Effective thermodynamics for a system with dry friction

Giacomo Gradenigo
Laboratoire Interdisciplinaire de Physique (LIPhy)

In collaboration with:

E. Bertin, E. Ferrero, J.-L. Barrat

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STATISTICAL MECHANICS OF FRICTIONAL ATHERMAL SYSTEMS ?

Grains with friction → Dissipation → ~~$e^{-\beta\mathcal{H}}$~~

Edwards
All packings where grains occupy the same volume are equiprobable

Frictional grains: change of configurations due to "extensive operations"

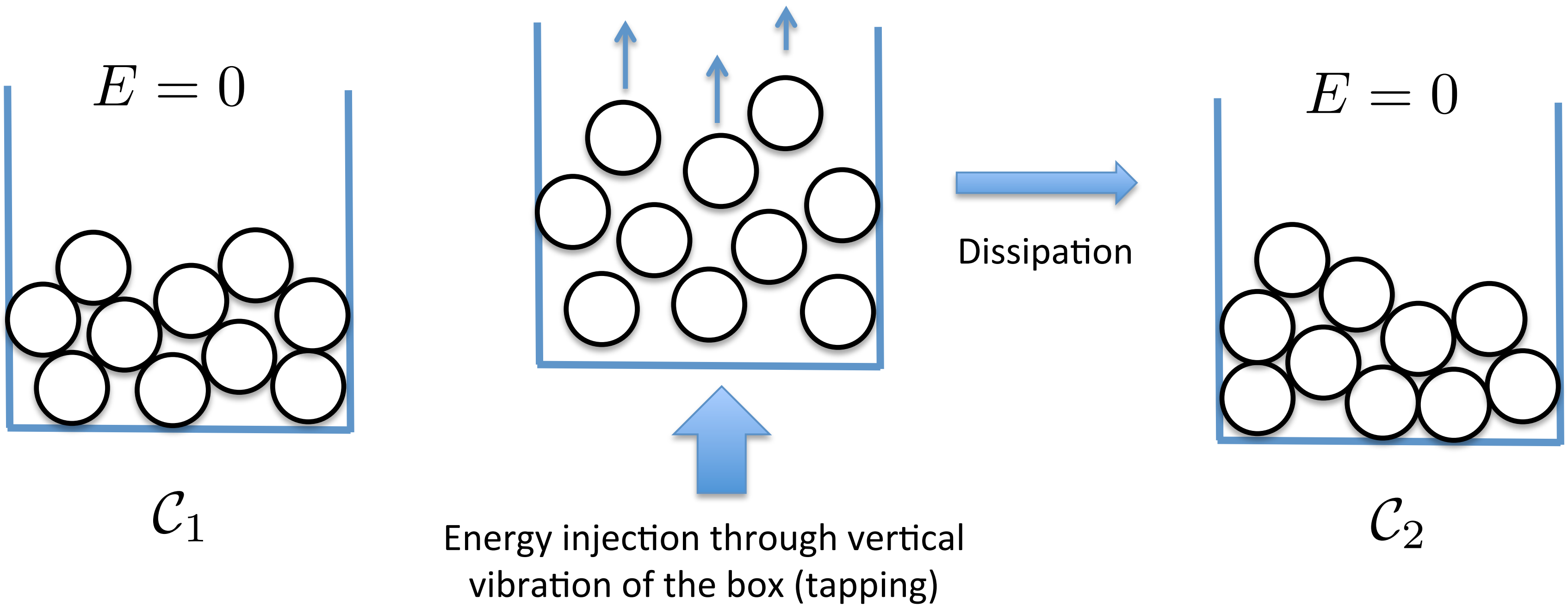
$$S \sim \log \int d\mathbf{q}\; \delta(V - \mathcal{W}(\mathbf{q}))$$

$$\frac{1}{T} = \frac{\partial S}{\partial E} \implies \frac{1}{X} = \frac{\partial S}{\partial V}$$

Compactivity

S.F Edwards, C.C. Mounfield, Physica A (210), 1994

TEST OF EDWARDS ASSUMPTION



DYNAMICAL SAMPLING

Average over states collected via tapping

$$\langle \mathcal{O} \rangle_{\mathrm{emp},V} = \mathcal{N}^{-1} \sum_i \mathcal{O}(\mathcal{C}_i)\ \delta(V - W(\mathcal{C}_i))$$

THERMODYNAMIC SAMPLING

Thermodynamic average with Edwards measure

$$\langle \mathcal{O} \rangle_X = \mathcal{Z}^{-1} \sum_{\mathcal{C}} \mathcal{O}(\mathcal{C})\ e^{-W(\mathcal{C})/X}$$

AMORPHOUS PACKINGS & GLASSES

Number of blocked structures in frictional granular assemblies at given Volume

$$\log[\mathcal{N}_{\mathrm{blocked}}(V)] \sim N$$

Number of energy minima in models of glasses at given Energy

$$\log[\mathcal{N}_{\mathrm{minima}}(E)] \sim N$$

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Edwards' Measures for Powders and Glasses

Alain Barrat,[1] Jorge Kurchan,[2] Vittorio Loreto,[3] and Mauro Sellitto[4]

A statistical mechanics approach to the inherent states of granular media

Antonio Coniglio[a,*], Mario Nicodemi[a,b]

VOLUME 90, NUMBER 19 PHYSICAL REVIEW LETTERS week ending 16 MAY 2003

Possible Test of the Thermodynamic Approach to Granular Media

David S. Dean and Alexandre Lefèvre

TEST OF EDWARDS IN ISING MODEL

A. Lefèvre & D. Dean, *J. Phys. A: Math. Gen.* **34** (2001)

TAPPING DYNAMICS

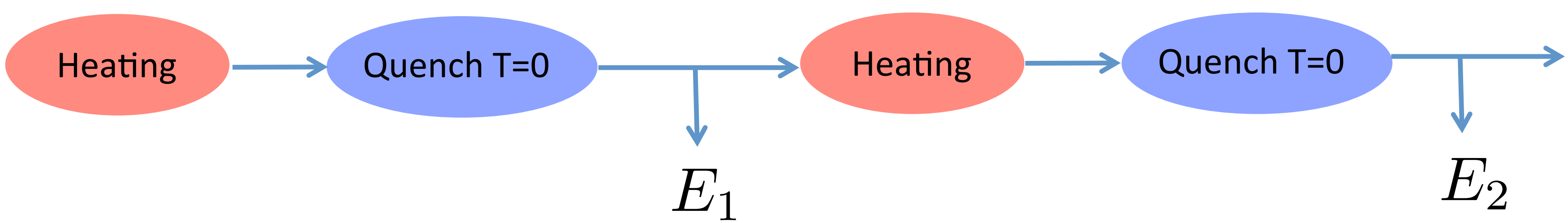
1) Heating: all spins are flipped with probability p

$p \in [0, 1/2[$

2) Quench at T=0: only spin flips which lower the energy are allowed

''BLOCKED CONFIGURATIONS''

Energy cannot be lowered with a single spin flip



Average over states colled via tapping dynamics

$$\langle \mathcal{O} \rangle_{\mathrm{emp},E} = \mathcal{N}^{-1} \sum_i \mathcal{O}(\mathcal{C}_i)\ \delta(E - E_i))$$

Edwards measure

$$\langle \mathcal{O} \rangle_E = \mathcal{Z}^{-1} \sum_{\{\boldsymbol{\sigma} | \boldsymbol{\sigma} \in \mathrm{blocked}\}} \mathcal{O}(\boldsymbol{\sigma})\ e^{-\beta_{Ed} E(\boldsymbol{\sigma})}$$

$$\beta_{Ed} = \left[\frac{\partial S}{\partial E} \right]_{\mathrm{blocked}}$$

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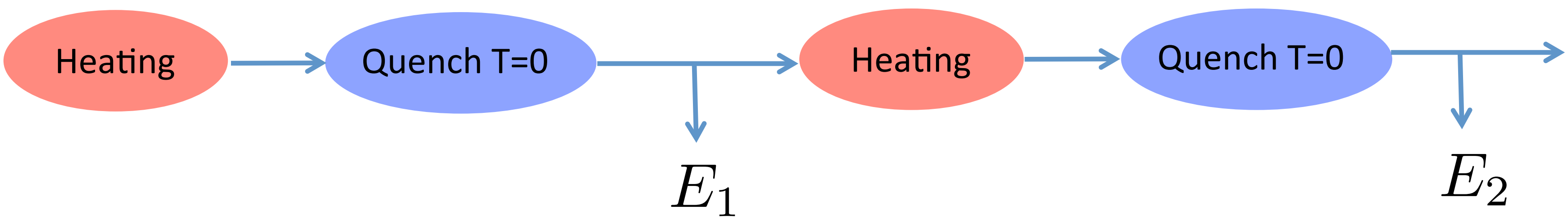

$$\mathcal{Z} = \sum_{\boldsymbol{\sigma}} e^{\beta_{Ed} \sum_i \sigma_i \sigma_{i+1}} \prod_i \Theta(\sigma_{i-1}\sigma_i + \sigma_i \sigma_{i+1})$$

$$x \geq 0 \;\rightarrow\; \Theta(x) = 1$$
$$x < 0 \;\rightarrow\; \Theta(x) = 0$$

Edwards measure

$$\langle \mathcal{O} \rangle_E = \mathcal{Z}^{-1} \sum_{\{\boldsymbol{\sigma} | \boldsymbol{\sigma} \in \text{blocked}\}} \mathcal{O}(\boldsymbol{\sigma}) \; e^{-\beta_{Ed} E(\boldsymbol{\sigma})}$$

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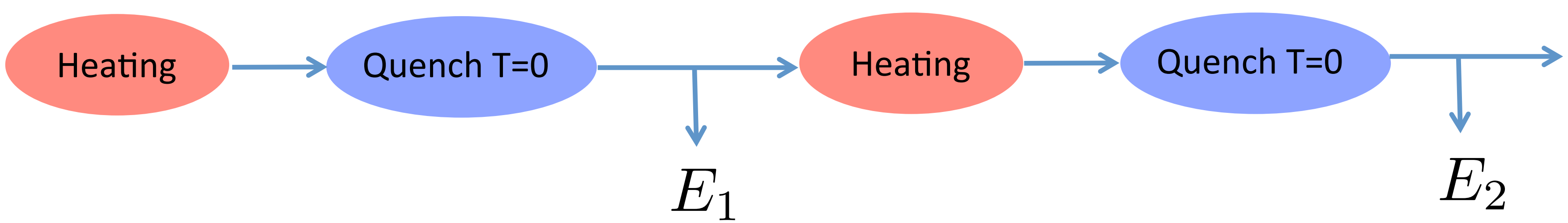


J. Berg, S. Franz, M. Sellitto, *EPJ B* (2002)

- Same test on a different 1D spin model (Friedrickson-Andersen)

- Disagreement between dynamical averages and Edwards effective theory

HARMONIC CHAIN WITH DRY FRICTION

True Dynamics

- DEFINITION OF THE MODEL
- TAPPING DYNAMICS
- BLOCKED CONFIGURATIONS
- RELEVANT OBSERVABLES

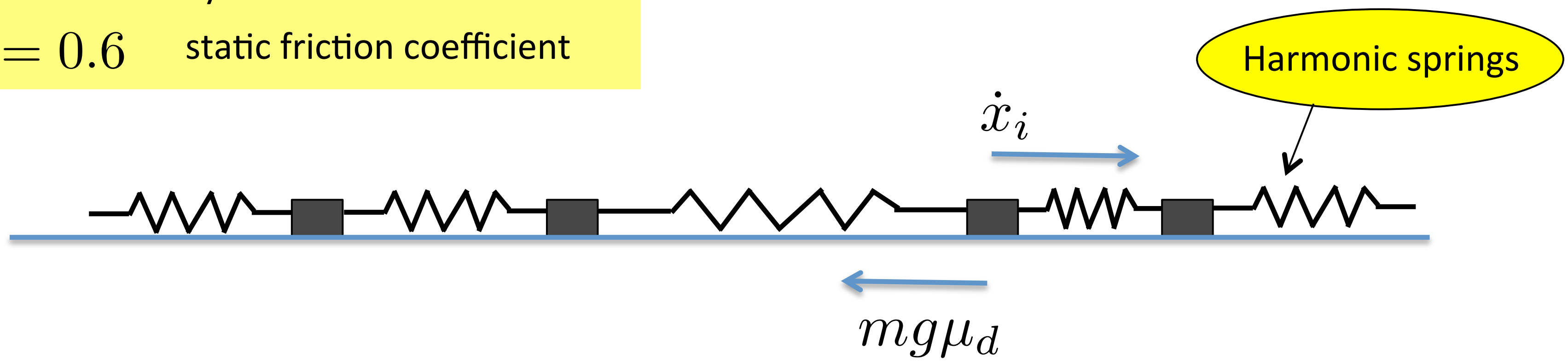
Effective Thermodynamics

- DEFINITION OF THE EFFECTIVE THEORY
- PREDICTIONS EFFECTIVE THEORY
- COMPARISON BETWEEN EFFECTIVE THEORY AND DRIVEN ATHERMAL DYNAMICS

HARMONIC CHAIN WITH DRY (Coulomb) FRICTION

$\mu_d = 0.5$ dynamic friction coefficient

$\mu = 0.6$ static friction coefficient



-Equations of motion (dynamic friction)

$$m\,\ddot{x}_i = \underbrace{-mg\mu_d\ \mathrm{sgn}(\dot{x}_i)}_{\text{Dynamic friction: energy dissipation}} + (x_{i+1} + x_{i-1} - 2x_i) + \underbrace{F_i(t)}_{\text{External Force: energy gain}}$$

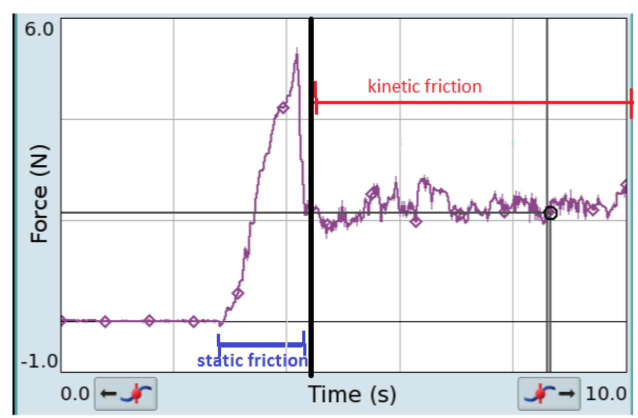
Dynamic friction: **energy dissipation**

External Force: **energy gain**

-Condition to start moving (static friction)

$$\dot{x}_i = 0$$

$$|(x_{i+1} + x_{i-1} - 2x_i + F(t))| > \mu\, mg$$



TAPPING DYNAMICS

1) External force **switched on** for a fixed duration τ: energy injection

$$m\,\ddot{x}_i = -mg\mu_d\,\mathrm{sgn}(\dot{x}_i) + (x_{i+1} + x_{i-1} - 2x_i) + F\,n_i$$

$$p(n_i) = (1-\rho)\,\delta(n_i) + \rho\,\delta(1-n_i)$$

Annealed disorder: for each ''tap'' the particles pulled are different

2) External force **switched off**: relaxation to mechanically stable (blocked) configuration, all particles are at rest

$$m\,\ddot{x}_i = -mg\mu_d\,\mathrm{sgn}(\dot{x}_i) + (x_{i+1} + x_{i-1} - 2x_i)$$

Dynamics is arrested

$$\dot{x}_i = 0 \qquad \forall i$$

$$|x_{i+1} + x_{i-1} - 2x_i| < \mu m g$$

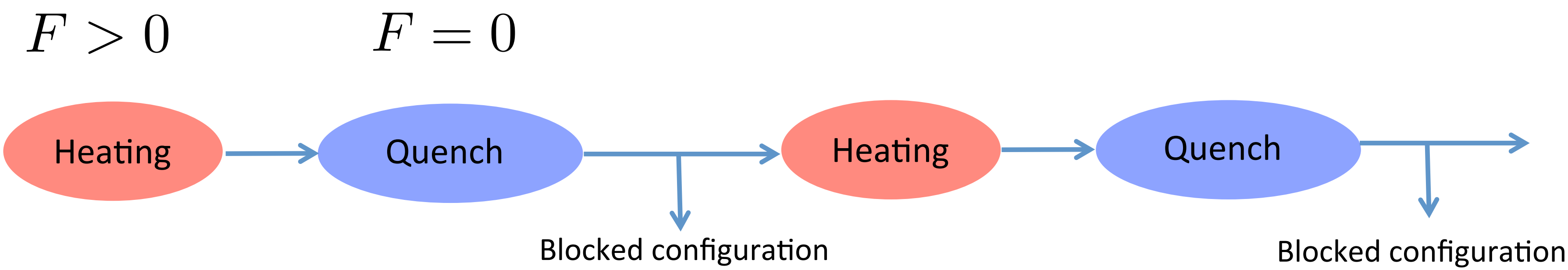
$$e = \frac{1}{N}\sum_{i=1}^{N}\frac{\xi_i^2}{2}$$

Energy of the mechanically stable configurations

$$\xi_i = x_i - x_{i-1} - \ell_0$$

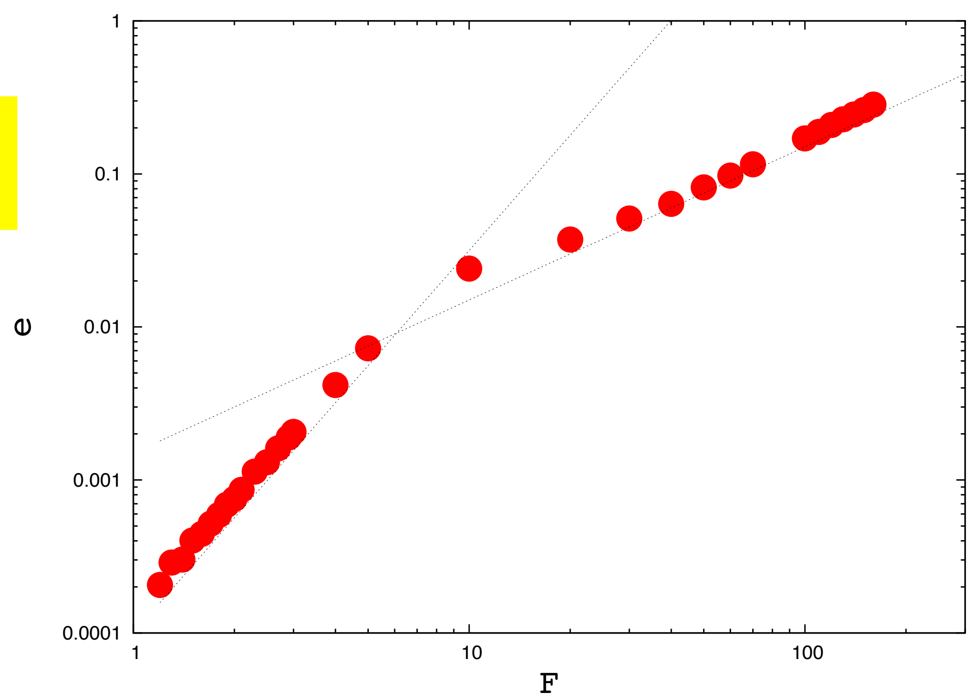
Spring elongation

HARMONIC CHAIN WITH DRY (Coulomb) FRICTION



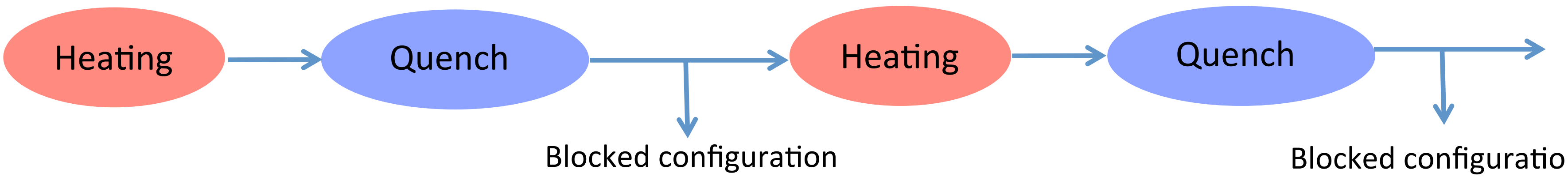
After few cycles the energy of blocked configurations fluctuates around a stationary value

Energy of mechanically stable (blocked) configurations

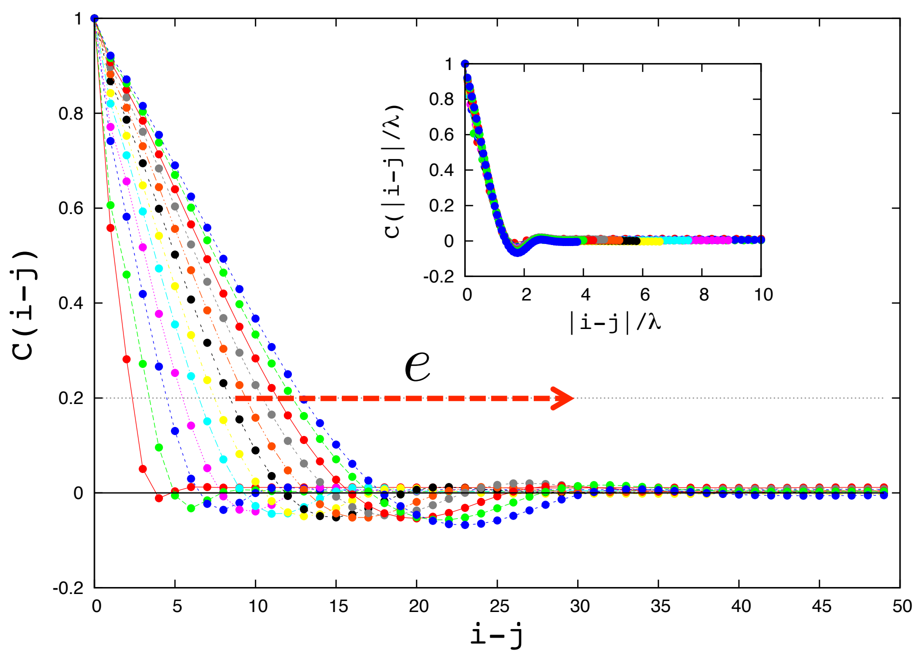


SPRING-SPRING CORRELATION
(IN MECHANICALLY STABLE CONFIGURATIONS)

$F > 0$ $F = 0$



$\langle \xi_m \xi_n \rangle_{\mathrm{emp},F}$ Spring-spring correlation



$$\langle \xi_m \xi_n \rangle \sim C(|n-m|/\ell(e))$$

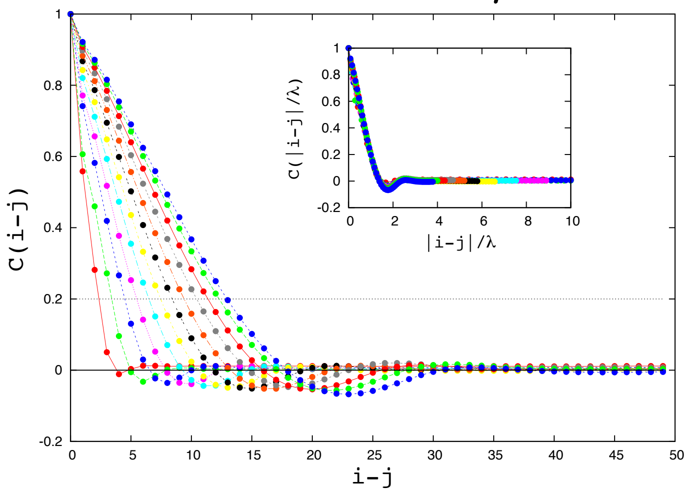
$$\ell(e) \sim e$$

Extent of correlation between springs grows as the energy stored by the springs

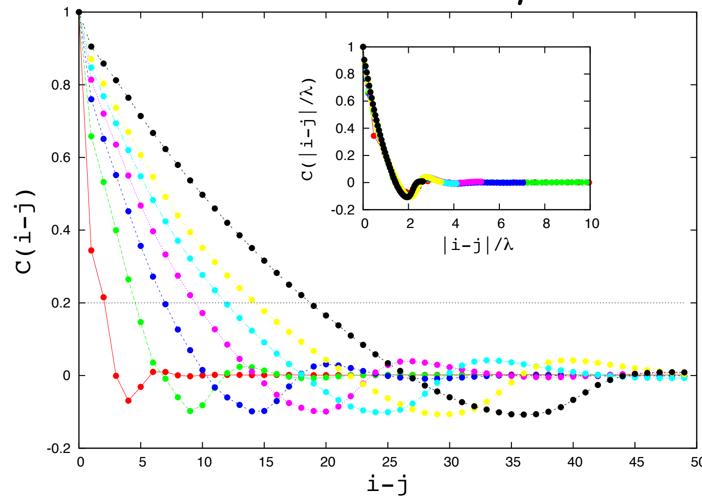
$\langle \xi_m \xi_n \rangle_{\mathrm{emp},F}$

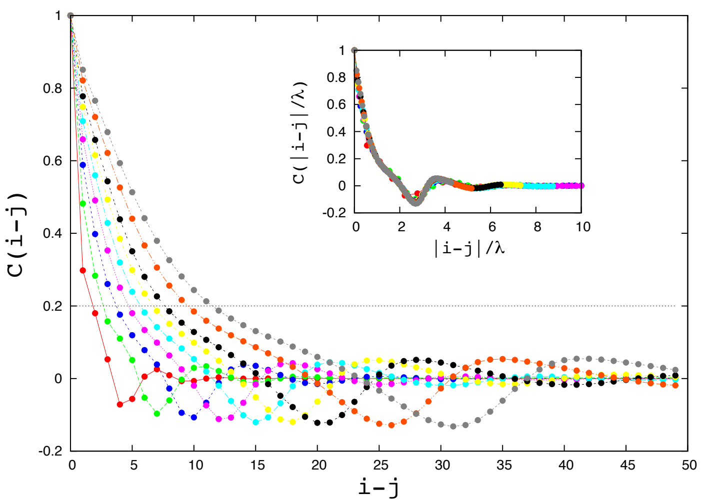
$\langle \xi_m \xi_n \rangle \sim C(|n-m|/\ell(e))$

$f_i = F \qquad \rho = 0.3$

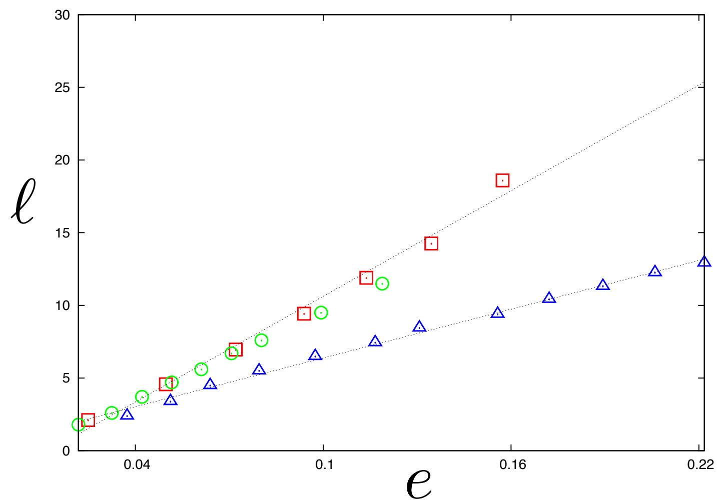


$f_i = F \qquad \rho = 0.8$





$f_i \sim e^{-\frac{(f-F)^2}{\sigma}} \qquad \rho = 1.0$



$\ell(e) \sim e$

Linear increase of correlation length for all the ''tapping'' dynamics we used

EFFECTIVE THERMODYNAMICS ''Á LA EDWARDS''

''Given a certain situation attained dynamically, physical observables are obtained by averaging over the ***usual equilibrium distribution*** at the corresponding volume, energy, etc. but restricting the sum to 'blocked' configurations.'' (Barrat, Kurchan, Loreto, Sellitto)

$$\boldsymbol{\xi} = \{\xi_1, \ldots, \xi_N\}$$ Springs elongations

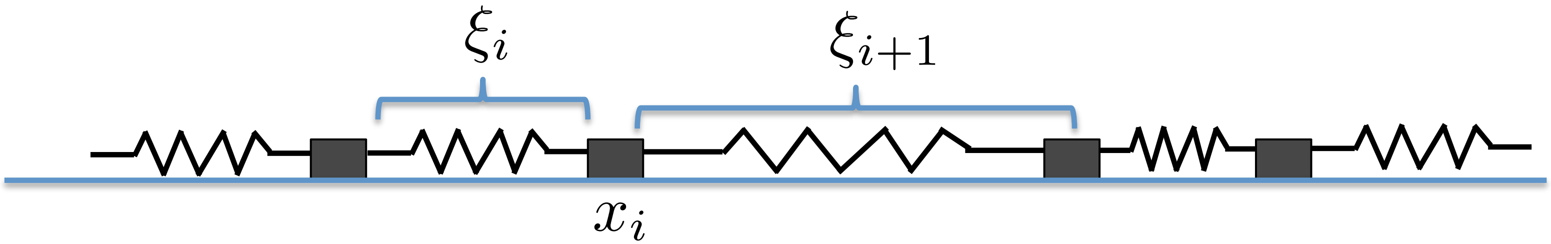
$$\mathcal{Z} = \int \mathcal{D}\boldsymbol{\xi} \; e^{-\beta_{Ed} E[\boldsymbol{\xi}]} \; \delta[\mathcal{F}(\boldsymbol{\xi}) - 1]$$

Mechanical stability $\mathcal{F}(\boldsymbol{\xi}) = 1$

Otherwise $\mathcal{F}(\boldsymbol{\xi}) = 0$

$$E[\boldsymbol{\xi}] = \sum_{i=1}^{N} \frac{\xi_i^2}{2} \qquad \beta_{Ed} = \left[\frac{\partial S}{\partial E}\right]_{\mathcal{F}(\boldsymbol{\xi})=1}$$

EFFECTIVE THERMODYNAMICS "Á LA EDWARDS"



$$\mathcal{Z} = \int d\xi_1 \ldots d\xi_N \; e^{-\beta_{Ed} \sum_{i=1}^{N} \frac{\xi_i^2}{2}} \prod_{i=1}^{N} \Theta(\mu - |\xi_{i+1} - \xi_i|)$$

$$\mathcal{Z} = \int d\xi_1 \ldots d\xi_N \; \prod_{i=1}^{N} T(\xi_i, \xi_{i+1})$$

$$T(x,y) = e^{-\beta_{Ed} \frac{x^2}{4}} \Theta(\mu - |x-y|) e^{-\beta_{Ed} \frac{y^2}{4}} \qquad \begin{array}{l} x \geq 0 \;\; \rightarrow \;\; \Theta(x) = 1 \\ x < 0 \;\; \rightarrow \;\; \Theta(x) = 0 \end{array}$$

Transfer Operator Formalism

$$\mathcal{T}[f](x) = \int_{-\infty}^{\infty} dy \; T(y,x) f(x)$$

$$\mathcal{Z} = \mathrm{Tr}[\mathcal{T}^N]$$

‘’THERMODYNAMIC’’ POTENTIALS

$$T(x,y) = e^{-\beta_{Ed}\frac{x^2}{4}}\Theta(\mu - |x-y|)e^{-\beta_{Ed}\frac{y^2}{4}}$$

$$T(x,y) \in L^2(X \text{ x } Y)$$

Hilbert-Schmidt integral operator
maximum isolated eigenvalue

$$\mathcal{Z} = \mathrm{Tr}[\mathcal{T}^N]$$

$$f(\beta_{Ed},\mu) = -\frac{1}{\beta_{Ed}}\log[\lambda_{\max}(\beta_{Ed},\mu)]$$

$$e = \partial_{\beta_{Ed}}(\beta_{Ed} f)$$

$$e = -\lambda_{\max}^{-1}\langle\lambda_{\max}|\partial_{\beta_{Ed}}\mathcal{T}|\lambda_{\max}\rangle$$

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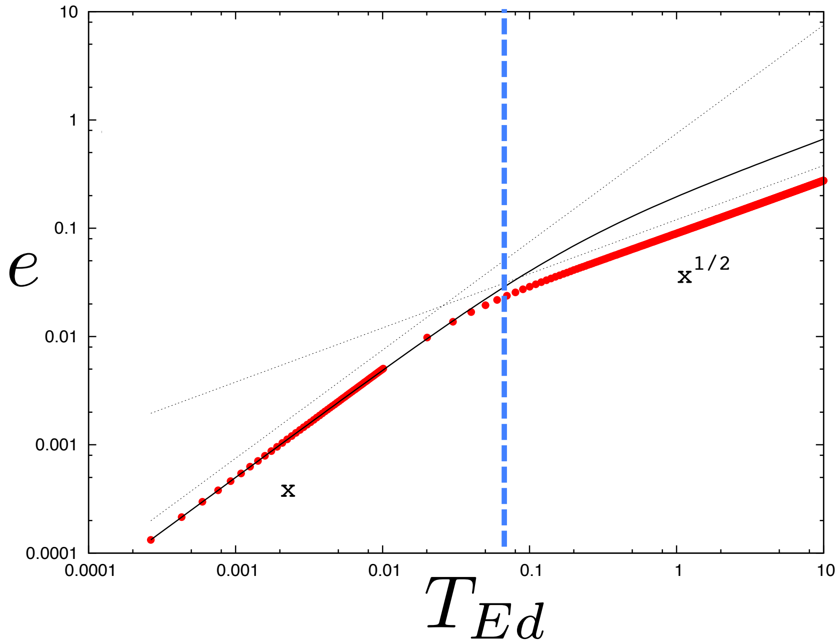


$$e = \partial_{\beta_{Ed}}(\beta_{Ed} f)$$

$$e = -\lambda_{\max}^{-1}\langle\lambda_{\max}|\partial_{\beta_{Ed}}\mathcal{T}|\lambda_{\max}\rangle$$

Equilibrium-like behaviour

$$T_{Ed} \ll \mu^2 \rightarrow T(x,y) \sim e^{-\beta_{Ed}\frac{x^2+y^2}{4}}$$

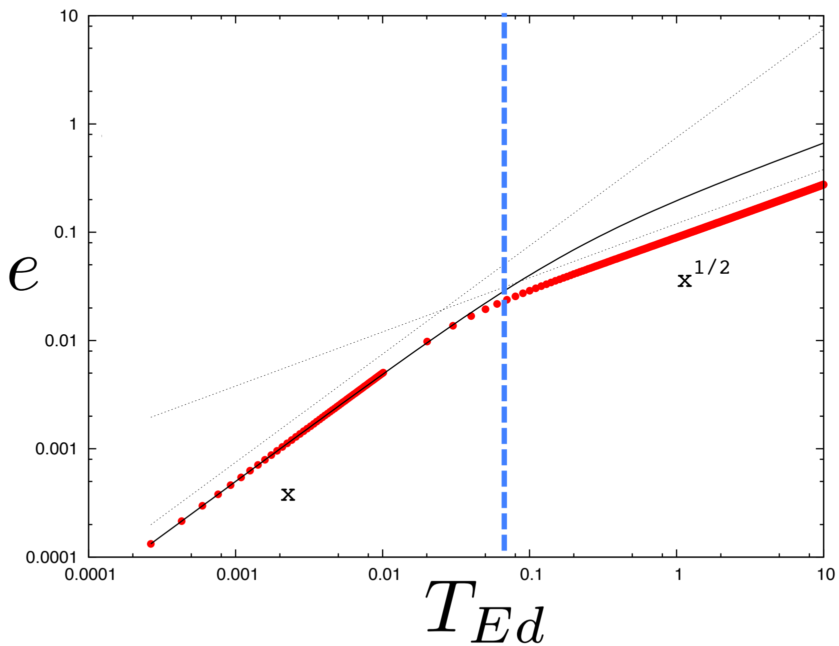
''THERMODYNAMIC'' POTENTIALS

$$T(x,y) = e^{-\beta_{Ed}\frac{x^2}{4}}\Theta(\mu - |x-y|)e^{-\beta_{Ed}\frac{y^2}{4}}$$

Smooth approximation of the constraint

$$\Theta(\mu - |x-y|) \sim \frac{1}{\sqrt{\pi}}\exp\left(-\frac{|x-y|^2}{4\mu^2}\right)$$

$$\mathcal{Z} = \int d\xi_1 \ldots d\xi_N \quad \exp\left[-\boldsymbol{\xi}\cdot A\boldsymbol{\xi}\right]$$



$$e = \frac{1}{2}\frac{\mu\, T_{Ed}}{\sqrt{2T_{Ed}+\mu^2}}$$

Equilibrium-like behaviour

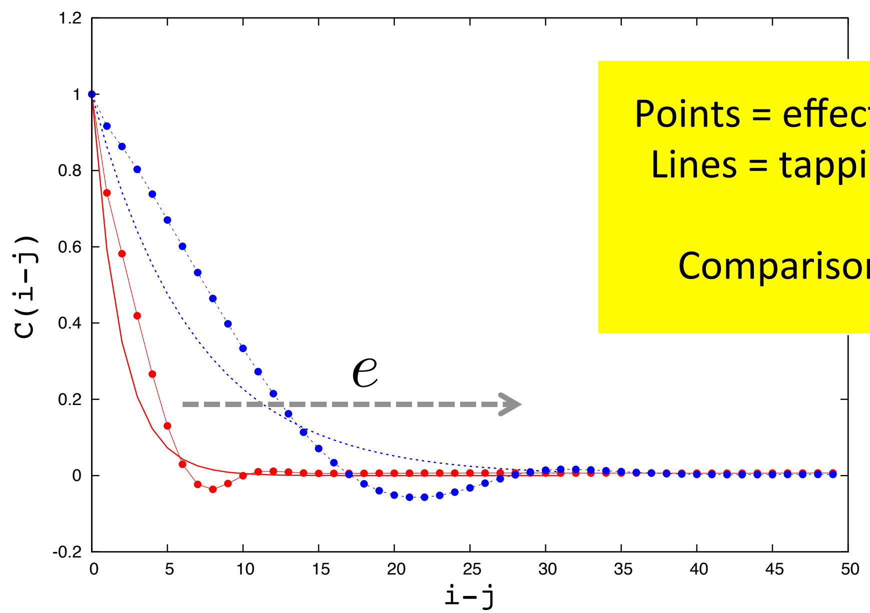
$$T_{Ed} \ll \mu^2 \longrightarrow e \sim \frac{T_{Ed}}{2}$$

SPRING-SPRING CORRELATION FUNCTION

The operator (real, symmetric kernel) has an orthonormal basis

$$\mathcal{T}[f_b](x) = \lambda_b f_b(x) \qquad \int_{-\infty}^{\infty} f_b(x) f_a(x) = \delta_{a,b}$$

$$\lim_{N\to\infty} \langle \xi_m \xi_n \rangle_{Ed} = \sum_{b\in Sp(\mathcal{T})} \left(\frac{\lambda_b}{\lambda_{\max}} \right)^{n-m} \left| \int_{-\infty}^{\infty} dx \; x f_b(x) f_{\lambda_{\max}}(x) \right|^2$$



Points = effective thermodynamics
Lines = tapping dynamics (ρ=0.3)

Comparison is at fixed energy

CORRELATION FUNCTION $C(|n-m|) \sim \exp(-|n-m|/\ell(e))$

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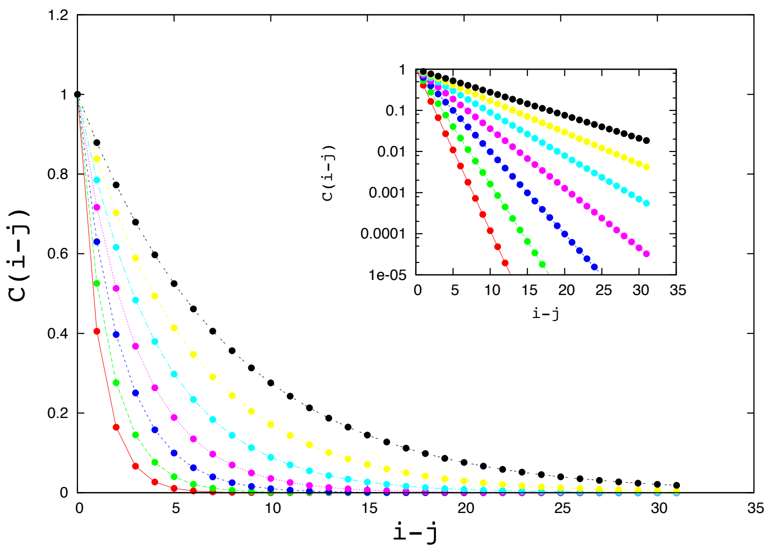
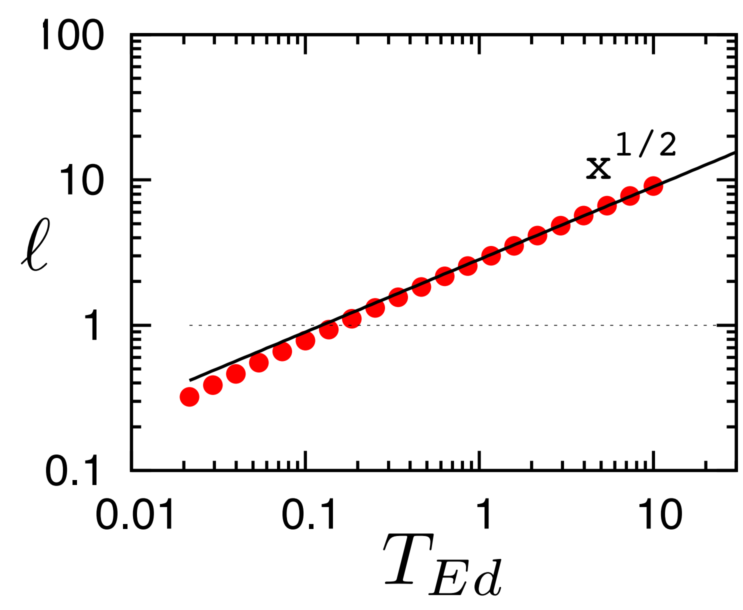
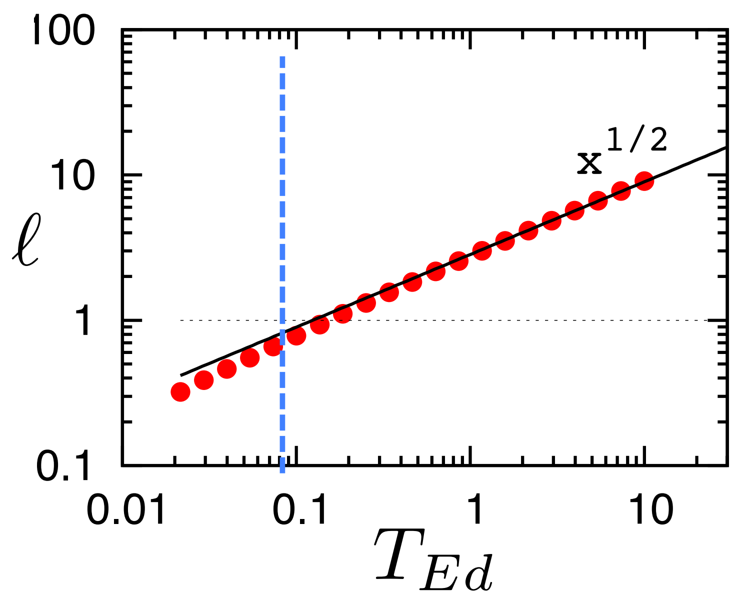
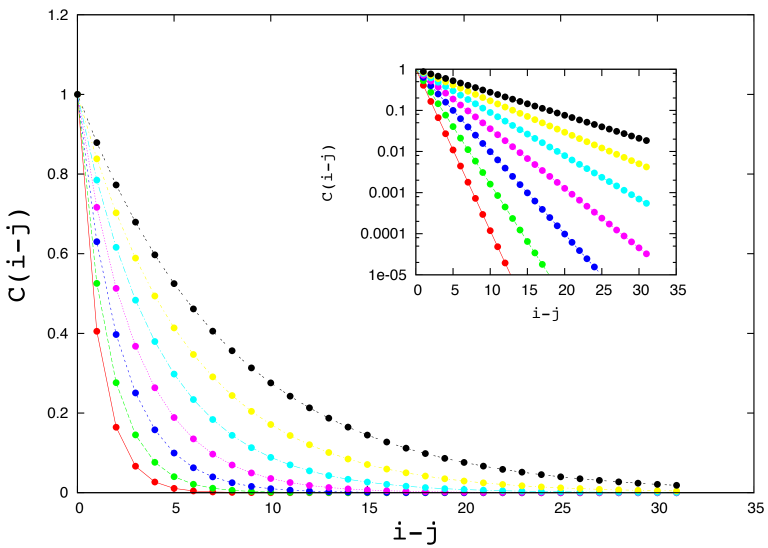


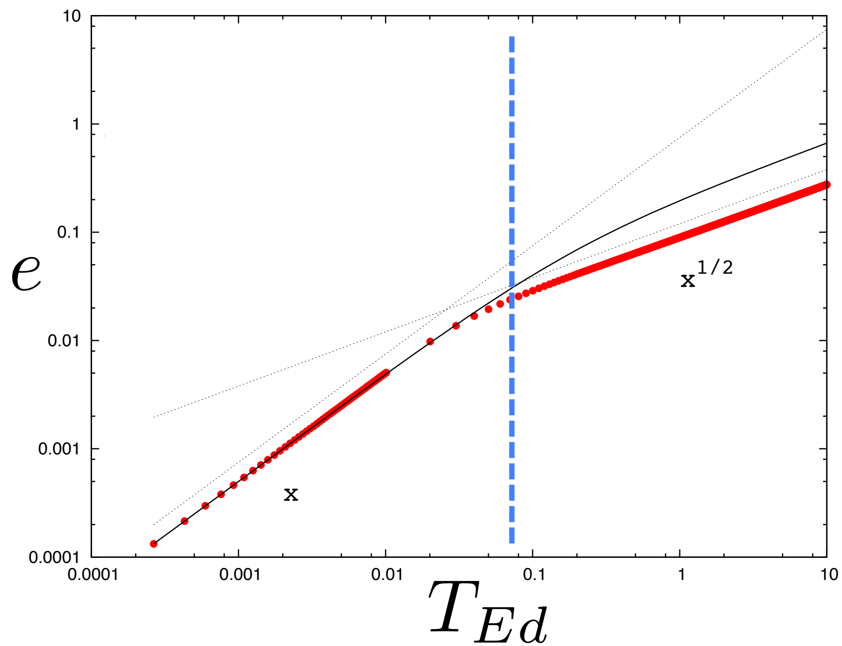


CORRELATION FUNCTION $C(|n-m|) \sim \exp(-|n-m|/\ell(e))$



Correlations appear in the "out-of-equilibrium" regime





DRIVEN ATHERMAL DYNAMICS

$$m\ \ddot{x}_i = F_{\rm diss} + F_{\rm el} + F_{\rm ext}$$

$$\downarrow$$

Blocked configurations

$$\langle \xi_m \xi_n \rangle \sim G(|n-m|/\ell(e))$$

$$\ell(e) \sim e$$

$$e \sim \sqrt{?}$$

EFFECTIVE THERMODYNAMICS

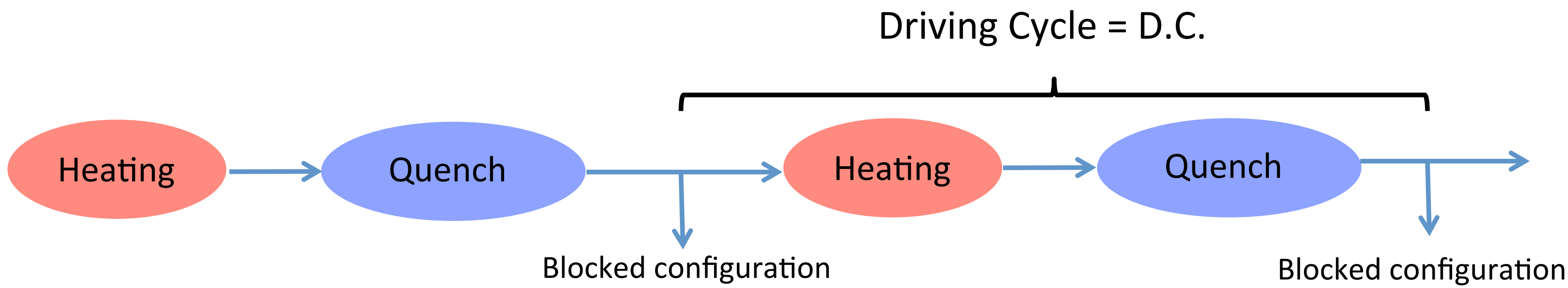
$$\mathcal{Z} = \int_{\boldsymbol{\xi} \in \rm blocked} \mathcal{D}\boldsymbol{\xi}\ e^{-\beta_{Ed} E[\boldsymbol{\xi}]}$$

$$\langle \xi_m \xi_n \rangle \sim C(|n-m|/\ell(e))$$

$$\ell(e) \sim e$$

$$e \sim \sqrt{T_{Ed}}$$

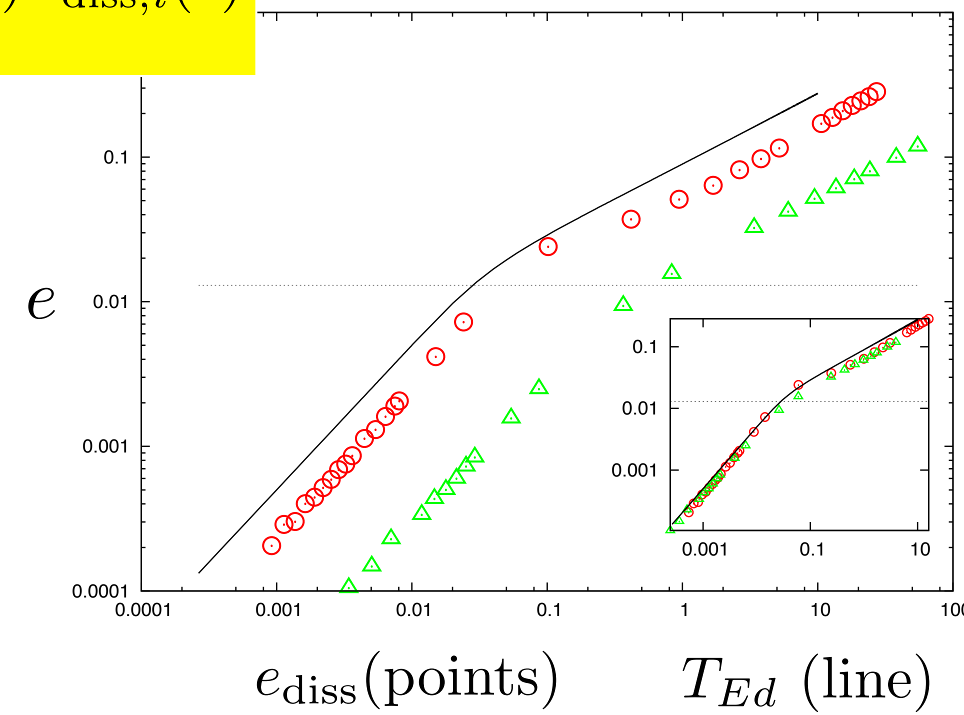
EDWARDS PARAMETER = DISSIPATED ENERGY



$$e_{\text{diss}} = -\frac{1}{N}\sum_{i=1}^{N}\int_{D.C.} ds\ v_i(s) F_{\text{diss},i}(s)$$

$$e_{\text{diss}} \ll 1 \qquad e \sim e_{\text{diss}}$$

$$e_{\text{diss}} \gg 1 \qquad e \sim \sqrt{e_{\text{diss}}}$$



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$\downarrow$

Blocked configurations

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$$\ell(e) \sim e$$

$$e \sim \sqrt{e_{\rm diss}}$$

EFFECTIVE THERMODYNAMICS

$$\mathcal{Z} = \int_{\boldsymbol{\xi}\in\text{blocked}} \mathcal{D}\boldsymbol{\xi}\; e^{-\beta_{Ed} E[\boldsymbol{\xi}]}$$

$$\langle \xi_m \xi_n \rangle \sim C(|n-m|/\ell(e))$$

$$\ell(e) \sim e$$

$$e \sim \sqrt{T_{Ed}}$$

$$e_{\rm diss} = T_{Ed}$$

CONCLUSIONS

- We presented a 1D model where the effective thermodynamics '' à la'' Edwards works pretty well

- The model is simple and realistic: 1)''blocked'' configurations are truly mechanically stable configurations; 2) Dry friction; 3) Dynamics is realistic

- The effective theory can be solved exactly by transfer operators

- The Edwards parameter T_{Ed} has a clear physical interpretation: the energy dissipated in a driving cycle

PERSPECTIVES

- What happens in D=2?

- Diagonalization of (M^L x M^L) matrices: GPU
- M >> 1, discretization of continuous variable