 attacker/convincer (cooperativity),
  then large scale outbreak comes much more sudden:
  $\rightarrow$ “seeds” have no effect,
  $\rightarrow$“epidemic” can only start with high density of infected ones
- Pushing fluid through some porous rock (“invasion percolation”)
  small surface tension:
  $\rightarrow$ percolative growth, fractal wetted cluster
  large surface tension:
  $\rightarrow$ branch tipe cannot grow, “bays” are filled in
  $\rightarrow$ rough, but non-fractal surface, compact wetted region
- Deposition of atoms/molecules onto some substrate
- Random field Ising system at $T = 0, H \neq 0$: as H passes through zero, the “front” between $+$ and $-$ spins is pushed from one side to the other

large disorder: $\rightarrow$ fractal fronts
small disorder: $\rightarrow$ rough, but non-fractal

In all cases we consider only vicinity of critical point, i.e.

epidemic is near threshold of outbreak
surfaces are (nearly) pinned / move infinitely slowly

Claims:

- All these are realizations of percolation
- In all of them, percolation can be first order (discontinuous)

Different:

Cooperation between different spreading agents:

- Co-epidemics ("syndemics"):

  Spreading of 2 diseases A, B that "support" each other:

  - Spanish flu & TB
  - HIV & herpes
  - HIV & hepatitis

- Computer malware :
  If malware A has broken some firewall, then malware B has easy game

- Failures in multiplex networks:
  If utility A is no longer locally available (e.g. electricity in black-out),
  then this might mean that other utilities B, C, ... (information flow, public transport, ...) might also be no longer accessible,
  which might then lead to more lack of A, etc. ....

Both type of cooperativity are mathematically rather different:

- Cooperativity among neighbors leads to n-point interactions
  Cooperativity among spreading agents leads to multicomponent order parameters.

- Cooperativity among neighbors shows tricritical points,
  while no tricritical points are seen in (at least some models with)
  cooperativity among spreading agents, such as co-epidemics

- In both cases P and S are no longer equivalent order parameters ,
  but details (P is continuous, but S not or inverse,
  or S shows both a jump and an anomalous power law)
  can be very different.

Very simple model for coinfections:

- SIR type: susceptible $\rightarrow$ infected $\rightarrow$ immune
- mean field: described by chemical rate equations
- symmetric between both diseases
- same recovery rate for both diseases & for multiple infection
- infection probability differs for primary (α) and secondary (β) infections; cooperative, if

$$C \equiv \beta/\alpha > 1$$

9 coupled ODE's $\to$ (symmetry) $\to$ 3 coupled ODE's
Define:

$$X(t) = [A] + [AB] + [Ab] = [B] + [AB] + [aB]$$

$$Y(t) = \alpha S(t) + \beta([A] + [a] = [B] + [b])$$

$$\begin{aligned} \dot{S} &= -2\alpha S X \\ \dot{X} &= (Y-1)X. \\ \dot{Y} &= [2(\beta - \alpha)\alpha S - \beta Y]X \end{aligned} \quad (1)$$

$$(2)$$

Initial condition: $S(0) = 1 - \epsilon, [A(0)] = [B(0)] = \epsilon/2$
Main order parameter: $R = 1 - S(\infty)$ = fraction of the population that got at least one disease

$\to$ rich phenomenology;
First order transition (“backward bifurcation”), if $C > 2$

Mean field artifact:
Interesting phenomena happen only, when ϵ small, but > 0.
When $\epsilon \to 0$, A and B never meet

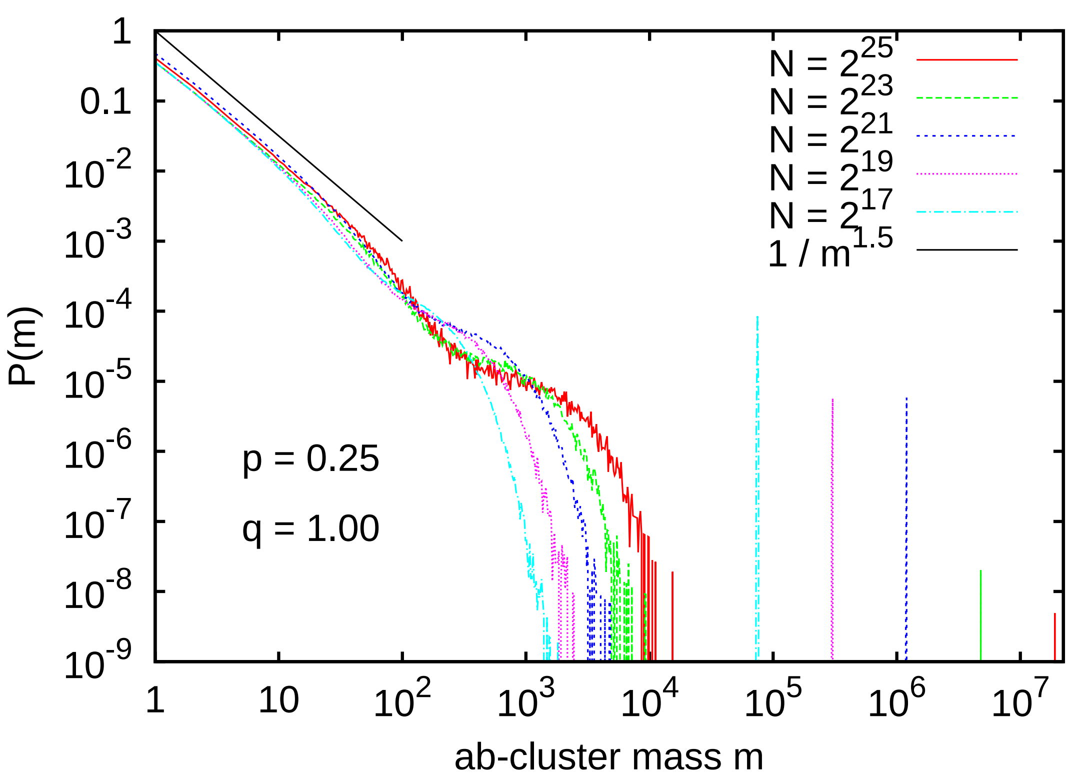
More realistic:

stochastic simulations on lattices and graphs!

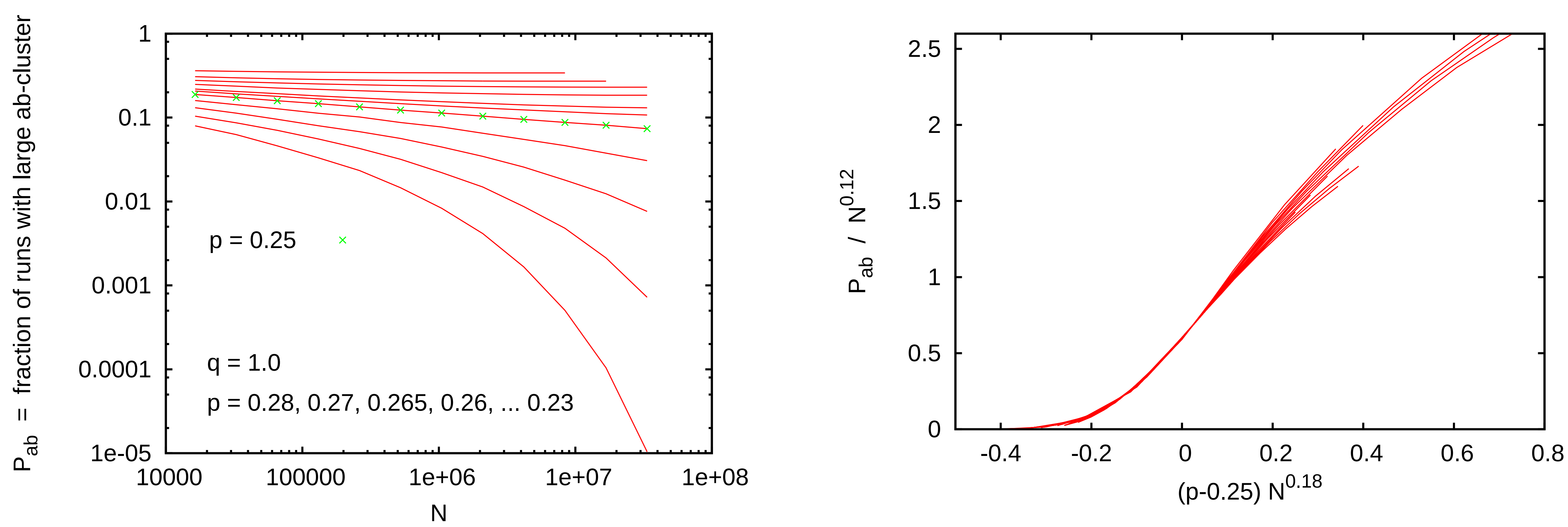
- Erdös-Renyi: true first order transition for large C, second order for small C
- 2-d lattice, finite range infection: only second order
- 4-d lattice: first order transitions for large C, maybe even for all $C > 1$?
- 3-d lattice: depends on
  - Microscopic details: latency / no latency
  - Lattice type: sc / bcc
- 2-d lattice with power-behaved infection prob.
  $P_{\mathrm{infect}}(r) \sim r^{-a}$:

  first order transitions for $a < a_0$, a_0 with $a_0 \approx 1.5$

Erdös-Renyi:
At $p = p_c = 1/\langle k \rangle$:

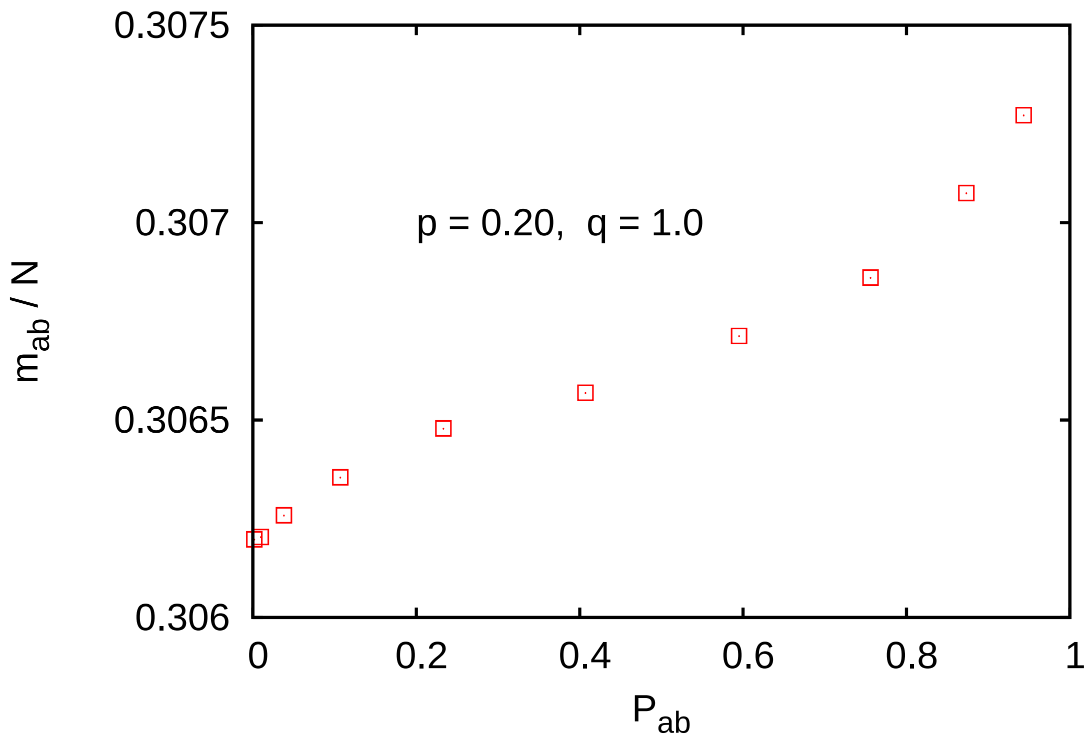


At $p < p_c$:
Multiple seeds!

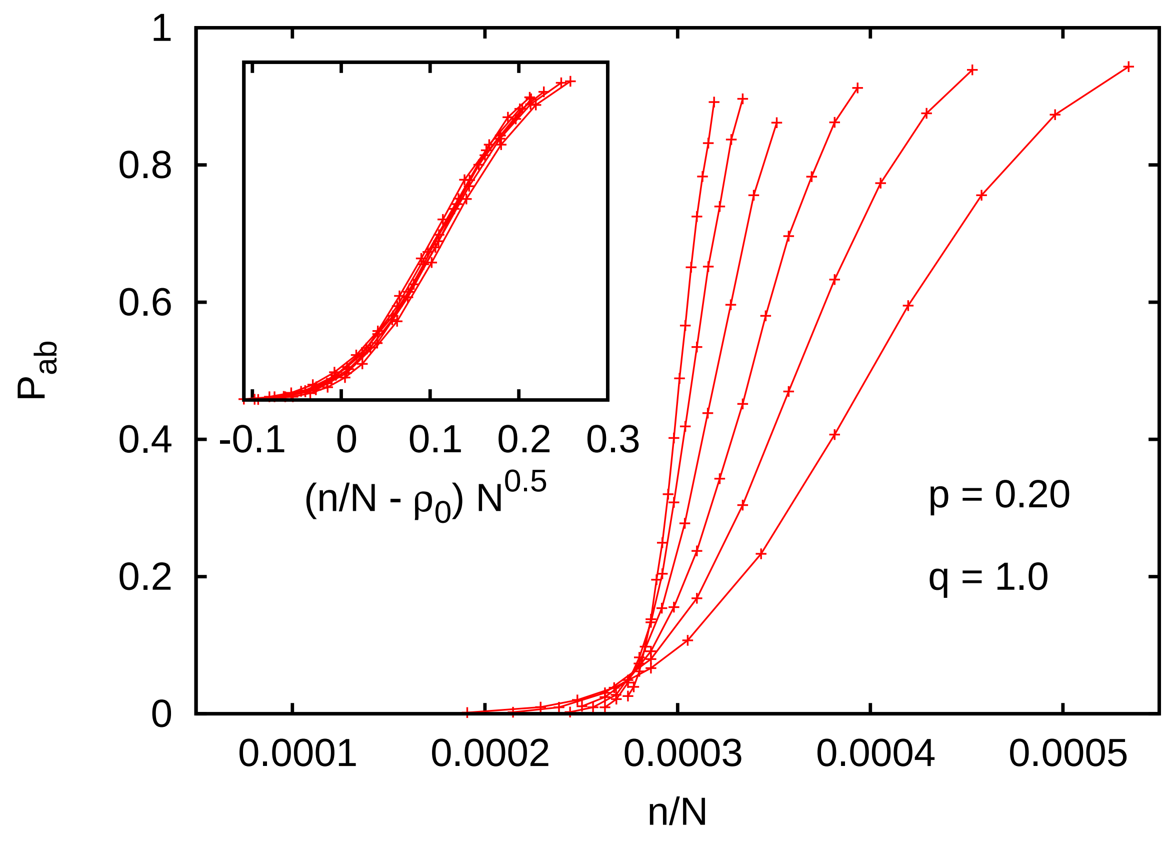


P_ab = fraction of runs with large ab-cluster
1
0.1
0.01
0.001
0.0001
1e-05
10000
100000
1e+06
1e+07
1e+08
N
p = 0.25
q = 1.0
p = 0.28, 0.27, 0.265, 0.26, ... 0.23
P_ab / N^0.12
2.5
2
1.5
1
0.5
0
-0.4
-0.2
0
0.2
0.4
0.6
0.8
(p-0.25) N^0.18

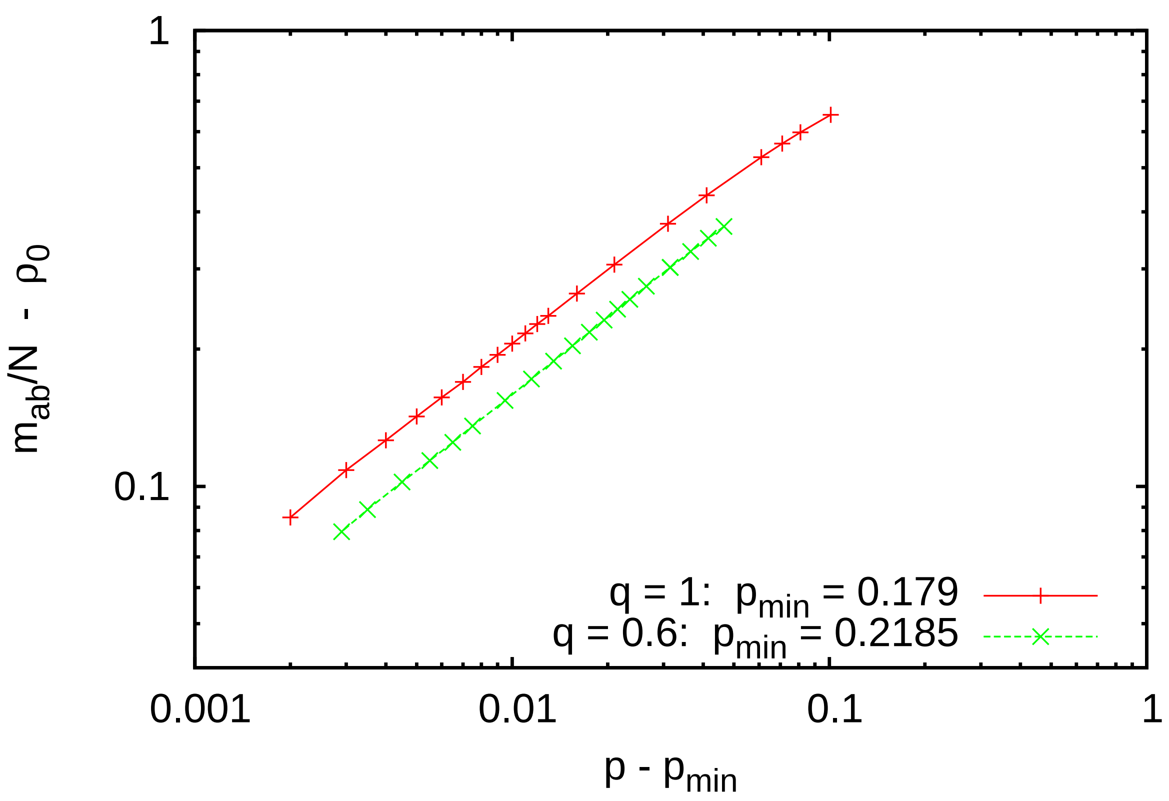
Cluster masses weakly dependent on seed size:



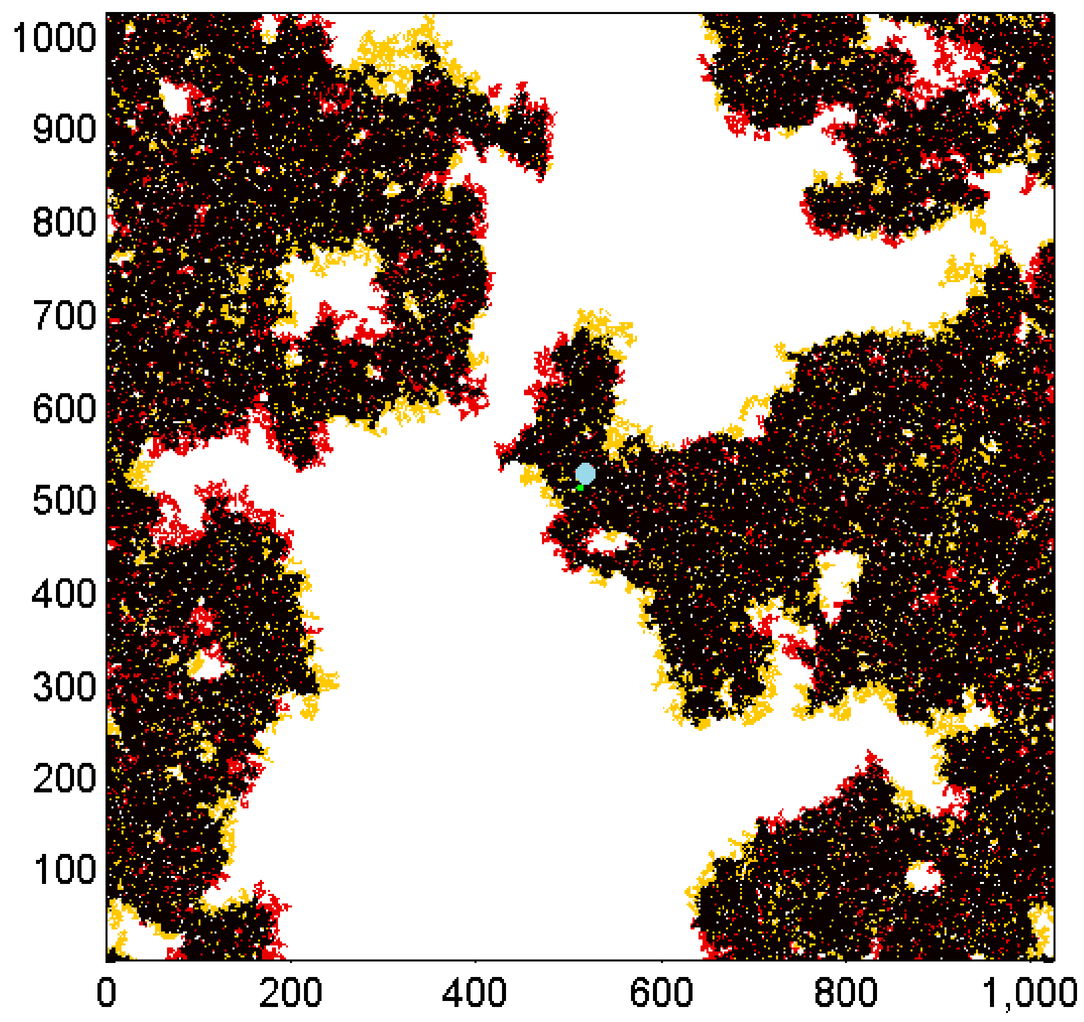
Clusters are infected at fixed **relative** seed size:



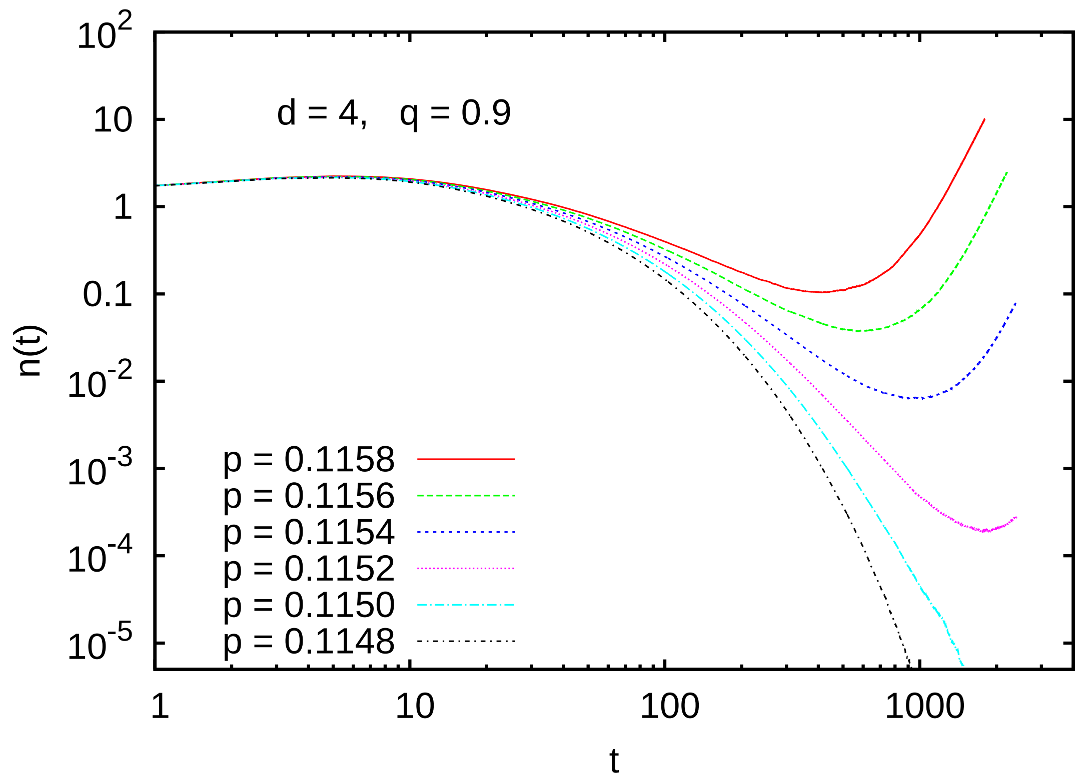
Relative cluster masses vanish with power law when $p \to p_{\min}$



d = 2:

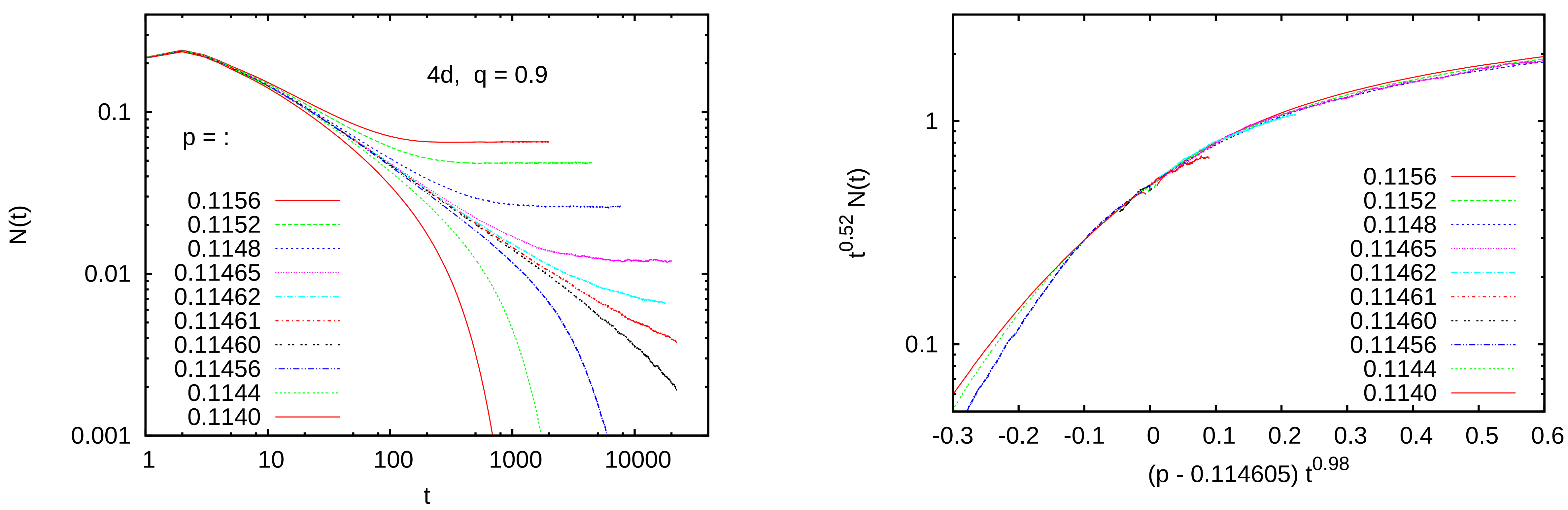


d = 4:



Use hyperplane seed!

$\to$ rough pinned interface!?



Analogous for height

$\rightarrow$ all critical exponents,
e.g. interface velocity:
$v \sim (p - p_c)^{\beta}, \quad \beta = 0.53(3)$

Different from ordinary rough pinned surfaces ($\beta \approx 0.84$)

Complex (synergetic, cooperative) contagion on random graphs

Assume random graph G:

arbitrary degree distribution,
sparse
no small loops: locally tree-like
$N \to \infty$, $z = \langle k \rangle = 2L/N$ finite

Ordinary (SIS) epidemic:

- start with some "seed" (group of infected sites), rest is susceptible
- each infected site remains *infectious* for one time step
- during this time it can infect each neighbor with probability p

Can this lead to an ∞ epidemic, i.e. to a cluster with size $S \times N$, with $S > 0$?

Self consistency criterium:

If a node is in such a cluster, then it must have at least one neighbor which is also in the cluster.

$P(k)$ = degree distribution
probability that a neighbor of a randomly chosen node is *not* in such a cluster =

$$= 1 - S = z^{-1} \sum_{k=1}^{\infty} k P_k (1 - pS)^{k-1}$$

or

$$F_{\mathrm{OP}}(S) \equiv \sum_{k=1}^{\infty} k P_k \{(1 - pS)^{k-1} + (S - 1)\} = 0.$$

Threshold for ∞ epidemic:

$$F_{\text{OP}}(0) = F'_{\text{OP}}(0) = 0, \;\; F''_{\text{OP}}(0) > 0$$

$$\rightarrow p_c = \frac{\langle k \rangle}{\langle k(k-1) \rangle}$$

(Bollobas, Newman, ...)

Assume now that nodes have *memory*:

After a first encounter with an infective neighbor, nothing happens
but the node is then more likely to succumb the next time

- If one guy tells you that this is a good movie, you might not believe – but if two, or three, or four tell you ?
- At a first attack a besieged city might not fall, but is weakened – and the next attack?

$p_n = \text{prob}\{\text{infection happens during } n+1\text{-st attack}\}$

Cooperative (synergistic) attacks:
$p_{n+1} > p_n$, immunity is weakened

Antagonistic:
$p_{n+1} < p_n$: "Whatever does not kill us makes us stronger"

On random graphs as before:

$(1 - pS)^{k-1}$ was
prob{none of the $k-1$ neighbors of a site that could pass on the epidemic is successfull in infecting it}

Now write

$$(1 - pS)^{k-1} = \sum_{n=0}^{k-1} \binom{k-1}{n} [(1-p)S]^n (1-S)^{k-n-1}.$$

$n-$th term: only n of these neighbors were infected themselves,
the chance that none of them is successfull $= (1-p)^n$

Now:

n neighbors together are successfull with probability q_n:

$q_1 = p_0$
$q_2 = p_0 + (1 - p_0)p_1$
$\ldots$
$q_{k+1} = q_k + (1 - q_k)p_k$

Replace

$(1 - pS)^{k-1} \rightarrow \sum_{n=0}^{k-1} \binom{k-1}{n} q_n S^n (1 - S)^{k-n-1}$.

$$F_{\mathrm{GEP}}(S) = \sum_{k=1}^{\infty} k P_k \sum_{n=0}^{k-1} \binom{k-1}{n} \times \quad (3)$$

$$\{[1-q_n] S^n (1-S)^{k-n-1} + (S-1)\} \quad (4)$$

("generalized epidemic process")

Criticality:
$F_{\text{GEP}}(0) = F'_{\text{GEP}}(0) = 0, \quad F''_{\text{GEP}}(0) >= 0$

First order transition:
$F_{\text{GEP}}(S) = F'_{\text{GEP}}(S) = 0, \quad F''_{\text{GEP}}(S) = 0$ for some $S > 0$

If $F''_{\text{GEP}}(0) = 0$ in addition to $F_{\text{GEP}}(0) = F'_{\text{GEP}}(0) = 0$:
tricritical point: phase transition switches second $\rightarrow$ first order

$\rightarrow$
$q_2 = 2q_1$ (Dodds & Watts, PRL 2004)

Do first order transitions & tricritical points also exist on lattices?

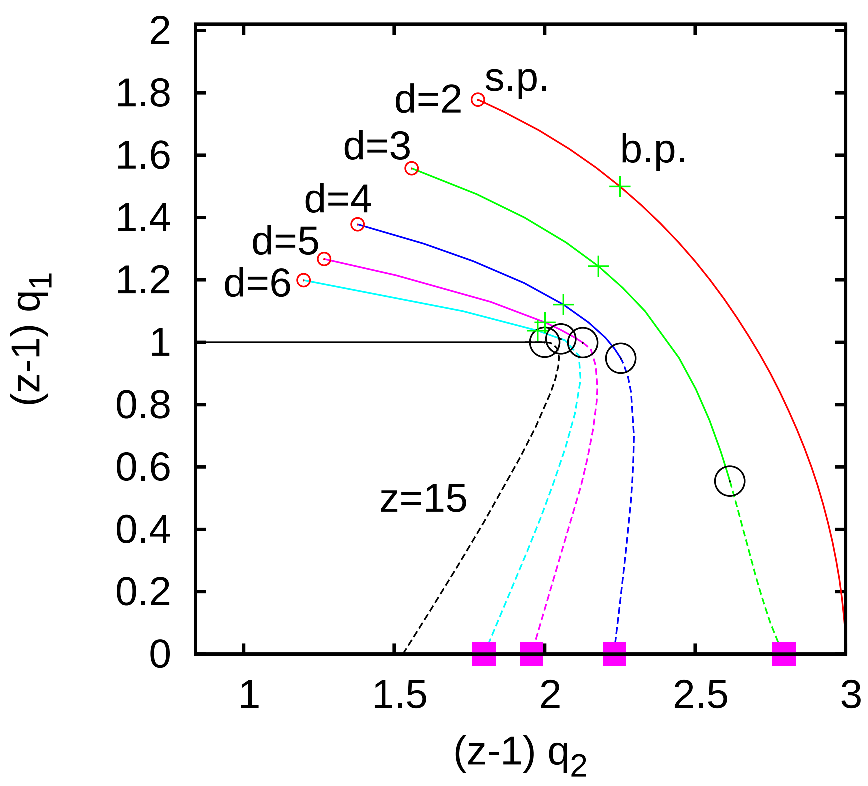
H.-K. Janssen et al., PRE 2004: Yes!

Perturbative RG group $+\epsilon$-expansion:

- $d_c = 5$: upper critical dimension
- for $d < d_c$: (asymptotic) power expansions of critical exponents in term of $\epsilon = d_c - d$
- First order percolation = rough but non-fractal pinned surfaces
- These surfaces show scaling (pinning transition is second order as far as surfaces are concerned),
  but the clusters behind them are compact (pinning transition is first order from percolation point of view)

Numerical Test: Simulations

Assume $p_1 = p_2 = p_3 = \ldots$: After the first attack, no further increases of p
(this should not make any qualitative difference from the more general case, in view of



d=2
s.p.
b.p.
d=3
d=4
d=5
d=6
z=15
(z-1) q1
(z-1) q2

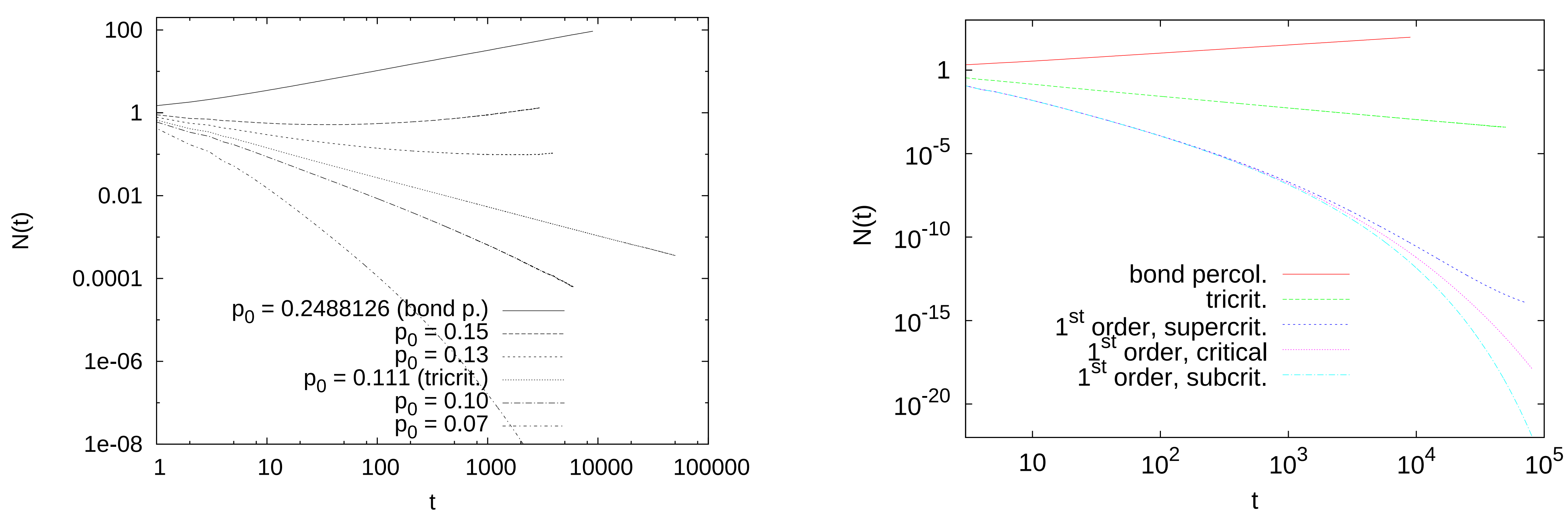
Several types of simulations:

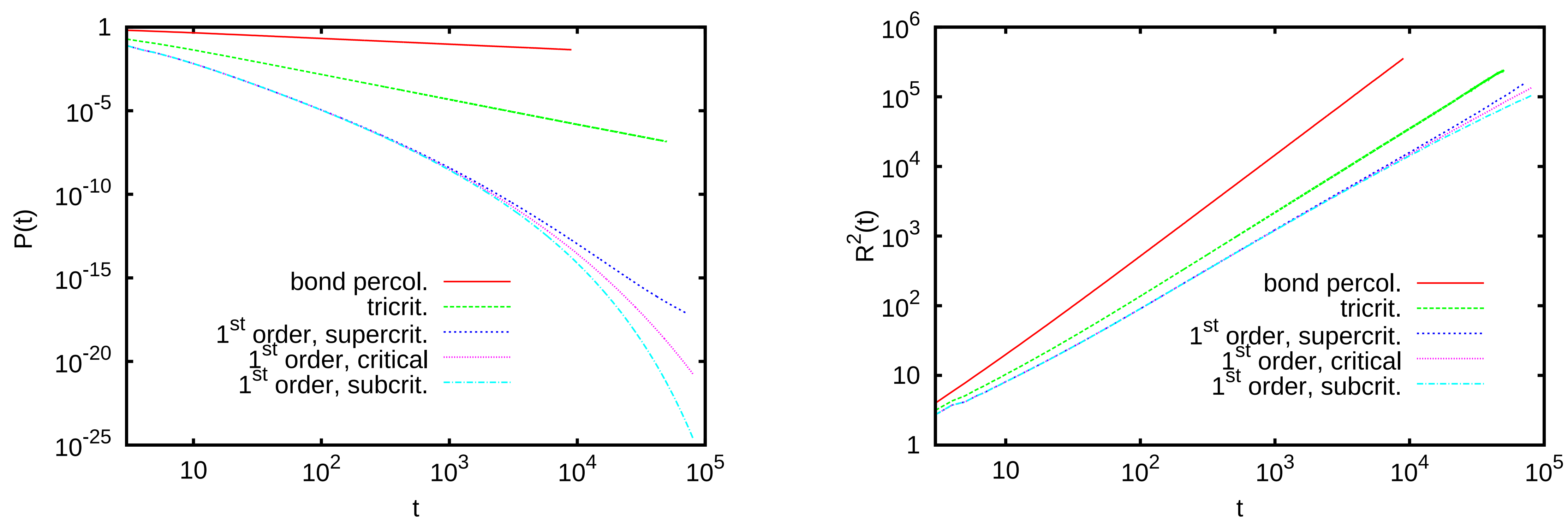
- point seeds, $2 \leq d \leq 6$
- (hyper-) plane seeds, plane orientation (1,0,0,...)
- (hyper-) plane seeds, plane orientation (1,1,1,...) ("tilted")

Determination of transition lines:

At (tri-)critical points: power laws, i.e. straight lines on log-log plots for $t \to \infty$

E.g. $d = 3$:


N(t)
100
1
0.01
0.0001
1e-06
1e-08
1
10
100
1000
10000
100000
t
p0 = 0.2488126 (bond p.)
p0 = 0.15
p0 = 0.13
p0 = 0.111 (tricrit.)
p0 = 0.10
p0 = 0.07
N(t)
1
10-5
10-10
10-15
10-20
10
102
103
104
105
t
bond percol.
tricrit.
1st order, supercrit.
1st order, critical
1st order, subcrit.



P(t)
t
bond percol.
tricrit.
1st order, supercrit.
1st order, critical
1st order, subcrit.
R2(t)

At critical point: o.k. (exponents well known)

At tricritical point:

$N(t) \sim t^{-\eta}$ with $\eta = -0.70(1)$
while Janssen et al.: $\eta > 0$

$d = 2$**:**

No tricritical point!
Rough pinned surfaces are always in the percolation universality class

Same conclusion: B. Drossel and K. Dahmen, EPJ B **3**, 485 (1998)

Recent opposite conclusion:
N.J. Zhou, B. Zheng, and Y.Y. He, PR B **80**, 134425 (2009)
N.J. Zhou and B. Zheng, PR E **82**, 031139 (2010)
X.P. Qin, B. Zheng, and N.J. Zhou, arXiv:1202.6486 (2012)

$d \geq 4$:
Janssen et al.: ok

Beyond tricriticality (first order / rough surfaces):

In spite of strong overhangs, critical exponents in first order regime (pinned surfaces) agree with Edwards-Wilkinson model (no overhangs; P. Le Doussal, K.J. Wiese, and P. Chauve, PRB **66**, 174201 (2002)

$d = 3$: Universality class of pinned surfaces is unclear