present case the simple equation of motion (5.3) is not appropriate anymore. The correct approach is to use the Newton second law and write an expression for the Lagrangian acceleration of the test particles

$$\rho_p \frac{d^2 \boldsymbol{x}}{dt^2} = \boldsymbol{F}, \tag{5.30}$$

where ρ_p is the test particle density, and $\boldsymbol{F}$ the force per unit volume acting on it. In the simplest case $\boldsymbol{F}$ may be represented by the sum of two terms. The first term is the force density exerted by the fluid in the position occupied by the test particle. This is clearly given by the fluid density times the acceleration that would be impressed to a hypothetical fluid particle placed in $\boldsymbol{x}(t)$, in the spirit of the Archimede principle. The second term is of phenomenological origin and represents the linear Stokes drag felt by a test particle whose istantaneous velocity $\dot{\boldsymbol{x}}$ is different from the local fluid velocity $\boldsymbol{u}(\boldsymbol{x}, t)$. Thus

$$\frac{d^2 \boldsymbol{x}}{dt^2} = \delta \frac{D\boldsymbol{u}}{Dt} - \mu \left(\frac{d\boldsymbol{x}}{dt} - \boldsymbol{u} \right), \tag{5.31}$$

where $\delta = \rho_f / \rho_p$ is the ratio of the fluid to the particle density. The total derivative $D\boldsymbol{u}/Dt = \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$. The parameter μ is a coefficient weighting the effect of the Stokes drag [LL87] and is obviously related to the fluid viscosity. Equation (5.31) has been derived with naive arguments. A more detailed analysis [H59] yields the same result, up to the so-called Basset term which in many cases can be neglected. The main difference is in the coefficients δ and μ which now become

$$\delta \to \tilde{\delta} = \frac{3\delta}{2 + \delta}, \qquad \mu \to \tilde{\mu} = \frac{2\mu}{2 + \delta}.$$

This rescaling, however, does not change the qualitative behaviour of the system.

Additional effects can be added to the right-hand side of (5.31) such as, for example, the acceleration due to gravity, as given by a bouyancy term, or as the acceleration due to the effects of electromagnetic fields on charged particles and plasmas. However, we shall not consider these additional terms.

For $\delta = 1$, *i.e.* $\rho_f = \rho_p$, one recovers (5.3). Indeed in this case (5.31) reduces to

$$\frac{d}{dt}\left(\frac{d\boldsymbol{x}}{dt} - \boldsymbol{u} \right) = -\mu \left(\frac{d\boldsymbol{x}}{dt} - \boldsymbol{u} \right), \tag{5.32}$$

which may be immediately integrated to give

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{u} + [\boldsymbol{u}_0 - \boldsymbol{u}(\boldsymbol{x}(0), 0)] \exp[-\mu t], \tag{5.33}$$

where $\boldsymbol{u}_0 = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$ at the initial time, and $\boldsymbol{u}(\boldsymbol{x}(0), 0)$ is the Eulerian velocity at time $t = 0$ and spatial position $\boldsymbol{x}(0)$. Therefore, in this case (5.31) reduces to (5.3) after a transient fixed by the value of μ. Physically (5.31) describes the Lagrangian motion of a fluid particle whose initial velocity is different from that of the Eulerian flow in that point. After a transient, the particle behaves like every other particle in the fluid.

The parameter μ fixes the characteristic scale of the model. In fact, it is easy to see that for large values of μ the dominat part of (5.31) is given by (5.3) up to a time t_μ, which diverges as $\mu \longrightarrow \infty$. However, small regions exist, usually around the separatrices of (5.3), where the inertial part is relevant also at small time. On the other hand, for $t > t_\mu$ the inertial part can play a nontrivial role. In addition, one can see that a rescaling of time and lengths transforms (5.31) into an equivalent system with $\mu = 1$. Consequently, the value of μ affects only the velocity of the process, without altering the qualitative aspects of the dynamics at large times.

To discuss (5.31) we use again the velocity field (5.19) given by the projection of the ABC flow with $C = 0$ on the (x, y)-plane and $|A| = |B|$. In this case (5.3) does not have chaotic trajectories. Conversely, they may be present for (5.31). The value $\delta = 1$ separates two very different dynamical regimes. Linear stability analysis shows that for $\delta > 1$ — test particles less dense than the fluid — (5.31) has stable and attracting fixed points located at the centre of the convective cells. In this case, the trajectory of a tracer will tend to one of these fixed points, see fig. 28. On the contrary, for $0 < \delta < 1$ the fixed points are unstable and particles may undergo chaotic paths which lead to diffusion, fig. 29.

A numerical study shows [CFPrV90] that for $0 < \delta < 1$ the trajectories are indeed chaotic with a maximum Lyapunov exponent

$$\lambda_1 \sim \varepsilon^\beta, \qquad \beta \simeq 0.14, \tag{5.34}$$

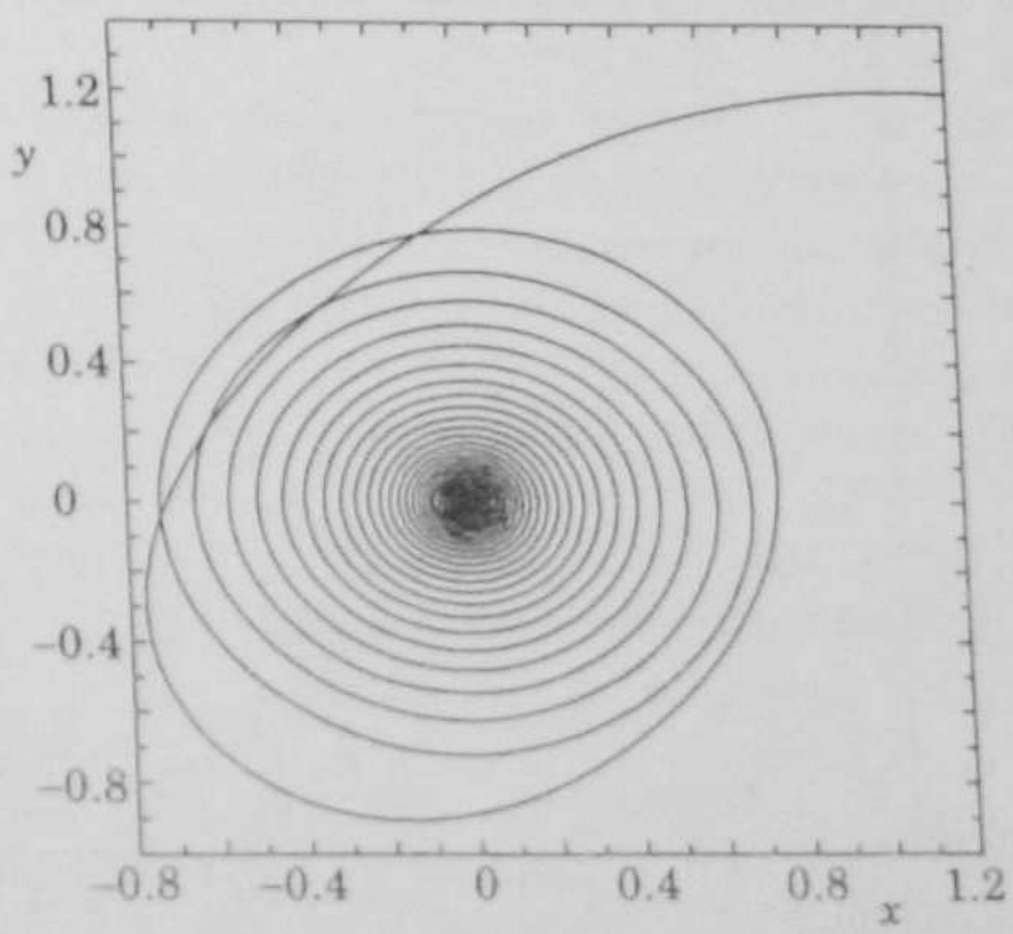

Fig. 28. – Example of trajectory of the system (5.31) for $\mu = 1$, $\delta = 1.2$ and a convecting field (5.19) wth $A = B$; the initial condition is $x_0 = y_0 = 1.2$ with zero initial velocity. The trajectory has been followed during 8×10^3 times steps $\Delta t = 0.01$ natural time units.

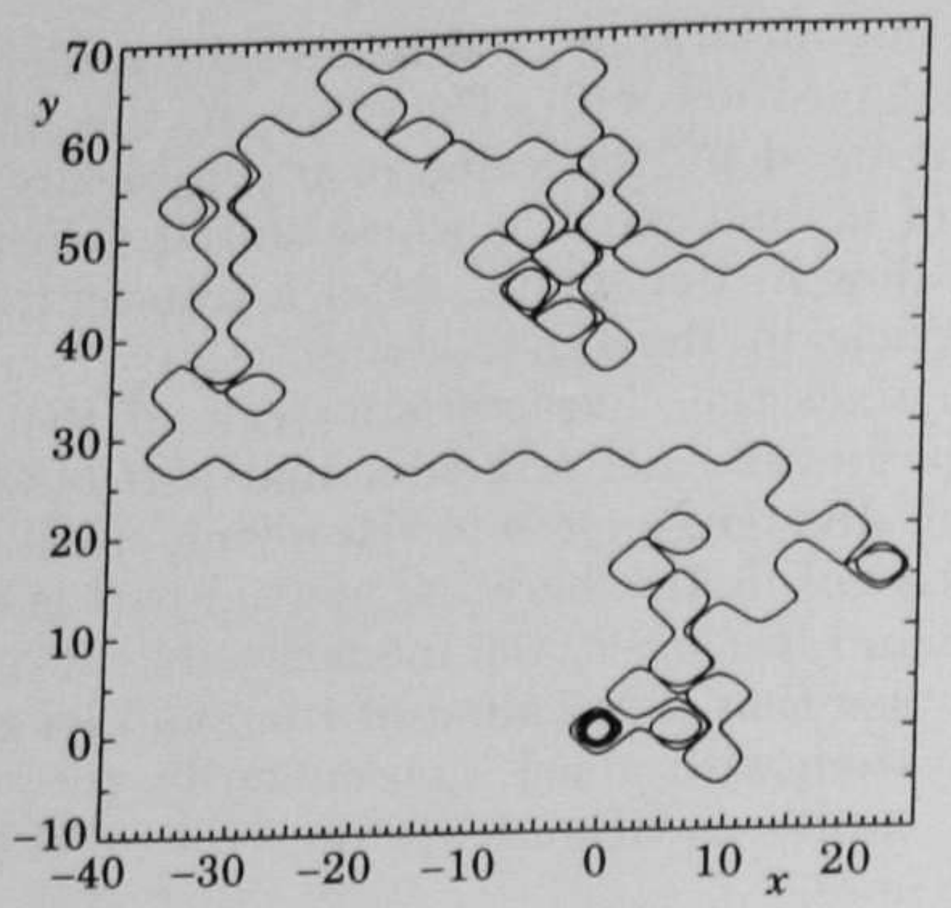

Fig. 29. – The same as fig. 28 with $\delta = 0.9$.

where $\varepsilon = 1 - \delta$. For small values of ε the transport on large distance is diffusive with effective diffusion coefficient, see fig. 30,

$$D_x \sim D_y \sim \varepsilon^{\alpha}, \qquad \alpha \simeq 0.25\,. \tag{5.35}$$

For larger values of ε the motion becomes ballistic in a direction which depends on the initial conditions.

It is interesting to compare these results to those obtained from (5.1). In that case, if $\chi \neq 0$, we have

$$\lambda_1 \sim \chi^{\beta'}, \qquad \beta' \simeq 0.3\,, \tag{5.36}$$

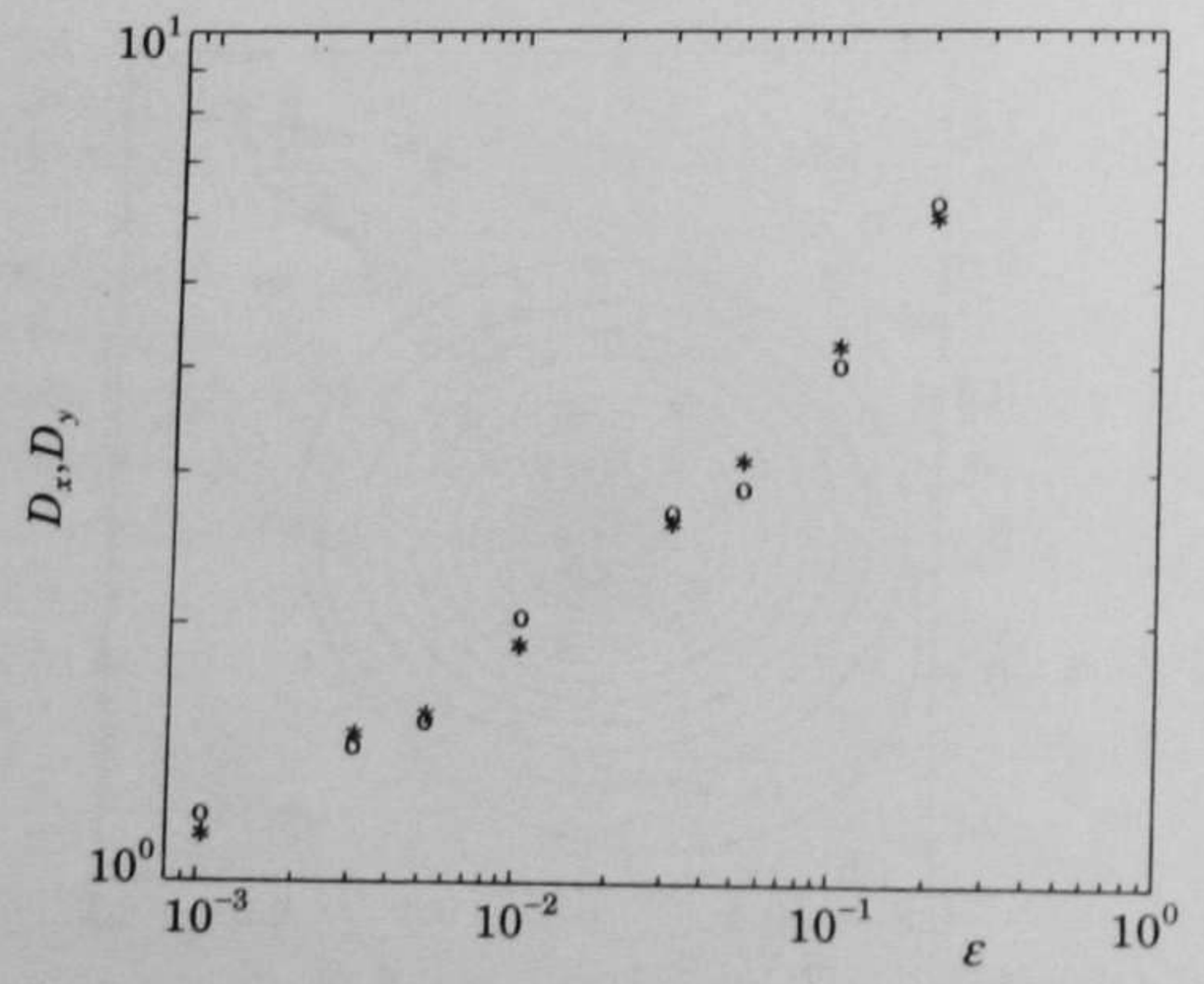

Fig. 30. – Diffusion coefficients (D_x(*) and D_y(○)) for system (5.31), (5.19) *vs.* the inertia parameter $\varepsilon = 1 - \delta$. Typical error bars are about three or four times as large as the symbols employed.

and (see (5.20))

$$D_x \sim D_y \sim \varepsilon^{\alpha'}, \qquad \alpha' \simeq 0.5. \tag{5.37}$$

Therefore the motion of particles denser than the surrounding fluid and the dynamics of fluid particles in the presence of additive noise indicates a strong similarity. Note in particular that $\alpha/\beta \simeq \alpha'/\beta'$.

Similar results are obtained for $|A| \neq |B|$ [CFPTV91].

Heuristically, this similarity is not completely surprising. Both problems may be thought of as a perturbation of (5.3). In both cases, a singular perturbation in the case of (5.31) and a random forcing in the case (5.1), one observes that the diffusion takes place along the separatrices of (5.3). Thus a perturbation of (5.3) seems to lead to a somewhat generic situation, in which standard diffusion process is observed.

6. – Technical details.

This section contains some technical or pedagogical subjects which we include for the sake of self-consistency.

6'1. *Hamiltonian systems.* – The evolution of a Hamiltonian system is ruled by the equations

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i}, \qquad i = 1, \ldots, n, \tag{6.1}$$

where $H(q_1, \ldots, q_n, p_1, \ldots, p_n, t)$ is the Hamiltonian and q_i and p_i are conjugate variables. The number n is traditionally called the number of degrees of freedom, while the dimension of the manifold where the motion develops is $N = 2n$ in the nonautonomous case, and $N = 2n - 1$ in the autonomous case. If H does not depend explicitly on time, then H is a constant of motion.

Let us remark that a Hamiltonian system is a dynamical system of particular type. A small perturbation of (6.1) destroys the Hamiltonian structure [LL83]. Therefore such a system has properties which are very different from those of a general, non-Hamiltonian system. Nevertheless, dynamical systems with the Hamiltonian structure have a relevant role in many applications, *e.g.*, celestial mechanics, particles accelerators, fluid mechanics and plasma physics [LL83].

A completely integrable system is a limit case of an autonomous Hamiltonian system, in which n independent constants of motion, F_i, exist and are in involution, *i.e.*

$$\{H, F_i\} = \{F_i, F_j\} = 0,$$

where $\{\}$ indicates the Poisson brackets. In such a system there is [A76], although it may not be easy to find, a canonical transformation

$$(q_i, p_i) \to (\theta_i, I_i),$$

which reduces the equations of the motion in the form

$$\frac{\mathrm{d}I_i}{\mathrm{d}t} = 0\,, \qquad \frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_i(\boldsymbol{I}) = \frac{\partial H\,(\boldsymbol{I})}{\partial I_i}\,. \tag{6.2}$$

The solution of (6.2) is

$$I_i(t) = I_i(0)\,, \qquad \theta_i(t) = \theta_i(0) + \omega_i(\boldsymbol{I}(0))\,t\,;$$

and the motion is confined on n-dimensional tori. A small perturbation destroys the complete integrability, *i.e.* if H has the form

$$H = H_0(\boldsymbol{I}) + \varepsilon H_1(\boldsymbol{I},\ \boldsymbol{\theta})\,, \tag{6.3}$$

then, apart from very particular cases, for $\varepsilon \ll 1$ the only surviving integral of the motion is H. This result, due to Poincaré, can be briefly sketched as follows, see [LL83]. Let us try to find the constants of motion, for the system (6.3), in the form

$$F = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \ldots., \qquad \text{where} \qquad F_0 = I_j\,.$$

We have to impose $\{H,\ F\} = 0$, in order to determine F_1, F_2, ... The equation for F_1 has the form

$$\{H_0,\ F_1\} = -\{H_1,\ F_0\}\,. \tag{6.4}$$

It is easy to see that (6.4) has no global solution. Let us consider the Fourier expansion of H_1 and F_1:

$$H_1 = \sum_k c_k(\boldsymbol{I}) \exp[i\boldsymbol{k}\cdot\boldsymbol{\theta}]\,, \qquad F_1 = \sum_k f_k(\boldsymbol{I}) \exp[i\boldsymbol{k}\cdot\boldsymbol{\theta}]\,, \tag{6.5}$$

where the vector $\boldsymbol{k} = (k_1,\ \ldots,\ k_n)$ has integer components. Inserting (6.5) into (6.4) and noting that

$$\{H_0,\ (\cdot)\} = \sum_{l=1}^{n} \omega_l(\boldsymbol{I}) \frac{\partial\,(\cdot)}{\partial\theta_l} \qquad \text{and} \qquad \{H_1,\ F_0\} = -i\sum_k k_j c_k(\boldsymbol{I}) \exp[i\boldsymbol{k}\cdot\boldsymbol{\theta}]\,,$$

one has

$$f_k(\boldsymbol{I}) = \frac{k_j c_k(\boldsymbol{I})}{\sum_{l=1}^{n} \omega_l(\boldsymbol{I})\,k_l}\,. \tag{6.6}$$

This equation for $f_k(\boldsymbol{I})$ shows that the problem admits no solution because the scalar product $\boldsymbol{\omega}\cdot\boldsymbol{k}$, that appears in the denominator of (6.6), can assume arbitrarily small values. This is the celebrated small divisors problem.

One of the most relevant results, in the field of Hamiltonian systems, is given by the KAM theorem (from Kolmogorov, Arnold and Moser) [K54, A63, M62]:

a) ε sufficiently small,

b) $\det \Omega \neq 0$, where $\Omega_{ij}(\boldsymbol{I}) = \partial^2 H_0 / \partial I_i \partial I_j$,

the measure of the invariant tori, on a fixed constant energy surface, is positive and goes to 1 for $\varepsilon \to 0$. Roughly speaking, one has that the motion on the invariant tori is ordered, *i.e.* similar to that of the integrable systems.

Let us note that the invariant tori have dimension n, while the available phase space has dimension $2n-1$. Thus in the case $n=2$ the tori separate regions that can exhibit different chaotic behaviours. On the contrary, for $n \geq 3$ the complementary of the set of invariant tori is connected. This fact allows for the Arnold diffusion: the system can move about the whole surface of constant energy by diffusing among the unperturbed tori [AA68]. The presence of the Arnold diffusion has been proved for particular systems, but it is believed to hold in a generic system. Unfortunately it is not easy to give theoretical estimates of the Arnold diffusion time scales, in the general case [PV84, MPV86]. One of the most relevant results in this direction is due to Nekhoroshev [N77]. We do not enter in details and defer the reader to the literature [BG86].

It is possible to show that the continuous time dynamics given by (6.1) can be reduced to a discrete map. For the sake of simplicity, we consider an autonomous Hamiltonian system with two degrees of freedom. Instead of studying the continuous flow

$$\Gamma(0) \to \Gamma(t),$$

where $\Gamma(t) = [\boldsymbol{q}(t), \boldsymbol{p}(t)] \in \boldsymbol{R}^4$, one can investigate the system (6.1) in terms of a discrete map in $\boldsymbol{R}^2$.

The energy is a constant of motion, so only 3 variables x_1, x_2 and x_3 are sufficient for a complete specification of the state of the system. Consider a plane S defined by $x_3 = h =$ constant and denote by $\boldsymbol{P}(0)$, $\boldsymbol{P}(1)$, $\boldsymbol{P}(2)$, ... the intersections of the flow $\Gamma(t)$ with the plane S and $\mathrm{d}x_3/\mathrm{d}t < 0$, at the successive times t_0, t_1, t_2, ... The plane S is called the *Poincaré section* of the flow (6.1). Due to the deterministic nature of the flow $\Gamma(0) \to \Gamma(t)$, the intersection $\boldsymbol{P}(k+1) \in \boldsymbol{R}^2$ can be obtained by the previous one via a map, called Poincaré map:

$$\boldsymbol{P}(k+1) = \boldsymbol{g}[\boldsymbol{P}(k)]. \tag{6.7}$$

The knowledge of $\boldsymbol{P}(k)$ is equivalent to the knowledge of $\Gamma(t_k)$.

The above argument can be repeated for a generic Hamiltonian system with n degrees of freedom. The energy conservation reduces the dimension of the phase space to $2n-1$. Introducing a section S, the intersection $\boldsymbol{P}(k+1) \in \boldsymbol{R}^{2n-2}$ of the trajectory $\Gamma(t)$ with S is related to $\boldsymbol{P}(k)$ via a map of the form (6.7). Moreover it is possible to show that the Poincaré map associated to Hamiltonian system is symplectic [LL83], *i.e.* the matrix $\boldsymbol{A}$, defined by $A_{ij} = \partial g_i / \partial P_j$, has the following property:

$$\boldsymbol{A}^T \boldsymbol{J} \boldsymbol{A} = \boldsymbol{J}, \quad \text{where} \quad \boldsymbol{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Another case in which one can reduce the continuous problem to the study of a symplectic map is given by time periodic Hamiltonians: $H(\boldsymbol{q}, \boldsymbol{p}, t+\tau) = H(\boldsymbol{q}, \boldsymbol{p}, t)$. In this case the analogous of $\boldsymbol{P}(k)$ is just $(\boldsymbol{q}(k\tau), \boldsymbol{p}(k\tau))$.

The Poincaré map simplifies the study of a continuous dynamical system because

a) the time is discrete;

b) the dimension of the phase space is reduced, if the system is autonomous;

c) it is easier to visualize whether the motion is regular or chaotic.

For instance, for $n = 2$, the track on the Poincaré section of a regular trajectory is given either by a set of isolated points (periodic motion) or by a regular curve (quasi-periodic motion). On the contrary, a chaotic trajectory leads to an unstructured spot. An example of these different features can be found in [HH64].

6·2. *Motion near a separatrix: Melnikov's method.* – Here we briefly discuss a simple method, due to Melnikov [M63], for studying the motion close to a separatrix of a quasi-integrable Hamiltonian system. This method yields a criterion for the existence of chaotic motion. We concentrate on the one-dimensional time-dependent Hamiltonian case, but this approach can be also applied to multi-dimensional systems [W88].

Consider a Hamiltonian function of the form

$$H(q, p, t) = H_0(q, p) + \varepsilon H_1(q, p, t),$$

where H_1 is periodic in time, with period τ. The evolution equation for $\boldsymbol{x} = (q, p)$ is

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}^{(0)}(\boldsymbol{x}) + \varepsilon \boldsymbol{f}^{(1)}(\boldsymbol{x}, t), \quad \text{with} \quad \boldsymbol{f}^{(1)}(\boldsymbol{x}, t+\tau) = \boldsymbol{f}^{(1)}(\boldsymbol{x}, t). \tag{6.8}$$

The unperturbed systems is integrable and is assumed to possess a hyperbolic fixed point $\tilde{\boldsymbol{x}}_0$ and a separatrix orbit $\boldsymbol{x}_0(t)$ such that $\lim_{t \to \pm\infty} \boldsymbol{x}_0(t) = \tilde{\boldsymbol{x}}_0$. See fig. 31 for an illustration of the phase space. The stable and the unstable orbits, $\boldsymbol{x}_0^{\mathrm{s}}$ and $\boldsymbol{x}_0^{\mathrm{u}}$, respectively, are smoothly joined. Let us consider the Poincaré map of the perturbed dynamics (6.8), *i.e.*

$$\boldsymbol{x}(t_0) \to \boldsymbol{x}(t_0 + \tau) = \boldsymbol{T}\boldsymbol{x}(t_0),$$

where t_0 denotes a shift in the initial condition. The hyperbolic fixed point, $\tilde{\boldsymbol{x}}_\varepsilon$, of the map $\boldsymbol{T}$ stays close to $\tilde{\boldsymbol{x}}_0$: $\tilde{\boldsymbol{x}}_\varepsilon = \tilde{\boldsymbol{x}}_0 + O(\varepsilon)$. Moreover the stable (unstable) orbit $\boldsymbol{x}_\varepsilon^{\mathrm{s}}(\boldsymbol{x}_\varepsilon^{\mathrm{u}})$, lying on the stable (unstable) manifold $W^-_{\varepsilon, t_0}(W^+_{\varepsilon, t_0})$, is close to $\boldsymbol{x}_0(t)$ for $t \to \infty\,(-\infty)$, that is

$$\begin{cases} \boldsymbol{x}_\varepsilon^{\mathrm{s}}(t, t_0) = \boldsymbol{x}_0(t - t_0) + \varepsilon \boldsymbol{x}_1^{\mathrm{s}}(t, t_0) + O(\varepsilon^2), & t \in [t_0, \infty], \\ \boldsymbol{x}_\varepsilon^{\mathrm{u}}(t, t_0) = \boldsymbol{x}_0(t - t_0) + \varepsilon \boldsymbol{x}_1^{\mathrm{u}}(t, t_0) + O(\varepsilon^2), & t \in [-\infty, t_0]. \end{cases} \tag{6.9}$$

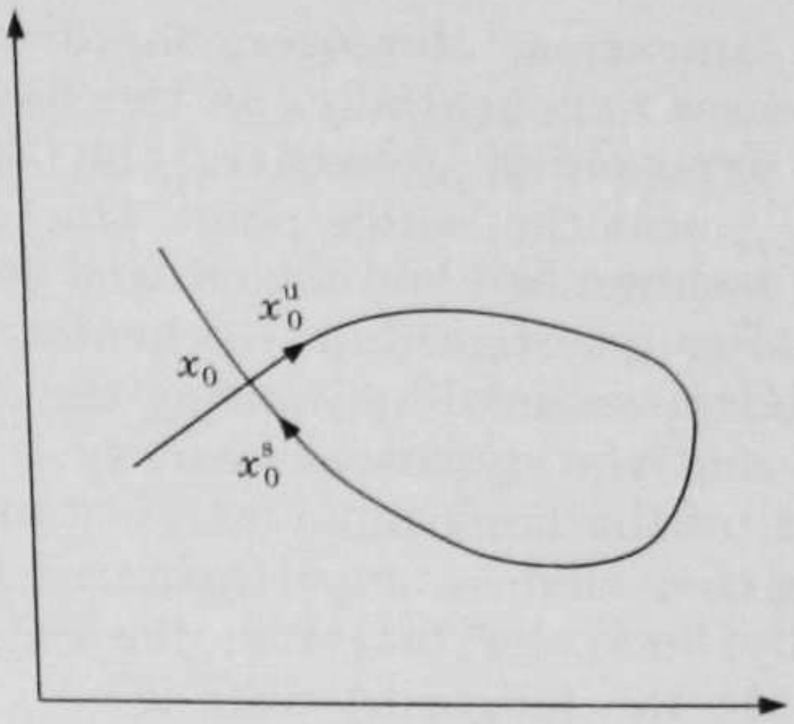

Fig. 31. – Phase space around an unperturbed hyperbolic fixed point, of a one-dimensional Hamiltonian system, with a homoclinic orbit.

In the generic case, fig. 32, W^+_{ε, t_0} intersects W^-_{ε, t_0} in a homoclinic point P, and it is not difficult to see that this implies a chaotic motion around the unperturbed separatrix. For simplicity we do not consider here the nongeneric case of a tangent contact. First of all let us observe that one intersection between W^+_{ε, t_0} and W^-_{ε, t_0} implies an infinite number of intersections. Indeed, the forward or backward iterations of the Poincaré map starting from P, bring to points $\boldsymbol{T}^{\pm n}(P)$, that, by construction, belong both to W^+_{ε, t_0} and to W^-_{ε, t_0}. So one has that W^+_{ε, t_0} intersects W^-_{ε, t_0} in an infinite number of homoclinic points, although W^+_{ε, t_0} and W^+_{ε, t_0} cannot have self-intersections.

«The intersections form a kind of trellis, a tissue, an infinite tight lattice; each of curves must never self-intersect, but it must fold itself in a very complex way, so as to return and cut the lattice an infinite number of times» (Poincaré [P99]).

The existence of one, and therefore infinite, homoclinic intersection implies chaos. This has been discussed in a rigorous way by Birkhoff [B27] and Holmes [H90]. Without entering into technical details, we observe that, because of the area conservation, the succesive loops formed between homoclinic

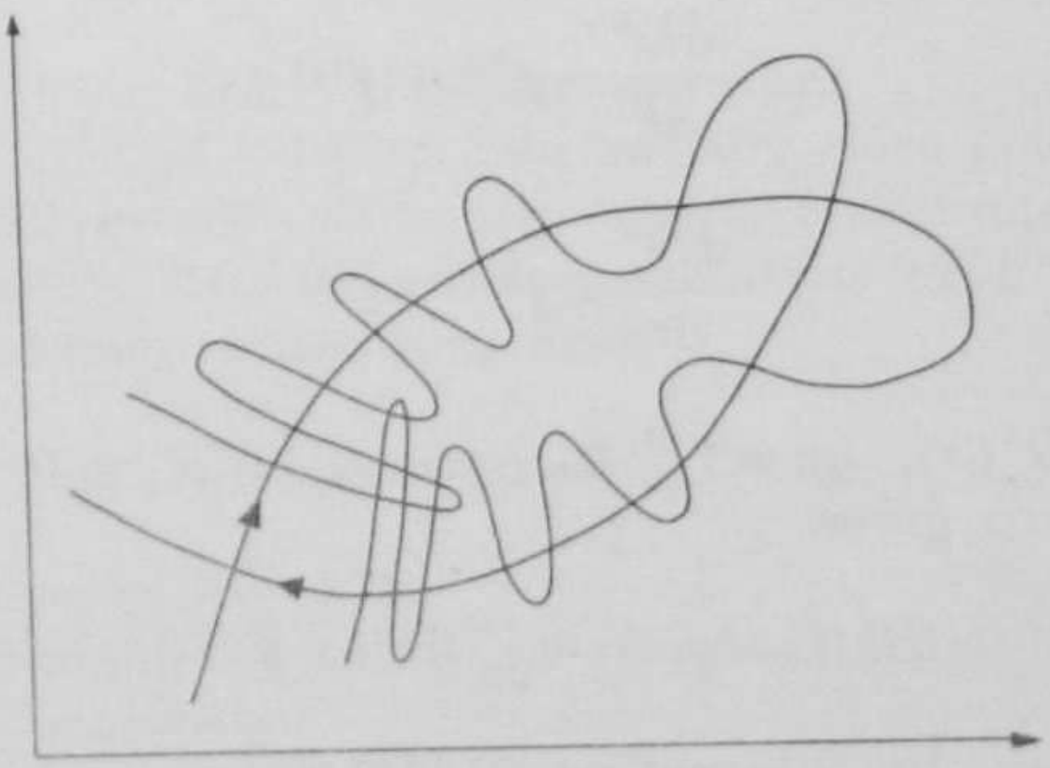

Fig. 32. – Some homoclinic intersections due to a perturbation of the hyperbolic fixed point of fig. 31.

points must have the same area. Moreover, the distance between succesive homoclinic points decreases exponentially, as the fixed point is approached. Therefore one has an exponential growth of the length of the loops and a strong bending of W^-_{ε, t_0} near the saddle point. One has that a small part of the plane around $\tilde{\boldsymbol{x}}_\varepsilon$ will be stretched and folded, and two points, initially close, will be found far apart after few iterations. Guckenheimer and Holmes [GH83] proved rigorously the above scenario, by showing the equivalence between the iterated horseshoe map and the dynamics near $\tilde{\boldsymbol{x}}_\varepsilon$.

The existence (or not) of the homoclinic intersection can be determined by a method, due to Melnikov, that is a perturbative computation using the properties (6.9). Basically, in order to prove the existence of a homoclinic point, one has to estimate the («signed») distance

$$\boldsymbol{d}(t,\ t_0) = \boldsymbol{x}^{\mathrm{s}}_\varepsilon(t,\ t_0) - \boldsymbol{x}^{\mathrm{u}}_\varepsilon(t,\ t_0) = \boldsymbol{x}^{\mathrm{s}}_1(t,\ t_0) - \boldsymbol{x}^{\mathrm{u}}_1(t,\ t_0)\,.$$

A direct computations gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}^{\mathrm{s},\,\mathrm{u}}_1(t,\ t_0) = \boldsymbol{M}(\boldsymbol{x}_0)\,\boldsymbol{x}^{\mathrm{s},\,\mathrm{u}}_1(t,\ t_0) + \boldsymbol{f}^{(1)}(\boldsymbol{x}_0(t-t_0),\ t)\,,$$

where

$$M_{ij}(\boldsymbol{x}_0) = \left.\frac{\partial f^{(0)}_i}{\partial x_j}\right|_{x_0(t-t_0)}\,.$$

Let us introduce the Melnikov distance $D(t,\ t_0) = \boldsymbol{N}\cdot\boldsymbol{d}$ which is the projection of $\boldsymbol{d}$ along a normal $\boldsymbol{N}$ to the unperturbed orbit $\boldsymbol{x}_0$ at time t, *i.e.* $\boldsymbol{N}(t,\ t_0) = (-f^{(0)}_2,\ f^{(0)}_1)$. One can write $D(t,\ t_0)$ in the form

$$D = D^{\mathrm{s}} - D^{\mathrm{u}}, \quad \text{with} \quad D^{\mathrm{s},\,\mathrm{u}} = \boldsymbol{f}^{(0)} \times \boldsymbol{x}^{\mathrm{s},\,\mathrm{u}}_1 \tag{6.10}$$

and $\boldsymbol{a}\times\boldsymbol{b} = a_1 b_2 - a_2 b_1$.

Taking the derivative of (6.10), using (6.8) and the conservative properties of the system, *i.e.* $\mathrm{Tr}\,\boldsymbol{M}(\boldsymbol{x}_0) = 0$, one obtains

$$\frac{\mathrm{d}D^{\mathrm{s}}}{\mathrm{d}t} = \boldsymbol{f}^{(0)} \times \boldsymbol{f}^{(1)}, \tag{6.11a}$$

$$\frac{\mathrm{d}D^{\mathrm{u}}}{\mathrm{d}t} = \boldsymbol{f}^{(0)} \times \boldsymbol{f}^{(1)}\,. \tag{6.11b}$$

By noting that $D^{\mathrm{s}}(\infty,\ t_0) = \boldsymbol{f}^{(0)}(\boldsymbol{x}_0(\infty - t_0)) \times \boldsymbol{x}^{\mathrm{s}}_1 = 0$, the integration of (6.11*a*) from t_0 to ∞ gives

$$D^{\mathrm{s}}(t_0,\ t_0) = -\int_{t_0}^{\infty} \boldsymbol{f}^{(0)} \times \boldsymbol{f}^{(1)}\,\mathrm{d}t\,.$$

In an analogous way, by integrating (6.11*b*) from $-\infty$ to t_0, one obtains

$$D^{\mathrm{u}}(t_0,\ t_0) = \int_{-\infty}^{t_0} \boldsymbol{f}^{(0)} \times \boldsymbol{f}^{(1)}\,\mathrm{d}t\,.$$

So finally one has

$$D(t_0, t_0) = -\int_{-\infty}^{\infty} \boldsymbol{f}^{(0)} \times \boldsymbol{f}^{(1)} \, \mathrm{d}t \, .$$

If $D(t_0, t_0)$, which is an explicitly computable function, changes sign at some t_0, then one has a homoclinic point and chaos appears around the separatrix.

Let us remark that, although the Melnikov method is based on a simple perturbation method, it is of considerable relevance.

The above analysis can be repeated for the case in which the unperturbed system has a heteroclinic orbit, which tends asymptotically to an unstable equilibrium point for $t \to -\infty$ and towards another one for $t \to +\infty$.

6·3. *Characteristic Lyapunov exponents and generalized Lyapunov exponents.* – The characteristic Lyapunov exponents (CLE) are a natural extension of the linear stability analysis to aperiodic motion in dynamical systems. Roughly speaking, they measure the typical rates of the exponential divergence of nearby trajectories. This sensitive dependence on initial conditions is one of the main characteristics of deterministic chaos, which renders the forecasting of the dynamics practically impossible since the initial state of the system cannot be known with an infinite precision [BGGS80, LL83, ER85].

Consider a differentiable dynamical system with an evolution law given, in the case of continuous time t, by the differential equation

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}[\boldsymbol{x}(t)] \, . \tag{6.12}$$

or, in the case of discrete times t, by the map

$$\boldsymbol{x}(t+1) = \boldsymbol{g}[\boldsymbol{x}(t)] \, . \tag{6.13}$$

The variable $\boldsymbol{x}$ and the differentiable functions $\boldsymbol{f}$ and $\boldsymbol{g}$ vary in a phase space which can be $\boldsymbol{R}^d$, or a compact manifold, or an infinite-dimensional space. Equations (6.12) and (6.13) generate a mapping of the phase space into itself.

$$\boldsymbol{x}(0) \to \boldsymbol{x}(t) = \boldsymbol{T}^t \boldsymbol{x}(0) \, , \tag{6.14}$$

where $\boldsymbol{T}$ is the nonlinear time evolution operator.

To study the separation between two initially close points, one introduces the tangent vector $\boldsymbol{z}(t)$ which can be regarded as an infinitesimal perturbation $\delta\boldsymbol{x}(t)$ of the trajectory $\boldsymbol{x}(t)$. The time evolution of $\boldsymbol{z}(t)$ is described by a mapping of the tangent space into itself

$$\boldsymbol{z}(0) \to \boldsymbol{z}(t) = \boldsymbol{D}_{x(0)} \boldsymbol{T}^t \boldsymbol{z}(0) \tag{6.15}$$

in terms of the linear operator $\boldsymbol{D}_x \boldsymbol{T}^t$.

The formal evolution law (6.15) is obtained from (6.13) and (6.14) as a linear differential equation

$$\frac{\mathrm{d}z_i(t)}{\mathrm{d}t} = \sum_{j=1}^{d} \left. \frac{\partial f_i}{\partial x_j} \right|_{x(t)} z_j(t) \, ,$$

or a map

$$z_i(t+1) = \sum_{j=1}^{d} \left.\frac{\partial g_i}{\partial x_j}\right|_{x(t)} z_j(t)\,, \tag{6.16}$$

respectively. In this last case, using the chain rule of differentiation, we have

$$\boldsymbol{D}_{x(0)}\boldsymbol{T}^t = \prod_{\tau=0}^{t-1} \boldsymbol{D}_{x(\tau)}\boldsymbol{T}.$$

The linear operator $\boldsymbol{D}_x\boldsymbol{T}^t$ is hence given by a product of matrices. For the sake of simplicity we only discuss the case of maps. However all the result also hold for the differential equations.

The study of the separation of nearby trajectories can then be reduced to the study of the properties of products of matrices. From the mathematical point of view, the most important result in this field is the Oseledec theorem [O68]. Let us consider an initial condition $\boldsymbol{x}(0)$ and the sequence of matrices $(\boldsymbol{A}(1), \boldsymbol{A}(2), \ldots)$ given by the linear operator $\boldsymbol{D}_{x(0)}\boldsymbol{T}^k$, *i.e.*

$$A_{ij}(k) = \left.\frac{\partial g_i}{\partial x_j}\right|_{x(k)},$$

and denote by $\boldsymbol{P}_N(\boldsymbol{x}(0))$ the product of the first N matrices of the sequence. The Oseledec theorem states:

Theorem.

Consider the transformation (6.13), $\boldsymbol{T}$ being a diffeomorphism of class C^1 of a compact connected Riemann manifold M onto itself, and μ an ergodic invariant measure of the system. Then, there exists a measurable subset $M_1 \subset M$, $\mu(M_1) = 1$, such that, for all $\boldsymbol{x}(0) \in M_1$,

$$\lim_{N\to\infty} \left(\boldsymbol{P}_N^{\dagger}(\boldsymbol{x}(0))\,\boldsymbol{P}_N(\boldsymbol{x}(0))\right)^{1/(2N)} = \boldsymbol{V}(\boldsymbol{x}(0))$$

exists. The matrix $\boldsymbol{V}(\boldsymbol{x}(0))$ has d real positive eigenvalues $\exp[\lambda_i(\boldsymbol{x}(0))]$ repeated according to their multiplicity. The exponents $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d$ are called *characteristic Lyapunov exponents.*

Moreover chosen $p \leq d$ «generic» random vectors $[\boldsymbol{z}^{(1)}(0), \ldots, \boldsymbol{z}^{(p)}(0)]$ one has

$$\lambda_1 + \ldots + \lambda_p = \lim_{t\to\infty} \frac{1}{t} \ln\left(\mathrm{VOL}_p[\boldsymbol{z}^{(1)}(t), \ldots, \boldsymbol{z}^{(p)}(t)]\right),$$

there $\mathrm{VOL}_p[\boldsymbol{z}^{(1)}, \ldots, \boldsymbol{z}^{(p)}]$ is the volume of the open parallelepiped generated by the p vectors $\boldsymbol{z}^{(i)}$ and each $\boldsymbol{z}^{(i)}(k)$ obeys (6.16).

The spectrum of Lyapunov exponents $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_d$ does not depend on the initial conditions if the dynamical system has an ergodic invariant measure. This if often not the case in low-dimensional symplectic systems, where

disconnected chaotic regions can exist. An impressive example can be found in oval billiards [BS78] where, for some values of the parameter, there are up to eight different stochastic regions, separated by invariant tori. In general, the Lyapunov exponents assume different values in each of the disconnected regions.

Roughly speaking the Oseledec theorem states that for almost all perturbation, *i.e.* almost all $\boldsymbol{z}(0)$, the distance between the trajectory and the perturbed one grows exponentially as

$$|\boldsymbol{z}(t)| \sim |\boldsymbol{z}(0)| \exp[\lambda_1 t][1 + O(\exp[-(\lambda_1 - \lambda_2)t])].$$

This relation leads us to introduce the response R to a perturbation in $\boldsymbol{x}(\tau)$ after a time t by the error growth rate

$$R_\tau(t) \equiv \frac{|\boldsymbol{z}(\tau + t)|}{|\boldsymbol{z}(\tau)|}.$$

The maximum Lyapunov exponent λ_1 can then be defined by averaging the logarithm of the response over the possible initial conditions along the trajectory

$$\lambda_1 = \lim_{t \to \infty} \frac{1}{t} \langle \ln R(t) \rangle, \tag{6.17}$$

where $\langle \cdot \rangle$ denotes the time average $\lim_{T \to \infty} (1/T) \int_t^{t+T} (\cdot)\, \mathrm{d}t$. The Oseledec theorem implies that for large time $\ln R(t)$ is a nonrandom quantity, in the sense that for almost all the initial conditions its value does not depend on the specific initial condition, so that the average in (6.17) can be neglected.

Since the typical growth of a generic tangent vector is given by the maximum Lyapunov exponent, it is clear the $R(t)$ alone cannot be used to extract the other Lyapunov exponents. To this end one introduces the n order response $R^{(n)}$ as [BGGS80]

$$R^{(n)}(t) \equiv \frac{|\boldsymbol{z}_1(t + \tau) \times \boldsymbol{z}_2(t + \tau) \times \ldots \times \boldsymbol{z}_n(t + \tau)|}{|\boldsymbol{z}_1(\tau) \times \boldsymbol{z}_2(\tau) \times \ldots \times \boldsymbol{z}_n(\tau)|},$$

where $\boldsymbol{z}_i$ are n nonparallel, generic tangent vectors. It is possible to show that the sum of the first n Lyapunov exponent is

$$\sum_{i=1}^{n} \lambda_i = \lim_{t \to \infty} \frac{1}{t} \langle \ln R^{(n)}(t) \rangle.$$

In other words the sum of the first $n \leq d$ Lyapunov exponent gives the typical rate of exponential growth of a n-dimensional volume in the tangent space.

In the case of continuous flows, it is easy to see that at least one of the Lyapunov exponents has to be zero, since $\boldsymbol{z}(t)$ cannot grow exponentially in time in the direction tangent to the flow.

Due to its global nature the maximum Lyapunov exponent cannot give any further information on intermittency properties, *i.e.* on the finite-time fluctuations of the growth rate. For these properties it is necessary to consider the full probability distribution of the response R.

A direct calculation of the probability distribution of R is a rather hard task. However, it can be reconstructed, under general conditions, from the knowledge of the moments $\langle R^q \rangle$. We then introduce the *generalized* Lyapunov exponent $L(q)$ of order q as [BPPV85, PV87a]

$$L(q) = \lim_{t\to\infty} \frac{1}{t} \ln \langle R(t)^q \rangle .$$

It is easy to see that

$$\lambda_1 = \left.\frac{\mathrm{d}L(q)}{\mathrm{d}q}\right|_{q=0}$$

and in the absence of fluctuations

$$L(q) = \lambda_1 q .$$

By general inequalities of probability theory, $L(q)$ is a concave function [F71].

The $L(q)$ give an indication on the large fluctuations of $R_\tau(t)$ at finite time t. Define indeed a local exponent parameter (LEP) γ as

$$R_\tau(t) \sim \exp[\gamma(\tau) t], \qquad t \gg 1 ,$$

and classify the trajectories of length t $\boldsymbol{x}(\tau)$, $\boldsymbol{x}(\tau+1)$, ..., $\boldsymbol{x}(\tau+t)$ according to their LEP.

In the limit $t\to\infty$ the probability of finding $\gamma \neq \lambda_1$ should vanish, as a consequence of the Oseledec theorem. Therefore for large t, the probability of having a trajectory of length t with a given LEP is peaked about the most probable value λ_1. If $L(q)$ is finite for all finite q, one can assume the following *ansatz*:

$$\mathrm{d}\mathscr{P}_t(\gamma) = \mathrm{d}\mu(\gamma) \exp[-S(\gamma) t] \qquad \text{with} \qquad S(\gamma) \geq 0 ,$$

where, for the Oseledec theorem, $S(\gamma) > 0$ for $\gamma \neq \lambda_1$ and $S(\lambda_1) = 0$. The function $\mu(\gamma)$ is a smooth function of γ.

The function $S(\gamma)$ is related to the generalized Lyapunov exponent $L(q)$. In fact, the moments $\langle R^q \rangle$ can be evaluated by averaging over the γ-distribution

$$\langle R(t)^q \rangle = \int \mathrm{d}\mu(\gamma) \exp[(q\gamma - S(\gamma)) t] \sim \exp[L(q) t] ,$$

For large t the integral can be calculated by the saddle point method

$$L(q) = \max_\gamma [q\gamma - S(\gamma)] . \tag{6.18}$$

The Legendre transformation (6.18) shows that each value of q selects a par-

ticular $\hat{\gamma}$ given by

$$q = \left.\frac{\mathrm{d}S(\gamma)}{\mathrm{d}\gamma}\right|_{\hat{\gamma}}.$$

We can thus obtain $S(\gamma)$ from $L(q)$ by inverting the Legendre transformation (6.18).

The probability distribution of $R(t)$ for $t \gg 1$ is usually close to a log-normal distribution

$$\mathscr{P}[R(t)] \simeq \frac{1}{R(t)\sqrt{2\pi\mu t}} \exp[-(\ln R(t) - \lambda_1 t)^2/(2\mu t)], \tag{6.19}$$

where

$$\mu = \lim_{t\to\infty} \frac{1}{t} \langle (\ln R(t) - \lambda_1 t)^2 \rangle .$$

Indeed writing $t = \tilde{t}\tau_c$, where τ_c is the typical correlation decay time, one has

$$R_\tau(t) = \prod_{k=1}^{\tilde{t}} \tilde{R}(k) \quad \text{with} \quad \tilde{R}(k) = R_{\tau+(k-1)\tau_c}.$$

Therefore, since $\ln\tilde{R}(k)$ are practically uncorrelated variables, one can use the central limit theorem for $\ln R_m(n)$ and, after a change of variables, one obtains (6.19). Under the hypothesis that $R(t)$ is exactly a log-normal variable one has

$$L(q) = \lambda q + \frac{\mu}{2} q^2 .$$

In general, even if the log-normal is a good approximation, (6.19) is correct only for small q. This is because the moments of the log-normal distribution grow very fast with q [C22, O70].

6·4. *Generalized fractal dimensions and multifractals.* – Fractal structures appear in many physical phenomena such as turbulence, random walks, chaotic dynamical systems [M82]. The concept of dimension plays a central role in the characterization of fractals. Usually the dimension of an object is defined as the number of independent directions for a point moving on it. In this case, it is called *topological* dimension d_T and is a positive integer number. It is equal or smaller than the dimension d of the space where the object is embedded. However a smooth line and a random-walk trajectory have the same topological dimension $d_T = 1$ but very different characteristics since the latter densely fills a $2d$ space. For this reason, it is necessary to introduce the fractal dimension d_F of a geometrical object, considering the scaling of the number $N(\varepsilon)$ of hypercubes of size ε necessary to cover the object

$$N(\varepsilon) \sim \varepsilon^{-d_F} \quad \text{for} \quad \varepsilon \to 0 .$$

A more precise definition requires sophisticated mathematical methods and leads to the introduction of the Hausdorff dimension [H19], which in some cases can be different from d_F. For a smooth geometrical object as a line or a sphere, $d_T = d_F$ but, for instance, a random walk has $d_F = 2$.

However, the scaling laws appearing in nature cannot be characterized by just one geometrical parameter. One has to consider the scaling properties of an appropriate density $\mu(\boldsymbol{x})$ (in many cases a probability density) over the object. One defines the coarse-grained measure

$$p_\varepsilon(x) = \int_{B(x,\,\varepsilon)} \mu(\boldsymbol{x})\, \mathrm{d}^d\boldsymbol{x}\,,$$

where $B(\boldsymbol{x}, \varepsilon)$ is a hypercube of linear size ε centred in the point $\boldsymbol{x}$ of the object. In general $p_\varepsilon(\boldsymbol{x})$ scales with an exponent α which depends on the particular point $\boldsymbol{x}$

$$p_\varepsilon(\boldsymbol{x}) \sim \varepsilon^\alpha$$

and, if the density is not uniform, $\alpha \neq d_F$. The object can be regarded as the superposition of different fractals

$$F(\alpha) = \left\{\boldsymbol{x} \text{ such that } p_\varepsilon(\boldsymbol{x}) \sim \varepsilon^\alpha \text{ for } \varepsilon \to 0\right\},$$

each one characterized by a different exponent α. The object is called multifractal [PF85, BPPV84, PV87a]. The fluctuations of the exponents α are ruled by a probability distribution which can be studied analysing the scaling law of the moments

$$\langle\, p_\varepsilon^q \,\rangle \equiv \sum_{k=1}^{N(\varepsilon)} [\, p_\varepsilon(\boldsymbol{x}(k))\,]^{q+1} \sim \varepsilon^{q d_{q+1}}, \qquad \text{for} \qquad \varepsilon \to 0\,,$$

where $\boldsymbol{x}(k)$ is centred in the k-th hypercube and the average is a weighted sum over the hypercubes, *i.e.*

$$\langle\, (\ldots) \,\rangle \equiv \sum (\ldots)\, p_\varepsilon(\boldsymbol{x}(k))\,.$$

The d_q are called generalized dimensions [G83, HP83] and it can be shown that $d_F \equiv d_0$. In a homogeneous fractal $d_q = d_F$ for all q, and in general standard arguments of probability theory assures that d_q is a nonincreasing function of q. The exponent $d_1 \leq d_0$ is the fractal dimension of the probability measure or *information dimension*.

The number of hypercubes of size ε necessary to cover a subset $F(\alpha)$ of the object should behave in the scaling hypothesis as

$$n(\alpha) \sim \varepsilon^{-f(\alpha)}\,,$$

where $f(\alpha) \leq d_F$ is the fractal dimension of the subset $F(\alpha)$ [HJKPS86]. Since the probability measure of a hypercube with centre $\boldsymbol{x} \in F(\alpha)$ scales as ε^α, the weight $\mathscr{P}(\alpha)$ of the corresponding subset should scale as

$$\mathscr{P}(\alpha) \sim \varepsilon^{H(\alpha)} \qquad \text{with} \qquad H(\alpha) = \alpha - f(\alpha)\,.$$

The function $H(\alpha \geq 0)$ is an entropy function, similar to the entropy function

$S(\gamma)$ introduced for the finite-time fluctuations of the Lyapunov exponent in subsect. **6**·3. The sum over the hypercubes can be estimated as

$$\sum_{k=1}^{N(\varepsilon)} [\, p_\varepsilon(\boldsymbol{x}(k))]^{q+1} \sim \int \mathrm{d}\alpha\, n(\alpha)\, \varepsilon^{\alpha(q+1)}\,.$$

In the limit $\varepsilon \to 0$, the integral is dominated by the saddle point value

$$\langle\, p_\varepsilon^q \,\rangle \sim \varepsilon^{\tilde{\alpha}q + H(\tilde{\alpha})}\,,$$

where $\tilde{\alpha}$ is given by the minimum condition

$$\min_\alpha [\alpha q - H(\alpha)] \quad \text{or} \quad \left.\frac{\mathrm{d}H(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=\tilde{\alpha}} = q\,.$$

The generalized dimensions are thus related to the $f(\alpha)$ function via the Legendre transformation

$$(q-1)\, d_q = \min_\alpha [\alpha q - f(\alpha)]\,. \tag{6.20}$$

From this formula, it is evident that each order q moment selects a particular exponent α. The less probable the α, the larger the corresponding entropy function $H(\alpha)$. In particular the minimum $H(\alpha) = 0$ — corresponding to the relation $\alpha = f(\alpha)$ — is reached for $\alpha = d_1$, selected by $q = 1$ in (6.20), while the maximum of the $f(\alpha)$ curve is given by the fractal dimension

$$d_\mathrm{F} \equiv d_0 = \max_\alpha f(\alpha)\,,$$

selected by $q = 0$ in (6.20).

In fig. 33 and 34 we show the typical shapes of $f(\alpha)$ as a function of α and of its Legendre transformation d_q as a function of q. In the limit $\varepsilon \to 0$, all the exponents different from d_1 cannot be detected, since their probability vanishes being $H > 0$. In this sense, d_1 is the most probable scaling exponent. This is a well-known result in the context of the information theory. Arguments borrowed from the Shannon-McMillan theorem [K57] show that the number

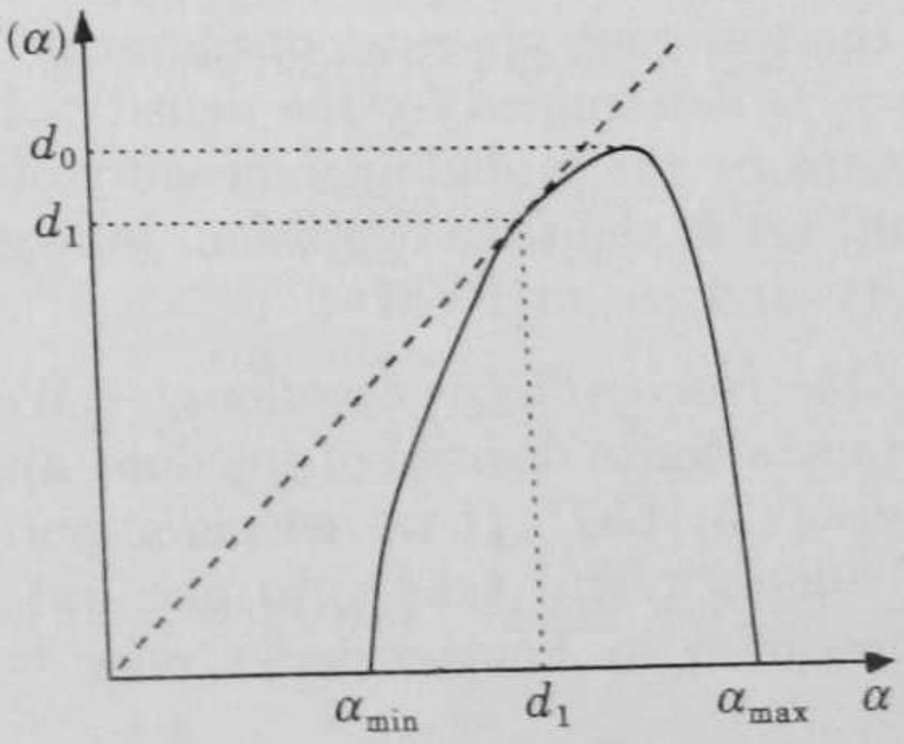

Fig. 33. – Typical shape of $f(\alpha)$ as a function of α. The dashed line has slope 1.

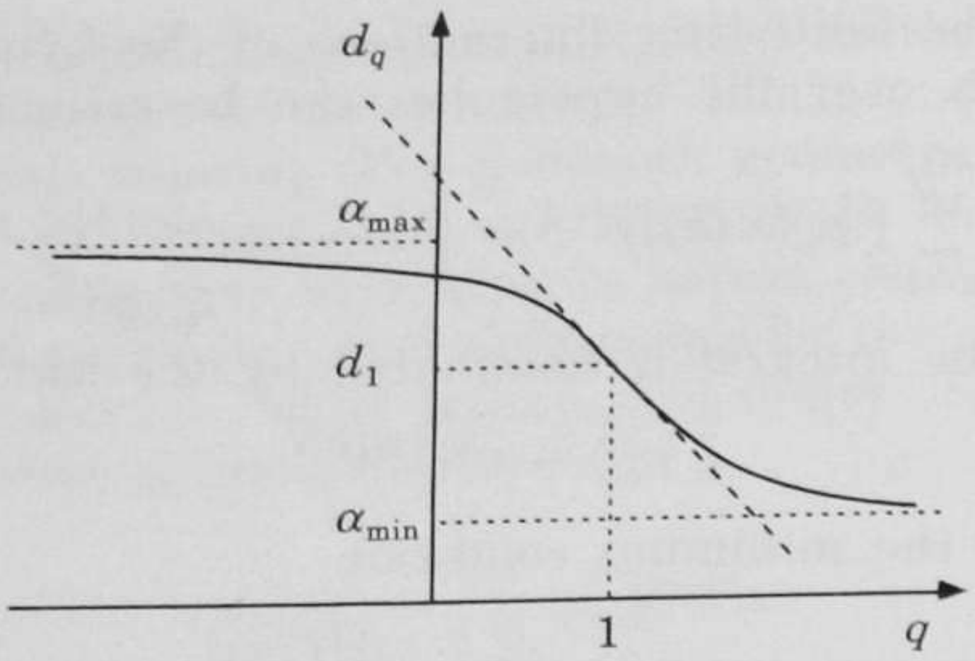

Fig. 34. – Typical shape of d_q as a function of q. The dashed line has slope $-\sigma^2/2$.

$N_{\text{eff}}(\varepsilon) \leq N(\varepsilon)$ of the hypercubes which give the leading contribution to the information

$$I(\varepsilon) = -\sum_{k=1}^{N(\varepsilon)} p_\varepsilon(\boldsymbol{x}(k)) \ln p_\varepsilon(\boldsymbol{x}(k)) \simeq -d_1 \ln \varepsilon$$

should scale as $N_{\text{eff}}(\varepsilon) \sim \varepsilon^{-d_1}$. In this sense, the information dimension is the fractal dimension of the probability measure.

The Legendre transformation becomes trivial in the limit $q \to \pm\infty$ where the minimum condition picks up the extreme values of the local exponents

$$\alpha_{\min} = \lim_{q\to\infty} d_q\,, \qquad \alpha_{\max} = \lim_{q\to-\infty} d_q\,.$$

If the α are random Gaussian variables of mean d_1 and variance σ^2 — log-normal distribution of the coarse-grained measure p_ε — the entropy function has a parabolic shape

$$H(\alpha) = \frac{(\alpha - d_1)^2}{2\sigma^2}$$

with Legendre transformation $d_{q+1} = d_1 - \sigma^2 q/2$. In typical cases, this form is a good approximation only for small q, *i.e.* around the maximum of the probability distribution (bottom of the $H(\alpha)$ curve).

We finally note that in practice appropriate probability measures p_ε are chosen according to the different physical phenomena. For instance, in fully developed turbulence p_ε is determined by the density of energy dissipation, in chaotic dynamical sysems by the probability measure obtained from frequency of visit on the invariant set of the time evolution, in aggregates of particles by the growing probability and so on [PV87a].

6·5. *Truncations of the Navier-Stokes equations.* – We briefly recall a standard procedure to obtain a finite-degrees-of-freedom approximation from the Navier-Stokes equations (2.5) [L87]. If we consider periodic boundary conditions on a square of edge 2π and take into account the incompressibility condition, the expansion of $\boldsymbol{u}$ in Fourier series may be written as

$$\boldsymbol{u}(\boldsymbol{x}) = \sum_{\boldsymbol{k}} \exp[i(\boldsymbol{k}\cdot\boldsymbol{x})]\, Q_k \frac{\boldsymbol{k}^\perp}{|\boldsymbol{k}|},$$

where $\boldsymbol{k} = (k_1,\ k_2)$, $\boldsymbol{k}^{\perp} = (k_2,\ -k_1)$ and $Q_k = Q^*_{-k}$ because $\boldsymbol{u}(\boldsymbol{x})$ is real. By expanding P and $\boldsymbol{f}$ in a similar way, one has the evolution equations for a truncation $\mathscr{L}$ of Q_k

$$\frac{\mathrm{d}Q_k}{\mathrm{d}t} = -i \sum_{k'+k''+k=0} \frac{(\boldsymbol{k}')^{\perp} \cdot (\boldsymbol{k}'')\,(|\boldsymbol{k}''|^2 - |\boldsymbol{k}'|^2)}{2|\boldsymbol{k}'||\boldsymbol{k}''||\boldsymbol{k}|} Q^*_{k'} Q^*_{k''} - \nu|\boldsymbol{k}|^2 Q_k + f_k,$$

where $\boldsymbol{k}'$, $\boldsymbol{k}''$, $\boldsymbol{k} \in \mathscr{L}$ and $\mathscr{L}$ is a set of wave vectors such that if $\boldsymbol{k} \in \mathscr{L}$, then $-\boldsymbol{k} \in \mathscr{L}$. Let us define, for instance, the modes 1, 2, ..., 9 corresponding to the following set:

$$\begin{aligned}
&\boldsymbol{k}_1 = (1,\ 1), && \boldsymbol{k}_2 = (3,\ 0), && \boldsymbol{k}_3 = (2,\ -1),\\
&\boldsymbol{k}_4 = (1,\ 2), && \boldsymbol{k}_5 = (0,\ 1), && \boldsymbol{k}_6 = (1,\ 0),\\
&\boldsymbol{k}_7 = (1,\ -2), && \boldsymbol{k}_8 = (3,\ 1), && \boldsymbol{k}_9 = (2,\ 2),
\end{aligned}$$

and pose

$$\begin{aligned}
&Q_{k_1} = Q_1, && Q_{k_2} = -iQ_2, && Q_{k_3} = Q_3,\\
&Q_{k_4} = iQ_4, && Q_{k_5} = Q_5, && Q_{k_6} = iQ_6,\\
&Q_{k_7} = iQ_7, && Q_{k_8} = Q_8, && Q_{k_9} = iQ_9.
\end{aligned}$$

After rescaling by a factor $\sqrt{10}$, letting $\nu = 1$ (that is equivalent to a change of time and length units) and assuming a forcing along the mode $\boldsymbol{k}_3$, the equations for the amplitude $Q_j\ (j = 1,\ \ldots,\ 9)$ become

$$\left\{\begin{aligned}
&\frac{\mathrm{d}Q_1}{\mathrm{d}t} = -2Q_1 + 4Q_2Q_3 + 4Q_4Q_5,\\
&\frac{\mathrm{d}Q_2}{\mathrm{d}t} = -9Q_2 + 3Q_1Q_3 + 9Q_5Q_8 + 3Q_7Q_9,\\
&\frac{\mathrm{d}Q_3}{\mathrm{d}t} = -5Q_3 - 7Q_1Q_2 + \frac{9}{\sqrt{5}}Q_1Q_7 - 5Q_4Q_8 + \mathrm{Re},\\
&\frac{\mathrm{d}Q_4}{\mathrm{d}t} = -5Q_4 - Q_1Q_5 + 5Q_3Q_8 + 7Q_6Q_9,\\
&\frac{\mathrm{d}Q_5}{\mathrm{d}t} = -Q_5 - 3Q_1Q_4 + \sqrt{5}\,Q_1Q_6 - Q_2Q_8,\\
&\frac{\mathrm{d}Q_6}{\mathrm{d}t} = -Q_6 - \sqrt{5}\,Q_1Q_5 - 3Q_4Q_9,\\
&\frac{\mathrm{d}Q_7}{\mathrm{d}t} = -5Q_7 - \frac{9}{\sqrt{5}}Q_1Q_3 + Q_2Q_9,\\
&\frac{\mathrm{d}Q_8}{\mathrm{d}t} = -10Q_8 - 8Q_2Q_5,\\
&\frac{\mathrm{d}Q_9}{\mathrm{d}t} = -8Q_9 - 4Q_2Q_7 - 4Q_4Q_6.
\end{aligned}\right. \tag{6.21}$$

Here $\mathrm{Re} \propto f_3$ is the Reynolds number, the only control parameter for (6.21). The studies done on (6.21), considering 5, 6, 7, 8 and 9 modes [BF79, L87], have reported the following behaviours for increasing Reynolds number:

a) stable fixed points,

b) Hopf bifurcation to periodic cyclic orbits,

c) periodic/aperiodic/chaotic orbits.

For large Re (case *c*)) the behaviours are strongly dependent on the truncation. For example, the 5 modes model becomes regular, while the 7 modes one remains chaotic at very large Reynolds number.

* * *

We are grateful to M. Feingold, U. Frisch, P. Ghilardi, F. M. Izrailev, A. R. Osborne, C. Presilla, A. Provenzale and G. M. Zaslavsky for useful discussions and suggestions.

Two of us (MF and AV) acknowledge the financial support of INFN (Iniziativa Specifica «Meccanica Statistica»).

REFERENCES

[ATI89] S. ADACHI, M. TODA and K. IKEDA: *J. Phys. A*, **22**, 3291 (1989).
[A72] J. K. AGGARWAL: *Notes on Nonlinear Systems* (Van Nostrand, 1972).
[A84] H. AREF: *J. Fluid Mech.*, **143**, 1 (1984).
[AB86] H. AREF and S. BALACHANDAR: *Phys. Fluids*, **29**, 3515 (1986).
[A63] V. I. ARNOLD: *Russ. Math. Survey*, **18**, 9 (1963).
[A65] V. I. ARNOLD: *C. R. Acad. Sci. Paris A*, **261**, 17 (1965).
[A76] V. I. ARNOLD: *Methodes Mathematiques de la Mécanique Classique* (Mir, Moscow, 1976).
[AA68] V. I. ARNOLD and A. AVEZ: *Ergodic Problems of Classical Mechanics* (Benjamin, New York, 1968).
[A83] W. ARTER: *Phys. Lett. A*, **97**, 171 (1983).
[BBLS87] A. BABIANO, C. BASDEVANT, B. LEGRAS and R. SADOURNY: *J. Fluid Mech.*, **183**, 379 (1987).
[BBS84] A. BABIANO, C. BASDEVANT and R. SADOURNY: *C. R. Acad. Sci. Paris*, **299**, 495 (1984).
[B52] G. K. BATCHELOR: *Proc. R. Soc. A*, **213**, 349 (1952).
[B59] G. K. BATCHELOR: *J. Fluid Mech.*, **5**, 113 (1959).
[B86] B. J. BAYLY: *Phys. Rev. Lett.*, **57**, 2800 (1986).
[BC87] B. J. BAYLY and S. CHILDRESS: *Phys. Rev. Lett.*, **59**, 1573 (1987).
[BG86] G. BENETTIN and G. GALLAVOTTI: *J. Stat. Phys.*, **44**, 293 (1986).
[BGGS80] G. BENETTIN, A. GIORGILLI, L. GALGANI and J. M. STRELCYN: *Meccanica*, **15**, 9 and 21 (1980).
[BS78] G. BENETTIN and J. M. STRELCYN: *Phys. Rev. A*, **17**, 2157 (1978).
[BPPV84] R. BENZI, G. PALADIN, G. PARISI and A. VULPIANI: *J. Phys. A*, **17**, 3521 (1984).
[BPPV85] R. BENZI, G. PALADIN, G. PARISI and A. VULPIANI: *J. Phys. A*, **18**, 2157 (1985).

[BR70] H. L. BERK and K. V. ROBERTS: in *Methods in Computational Physics*, Vol. 9, edited by B. ADLER, S. FERNBACH and M. ROTENBERG (New York, Academic Press, 1970), p. 87.

[B83] M. BERRY: in *Chaotic Behaviour of Deterministic Systems*, edited by G. JOOSS, R. H. G. HELLEMAN and R. STORA (North-Holland Publ. Comp., 1983).

[BPTV88] D. BESSIS, G. PALADIN, G. TURCHETTI and S. VAIENTI: *J. Stat. Phys.*, **51**, 109 (1988).

[B27] G. D. BIRKHOFF: *Dynamical Systems* (AMS Publications, 1927).

[BF79] C. BOLDRIGHINI and V. FRANCESCHINI: *Commun. Math. Phys.*, **64**, 159 (1979).

[C22] T. CARLEMAN: *C. R. Acad. Sci. Paris*, **1174**, 1680 (1922).

[CCTT87] J. CHAIKEN, C. K. CHU, M. TABOR and Q. M. TAN: *Phys. Fluids*, **30**, 687 (1987).

[C43] S. CHANDRASEKHAR: *Rev. Mod. Phys.*, **15**, 1 (1943).

[C70] S. CHILDRESS: *J. Math. Phys.*, **11**, 3063 (1970).

[CS89] S. CHILDRESS and A. M. SOWARD: *J. Fluid Mech.*, **205**, 99 (1989).

[C79] B. V. CHIRIKOV: *Phys. Rep.*, **52**, 263 (1979).

[C69] W. J. COCKE: *Phys. Fluids*, **12**, 2488 (1969).

[C71] W. J. COCKE: *Phys. Fluids*, **14**, 1624 (1971).

[C89] P. COLLET: *Workshop on Small Diffusivity Dynamos and Dynamical Systems* (Nice, 1989).

[C51] S. CORRSIN: *J. Appl. Phys.*, **22**, 469 (1951).

[C62] S. CORRSIN: in *Mécanique de la Turbulence*, edited by A. FAVRE (CNRS, Paris, 1962), p. 27.

[CFPV90a] A. CRISANTI, M. FALCIONI, G. PALADIN and A. VULPIANI: *J. Phys. A: Math. Gen.*, **23**, 3307 (1990).

[CFPV90b] A. CRISANTI, M. FALCIONI, G. PALADIN and A. VULPIANI: *Physica A*, **166**, 305 (1990).

[CFPTV91] A. CRISANTI, M. FALCIONI, A. PROVENZALE, P. TANGA and A. VULPIANI: *to appear* (1991).

[CFPrV90] A. CRISANTI, M. FALCIONI, A. PROVENZALE and A. VULPIANI: *Phys. Lett. A*, **150**, 79 (1990).

[DFGHMS86] T. DOMBRE, U. FRISCH, J. M. GREENE, M. HÉNON, A. MEHR and A. M. SOWARD: *J. Fluid Mech.*, **167**, 353 (1986).

[DM90] I. T. DRUMMOND and W. MÜNCH: *J. Fluid Mech.*, **215**, 45 (1990).

[DM91] I. T. DRUMMOND and W. MÜNCH: *J. Fluid Mech.*, **225**, 529 (1991).

[ER85] J.-P. ECKMANN and D. RUELLE: *Rev. Mod. Phys.*, **57**, 617 (1985).

[FPV88] M. FALCIONI, G. PALADIN and A. VULPIANI: *J. Phys. A: Math. Gen.*, **21**, 3451 (1988).

[FPV89] M. FALCIONI, G. PALADIN and A. VULPIANI: *Europhys. Lett.*, **10**, 201 (1989).

[FV89] M. FALCIONI and A. VULPIANI: *J. Phys. A: Math. Gen.*, **22**, 1201 (1989).

[FKP88] M. FEINGOLD, L. P. KADANOFF and O. PIRO: *J. Stat. Phys.*, **50**, 529 (1988).

[F71] W. FELLER: *An Introduction to Probability Theory and its Applications* (New York, John Wiley and Sons, 1971).

[FO88] J. M. FINN and E. OTT: *Phys. Rev. Lett.*, **60**, 760 (1988).

[FO90] J. M. FINN and E. OTT: *Phys. Fluids B*, **2**, 916 (1990).

[GF84] D. J. GALLOWAY and U. FRISCH: *Geophys. Astrophys. Fluid Dyn.*, **29**, 13 (1984).

[GF87] D. J. GALLOWAY and U. FRISCH: *J. Fluid Mech.*, **180**, 557 (1987).

[G83] P. GRASSBERGER: *Phys. Lett. A*, **97**, 227 (1983).

[GP83] P. GRASSBERGER and I. PROCACCIA: *Phys. Rev. A*, **28**, 2591 (1983).
[GH83] J. GUCKENHEIMER and P. HOLMES: *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (Springer Verlag, 1983).
[G90] M. C. GUTZWILLER: *Chaos in Classical and Quantum Mechanics* (Springer Verlag, 1990).
[HJKPS86] T. HALSEY, M. H. JENSEN, L. P. KADANOFF, I. PROCACCIA and B. I. SHRAIMAN: *Phys. Rev. A*, **33**, 1141 (1986).
[H19] F. HAUSDORFF: *Math. Ann.*, **79**, 157 (1919).
[H66] M. HÉNON: *C. R. Acad. Sci. Paris A*, **262**, 312 (1966).
[HH64] M. HÉNON and C. HEILES: *Astron. J.*, **69**, 73 (1964).
[HP83] H. G. E. HENTSCHEL and I. PROCACCIA: *Phisica D*, **8**, 435 (1983).
[H59] J. O. HINZE: *Turbulence* (New York, McGraw-Hill, 1959).
[H90] P. HOLMES: *Phys. Rep.*, **193**, 137 (1990).
[HHIM90] T. HORITA, H. HATA, R. ISHIZAKI and H. MORI: *Prog. Theor. Phys.*, **83**, 1065 (1990).
[IKTTY89] M. B. ISICHENKO, YA. L. KALDA, E. B. TATARINOVA, O. V. TEL'KOVSKAYA and V. V. YAN'KOV: *Soc. Phys. JETP*, **69**, 517 (1989).
[I90] F. M. IZRAILEV: *Phys. Rep.*, **196**, 299 (1990).
[KST90] P. A. KALUGIN, A. V. SOKOL and E. B. TATARINOVA: *Europhys. Lett.*, **13**, 417 (1990).
[K57] A. I. KHINCHIN: *Mathematical Foundation of Information Theory* (Dover Publ., New York, 1957).
[KD84] R. G. KLEVA and J. F. DRAKE: *Phys. Fluids*, **27**, 1686 (1984).
[K54] A. N. KOLMOGOROV: *Dokl. Akad. Nauk SSSR*, **98**, 527 (1954).
[K74] R. A. KRAICHNAN: *J. Fluid Mech.*, **64**, 737 (1974).
[K84] J. A. KROMMES: in *Handbook of Plasma Physics*, Vol. 2, edited by M. N. ROSENBLUTH and R. Z. SAGDEEV (Elsevier Science Publishers, 1984).
[KRS83] L. M. KUZNETSOVA, M. I. RABINOVICH and M. M. SUSHCHIK: *Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana*, **19**, 36 (1983).
[L45] H. LAMB: *Hydrodynamics* (New York, Dover Publ., 1945).
[LL87] L. D. LANDAU and L. LIFSHITZ: *Fluid Mechanics* (New York, Pergamon Press, 1987).
[L87] J. LEE: *Physica D*, **24**, 54 (1987).
[LH85] M. LESIEUR and J. HERRING: *J. Fluid Mech.*, **161**, 77 (1985).
[L73] D. C. LESLIE: *Developments in the Theory of Turbulence* (Oxford, Clarendon Press, 1973).
[LL83] A. J. LICHTENBERG and M. A. LIEBERMANN: *Regular and Stochastic Motions* (Berlin, Springer Verlag, 1983).
[L63] E. N. LORENZ: *J. Atm. Sci.*, **20**, 130 (1963).
[L62] J. L. LUMLEY: in *Mécanique de la Turbulence*, edited by A. FAVRE (CNRS, Paris, 1962), p. 17.
[LS89] R. LUPINI and S. SIBONI: *Nuovo Cimento B*, **103**, 237 (1989).
[MPV86] A. MALAGOLI, G. PALADIN and A. VULPIANI: *Phys. Rev. A*, **34**, 1550 (1986).
[M75] B. B. MANDELBROT: *J. Fluid Mech.*, **72**, 401 (1975).
[M82] B. B. MANDELBROT: *The Fractal Geometry of Nature* (New York, Freeman, 1982).
[MM75] J. E. MARSDEN and M. MCCRACKEN: *The Hopf Bifurcation and its Applications* (Cambridge, Mass., MIT Press, 1975).
[M90] J. C. MAXWELL: *The Scientific Papers*, Vol. 2 (Cambridge UK, Cambridge University Press, 1890).
[MO87] R. MCWILLIAMS and M. OKUBO: *Phys. Fluids*, **30**, 2840 (1987).
[M63] V. K. MELNIKOV: *Trans. Moscow Math. Soc.*, **12**, 1 (1963).

[M83] H. K. MOFFAT: *Rep. Prog. Phys.*, **46**, 621 (1983).
[MRS84] S. A. MOLCHANOV, A. A. RUZMAIKIN and D. D. SOKOLOFF: *Geophys. Astrophys. Fluid Dyn.*, **30**, 241 (1984).
[MY75] A. S. MONIN and A. M. YAGLOM: *Statistical Fluid Mechanics*, Vol. 2 (Cambridge, Mass., MIT Press, 1975).
[MS84] E. W. MONTROLL and M. F. SCHLESINGER: in *Studies in Statistical Mechanics*, Vol. 11, edited by A. W. MONTROLL and J. LEBOWITZ (Amsterdam, North-Holland, 1984).
[M62] J. MOSER: *Nachr. Akad. Wiss. Göttingen Math-Phys KC 2*, **1**, 15 (1962).
[N77] N. N. NEKHOROSHEV: *Russ. Math. Survey*, **32**, 1 (1977).
[O49] A. M. OBUKHOV: *Izv. Akad. Nauk SSSR, Ser. Geogr. Geofiz.*, **13**, 58 (1949).
[O70] S. A. ORSZAG: *Phys. Fluids*, **13**, 2211 (1970).
[OC91] A. R. OSBORNE and R. CAPONIO: *Phys. Rev. Lett.*, **64**, 1733 (1991).
[OKPB86] A. R. OSBORNE, A. D. KIRWAN, A. PROVENZALE and L. BERGAMASCO: *Physica D*, **23**, 75 (1986).
[O68] V. I. OSELEDEC: *Trans. Moscow Math. Soc.*, **19**, 197 (1968).
[OA88] E. OTT and T. M. jr. ANTONSEN: *Phys. Rev. Lett.*, **61**, 2839 (1988).
[OA89] E. OTT and T. M. jr. ANTONSEN: *Phys. Rev. A*, **39**, 3660 (1989).
[OA90] E. OTT and T. M. jr. ANTONSEN: *Phys. Rev. Lett.*, **64**, 699 (1990).
[OP91] M. OTTAVIANI and M. PETTINI: *Int. J. Mod. Phys. B*, **5**, 1243 (1991).
[O89] J. M. OTTINO: *The Kinematic of Mixing: Stretching, Chaos and Transport* (Cambridge Univ. Press, 1989).
[O90] J. M. OTTINO: *Ann. Rev. Fluid Mech.*, **22**, 207 (1990).
[OLRS88] J. M. OTTINO, C. W. LEONG, H. RISING and P. D. SWANSON: *Nature*, **333**, 419 (1988).
[PV87a] G. PALADIN and A. VULPIANI: *Phys. Rep.*, **156**, 147 (1987).
[PV87b] G. PALADIN and A. VULPIANI: *Phys. Rev. A*, **35**, 1971 (1987).
[PF85] G. PARISI and U. FRISCH: in *Turbulence and Predictability of Geophysical Flows and Climatic Dynamics*, edited by N. GHIL, R. BENZI and G. PARISI (North-Holland, 1985), p. 84.
[P88] R. A. PASMANTER: *Fluid Dyn. Res.*, **3**, 320 (1988).
[PV84] M. PETTINI and A. VULPIANI: *Phys. Lett. A*, **106**, 207 (1984).
[PVMDOB88] M. PETTINI, A. VULPIANI, J. H. MISGUICH, M. DE LEENER, J. ORBAN and R. BALESCU: *Phys. Rev. A*, **38**, 344 (1988).
[P99] H. POINCARÉ: *Les Methodes Nouvelles de la Mécanique Celeste*, Vol. 1-3 (Gautier Villars, Paris, 1899).
[P85] Y. POMEAU: *C. R. Acad. Sci. Paris*, **301**, 1323 (1985).
[P84] I. PROCACCIA: *J. Stat. Phys.*, **36**, 649 (1984).
[R22] L. F. RICHARDSON: *Weather Prediction by Numerical Process* (Cambridge Univ. Press, 1922).
[RS78] H. A. ROSE and P. L. SULEM: *J. Phys. (Paris)*, **39**, 441 (1978).
[RBDH87] M. N. ROSENBLUTH, A. L. BERK, I. DOXAS and W. HORTON: *Phys. Fluids*, **30**, 2636 (1987).
[S87] B. I. SHRAIMAN: *Phys. Rev. A*, **36**, 261 (1987).
[SG88a] T. H. SOLOMON and J. P. GOLLUB: *Phys. Rev. A*, **38**, 6280 (1988).
[SG88b] T. H. SOLOMON and J. P. GOLLUB: *Phys. Fluids*, **31**, 1372 (1988).
[S83] Y. S. SUN: *Celest. Mech.*, **30**, 7 (1983).
[S84] Y. S. SUN: *Celest. Mech.*, **33**, 111 (1984).
[T89] T. TAJIMA: *Computational Plasma Physics* (Addison-Wesley Publ. Comp. Inc., 1989).
[T53] G. I. TAYLOR: *Proc. R. Soc. A*, **219**, 186 (1953).
[T54] G. I. TAYLOR: *Proc. R. Soc. A*, **225**, 473 (1954).

[TFR84] TFR GROUP: *Plasma Phys. Contr. Fusion*, **26**, 1045 (1984).
[T88] D. J. TRITTON: *Physical Fluid Dynamics* (Oxford Science Publications, 1988).
[V89] A. VULPIANI: *Physica D*, **38**, 372 (1989).
[WBS90] L. P. WANG, T. D. BURTON and D. E. STOCK: *Phys. Fluids A*, **2**, 1305 (1990).
[WBS91] L. P. WANG, T. D. BURTON and D. E. STOCK: *Phys. Fluids A*, **3**, 1073 (1991).
[W85] N. O. WEISS: in *Solar System Magnetic Fields*, edited by E. R. PRIEST (D. Reidel P. C., 1985), p. 156.
[WK89] J. B. WEISS and E. KNOBLOCK: *Phys. Rev. A*, **40**, 2579 (1989).
[W88] S. WIGGINS: *Global Bifurcations and Chaos: Analytical Methods* (Springer Verlag, 1988).
[YGO90] L. YU, C. GREBOGI and E. OTT: in *Nonlinear Structures in Physical Systems*, edited by L. LAM and H. C. MORRIS (Springer Verlag, 1990), p. 223.
[Z81] G. M. ZASLAVSKY: *Phys. Rep.*, **80**, 157 (1981).
[Z57] YA. B. ZEL'DOVICH: *Sov. Phys. JETP*, **4**, 460 (1957).
[ZMRS84] YA. B. ZEL'DOVICH, S. A. MOLCHANOV, A. A. RUZMAIKIN and D. D. SOKOLOFF: *J. Fluid Mech.*, **144**, 1 (1984).
[ZR86] YA. B. ZEL'DOVICH and A. A. RUZMAIKIN: in *Nonlinear Phenomena in Plasma Physics and Hydrodynamics*, edited by R. Z. SAGDEEV (MOSCOW, M.I.R. Publ., 1986), p. 119.

Direttore responsabile: RENATO ANGELO RICCI

Questo periodico è iscritto all'Unione Stampa Periodica Italiana

Questo fascicolo è stato realizzato in fotocomposizione dalla tipografia MG, Città di Castello (PG) e stampato dalla tipografia Compositori, Bologna nel mese di marzo 1992

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