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Abstract:

Anomalous scaling laws appear in a wide class of phenomena where global dilation invariance fails. In this case, the description of scaling properties requires the introduction of an infinite set of exponents.

Numerical and experimental evidence indicates that this description is relevant in the theory of dynamical systems, of fully developed turbulence, in the statistical mechanics of disordered systems, and in some condensed matter problems.

We describe anomalous scaling in terms of multifractal objects. They are defined by a measure whose scaling properties are characterized by a family of singularities, which are identified by a scaling exponent. Singularities corresponding to the same exponent are distributed on a fractal set. The multifractal object arises as the superposition of these sets, whose fractal dimensions are related to the anomalous scaling exponents via a Legendre transformation. It is thus possible to reconstruct the probability distribution of the singularity exponents.

We review the application of this formalism to the description of chaotic attractors in dissipative systems, of the energy dissipating set in fully developed turbulence, of some probability distributions in condensed matter problems. Moreover, a simple extension of the method allows us to treat from the same point of view temporal intermittency in chaotic systems and sample to sample fluctuations in disordered systems.

We stress the phenomenological nature of the approach and discuss the few cases in which it was possible to reach a more fundamental understanding of anomalous scaling. We point out the need of a theory which should explain its origin and pave the way to a microscopic calculation of the probability distribution of the singularities.

Io fingo e suppongo che qualche corpo si muova all'insù secondo la nota proporzione et orizzontalmente con moto equabile . . .
Se poi le palle di piombo, di ferro, di pietra, non osservano quella supposta direzione, suo danno: noi diremo che non parliamo di esse.

I feign and assume that some bodies move vertically according to the known ratio, and horizontally by uniform motion . . .
If balls made of lead, iron or stone do not comply to this rule, it is all to their disadvantage: for we shall say that we are not talking of them.

Evangelista Torricelli

0. Introduction

Scaling invariance plays a fundamental role in many natural phenomena and is often related to the appearance of irregular forms which cannot be described by the usual differential geometry. A classical example is given by the Brownian motion the study of which led Jean Perrin [P06, P13] to understand the physical relevance of non-differentiable curves and surfaces.

The necessity of introducing a new class of geometrical objects, the fractals, has subsequently arisen in various different problems. Indeed, some aspects of 'fractality' were already present in the ideas of some scientists at the beginning of this century like Perrin himself, Besicovitch, Hausdorff, Wiener, Richardson, but the concept of 'fractal object' was explicitly formulated and made popular in the scientific community in recent years by Mandelbrot.

The main idea consists in the characterization of the scaling structure of an object by means of an index, the fractal dimension D_F which coincides for 'ordinary' shapes with the usual (topological) dimension D_T. Indeed, the dimensionality of objects can be defined in different ways. One can define it as a topological concept by counting the number of independent directions in which one can move around any given point (for a rigorous definition see, e.g., [ER85]). One can call this notion the topological dimension. On the other hand, we can define the fractal dimension as a 'capacity' measure by considering the number $N(l)$ of hypercubes of edge l necessary to cover an object embedded in a D-dimensional space in the limit $l \to 0$:

$$N(l) \propto l^{-D_F} . \tag{0.1}$$

It follows that $D_F \leq D$ and the object is called fractal if $D_F > D_T$.

The fractal dimension is purely geometrical, i.e. it only depends on the shape of the object. In general, one has to assign to the physical object a suitable probability measure $d\mu$, according to the particular phenomenon considered.

The measure $d\mu$ should scale with the resolution length. Let us define a coarse grained probability density

$$p_i(l) = \int_{\Lambda_i} d\mu(x) \tag{0.2}$$

as the 'mass' of the hypercube Λ_i of size l, with $i = 1, 2, \ldots N(l)$. The scaling rate is given by the information dimension D_I defined by Balatoni and Renyi [BR56]:

$$\sum_{i=1}^{N(l)} p_i \ln(p_i) \simeq D_I \ln(l) . \tag{0.3}$$

One can easily show that $D_I \leq D_F$ where the equality is valid only for a uniform distribution $p_i^* = 1/N(l) \propto l^{D_F}$ for each box Λ_i.

D_I is a more interesting index than D_F. It must be noted, in fact, that the number $N_R(l)$ of boxes containing the dominant contributions to the total mass, and thus the relevant part of the information, is:

$$N_R(l) \propto l^{-D_I} \tag{0.4}$$

as consequence of the Shannon–McMillan theorem (see e.g. [K57]).

If $D_I < D_F$, then the measure itself is called fractal [F82] since it is singular with respect to the uniform distribution p^*, i.e. p_i/p_i^* can diverge in the limit of vanishing l. The support of the measure is in this sense an inhomogeneous fractal object.

The principal aim of this review is to characterize different physical systems by means of the analysis of the probability measure singularities. The understanding of their structure can be actually achieved by extracting from experiments or simulations the 'mass' moment scaling

$$\langle p_i(l)^q \rangle \equiv \sum_{i=1}^{N(l)} p_i(l)^{q+1} \propto l^{q \cdot d_{q+1}} . \tag{0.5}$$

The d_q are the Renyi dimensions which generalize the information dimension $D_I = d_1$ as well as the fractal dimension $D_F = d_0$. If the fractal is homogeneous, then one can extract q out of the average operation in (0.5) and the Renyi dimensions are therefore all equal to the fractal dimension. On the contrary, if the d_q's are not constant, one speaks of anomalous scaling and, as the order q varies, the amount of the difference $d_q - D_F$ gives a first rough measurement of the inhomogeneity of the probability distribution. Such a behaviour arises in many different systems and was first pointed out by Mandelbrot [M74] in fully developed turbulence. Subsequently Parisi and coworkers [FP85, BPPV84] introduced the concept of multifractal object in the same context, realizing that the moment scaling indices can be related to the scaling of the probability distribution of the singularities. The object can be regarded in this approach, further developed by Halsey et al. [HJKPS86, JKLPS85], as an interwoven family of different homogeneous fractal sets $S(\alpha)$ on which the measure has a singularity of type α (i.e. $p_i(l) \propto l^\alpha$ for boxes Λ_i belonging to $S(\alpha)$). It is possible to relate the Renyi dimensions (which can be directly measured in experiments) to the fractal dimensions $f(\alpha)$ of the sets $S(\alpha)$ (or equivalently to the probability of picking up a singularity α with resolution scale l) via a Legendre transformation.

This is the heart of the method which has been also applied to the study of the moments of generic observables A computed on scale l:

$$\langle A(l)^q \rangle \propto l^{g(q)} .$$

Despite its name, anomalous scaling, i.e. a non-linear shape of the function $g(q)$, is the more common situation even if it contrasts with the standard ideas about critical phenomena where one usually considers only a finite number of scaling exponents.

Moreover the same object can be a multifractal with respect to a certain observable and a homogeneous fractal with respect to another one. The reader can find an explicit example of such a feature in section 2.

Up to now we have looked at multifractality as the manifestation of the spatial fluctuations of the observables. Nevertheless, temporal scaling features appear of great importance in the chaotic evolution of deterministic dynamical systems. In these cases one usually observes strong time variations in the degree of chaoticity. This intermittency phenomenon involves an anomalous scaling with respect to 'time dilations' identifying the parameter $\exp(-t)$ with the parameter l used in spatial dilations. A measure of the degree of intermittency requires the introduction of infinite sets of exponents which are analogous to the Renyi dimensions and can be related to a multifractal structure given by the dynamical system in the functional trajectory space.

In section 1 we introduce the multifractal formalism in the context of dissipative dynamical systems.

We define the Renyi dimensions of the natural measure generated by a deterministic evolution law on a chaotic attractor and describe the numerical algorithms for their calculation. Indeed, typical chaotic attractors can be regarded as multifractal objects. We discuss in detail how the scaling of the probability that a point of the attractor (representative of the state of the dynamical system) belongs to $S(\alpha)$ is determined by the fractal dimensionality $f(\alpha)$ of $S(\alpha)$ and we point out that there is a singularity hierarchy the top of which is given by the information dimension D_{I}.

If the attractor is homogeneous, the natural measure is not fractal and has only a singularity $\alpha = D_{\mathrm{F}} = D_{\mathrm{I}}$ with respect to the Lebesgue measure and $d_{\mathrm{q}} = D_{\mathrm{F}}$, $\forall q$.

In section 2 we apply this approach to the study of the statistical properties of fully developed turbulence where there is spatial intermittency of the energy dissipation $\varepsilon(x)$ implying anomalous scaling laws for the moments of the velocity differences, the so-called structure functions. Intermittency can be reproduced by models assuming that ε is concentrated on fractal structures with $D_{\mathrm{F}} \leq 3$.

One may then consider ε as a mass density and a non-homogeneous distribution corresponds to anomalous scaling laws as in chaotic attractors. This allows us to extend the results of section 1 to the characterization of the singularities of ε.

We introduce a multiplicative process (random β-model) for building-up a proper multifractal object and thus obtain a good fit to the experimental data with only one adjustable parameter on the basis of phenomenological assumptions.

We also emphasize that the multifractal nature of turbulence does not affect in a substantial way some phenomena like the separation of particle pairs whereas it is relevant in determining the number of degrees of freedom involved.

Section 3 shows how the degree of temporal intermittency of the chaoticity in a dynamical system can be measured by indices which are extracted either by an experimental signal (the Renyi entropies K_q) or by numerical calculations (the generalized Lyapunov exponents $L(q)$).

The multifractal approach can be extended to the study of 'temporal inhomogeneities' with slight modifications and it allows to reconstruct the probability distribution which rules the temporal fluctuations around the average degree of chaoticity measured by the Kolmogorov entropy K_1 and the characteristic Lyapunov exponents. We also give some examples of numerical calculations of the $L(q)$'s in dynamical systems with few degrees of freedom.

We finally discuss the case of one-dimensional chaotic maps which highlights the relation between the multifractal method and the thermodynamic formalism introduced for hyperbolic systems by Bowen, Ruelle, Sinai and Walters. In this framework we show that the appearance of edges in the Renyi entropies is an indication of phase transitions.

In section 4 the techniques developed for the characterization of the chaotic behaviour of dynamical systems will be used in the study of the trajectories given by the products of random transfer matrices. They can, e.g., describe the localization of the wave function of the Schroedinger equation in a random potential or the partition function of spin glasses. The multifractal approach is in this case the analogue of the usual statistical theory of finite volume fluctuations of the physical observables (we shall respectively consider the localization length and the free energy) among different replicas of the same system with respect to disorder. The calculation of the generalized Lyapunov exponents therefore gives the possibility to reconstruct the scaling properties of the probability distribution of the observables. The mass density is the density of replicas characterized by the same observable value corresponding to trajectories with the same degree of chaoticity, and in this sense the replicas define a multifractal object in the realization space.

In section 5 we analyse some critical phenomena (localization transition and conduction in random resistor networks at the percolation threshold) as well as some growth phenomena (diffusion limited aggregation) where a hierarchy of different exponents appears as a new interesting feature. We stress the fact that, approaching the critical point, the probability of finding scaling exponents different from that corresponding to the information dimension tends to zero as a power of correlation length.

In section 6 the reader will find some concluding remarks.

1. Chaotic attractors as inhomogeneous fractals

1.1. Why study attractors' dimensions?

Recently, it has been shown that deterministic evolution laws may lead to chaotic behaviours even in absence of external noise [LL83, ER85]. This phenomenon, called deterministic chaos, is essentially due to a sensitive dependence on initial conditions and has a great relevance in the description of many physical systems whose dynamics can be modelled by ordinary differential equations or maps:

$$\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{f}(\boldsymbol{x}(t)) \,, \qquad \boldsymbol{x}(i+1) = \boldsymbol{g}(\boldsymbol{x}(i)) \tag{1.1.1}$$

with $\boldsymbol{x}, \boldsymbol{f}, \boldsymbol{g} \in \mathrm{R}^F$.

One of the first examples was given by Lorenz [L63] who showed that a dynamical system of just three differential equations can be chaotic.

In this section we limit ourselves to the study of the attractors of dissipative systems but many results can also be applied to generic chaotic signals. Indeed, after a transient, a dissipative system usually evolves in the neighbourhood of a set called attractor (for a rigorous definition see, e.g., [ER85]).

The concept of dimension is relevant for the dynamics because it provides a precise way to estimate the number n_f of independent relevant variables involved in the evolution.

To be explicit, let us consider fixed points, limit cycles, tori, where n_f is the dimension of the attractor.

A trivial example is given by a dynamical system with a stable fixed point x_0, where for large times $x(t) \to x_0$ (fig. 1a). The attractor is a zero-dimensional set and $n_f = 0$. On the other hand, a limit cycle (fig. 1b) is a one-dimensional set and $n_f = 1$ since $x(t)$ asympotically evolves on a line. Generally speaking, for a quasi-periodic motion with n incommensurate frequencies, the attractor is an n-dimensional torus and $n_f = n < F$.

On the contrary, the spatial features of deterministic chaos are much more complex as numerical experiments indicate that the points generated by the time evolution (1.1.1) cover a strange set with a selfsimilar structure (see fig. 2). This chaotic attractor usually is a fractal object [M82] in the phase space. What is in this case the effective number of degrees of freedom? The fractal dimension D_F of the attractor gives a first estimate of n_f since one has the lower bound:

$$n_f \geq [D_F] + 1 \tag{1.1.2}$$

where $[(\cdot)]$ is the integer part of $(\cdot)$.

a)

b)

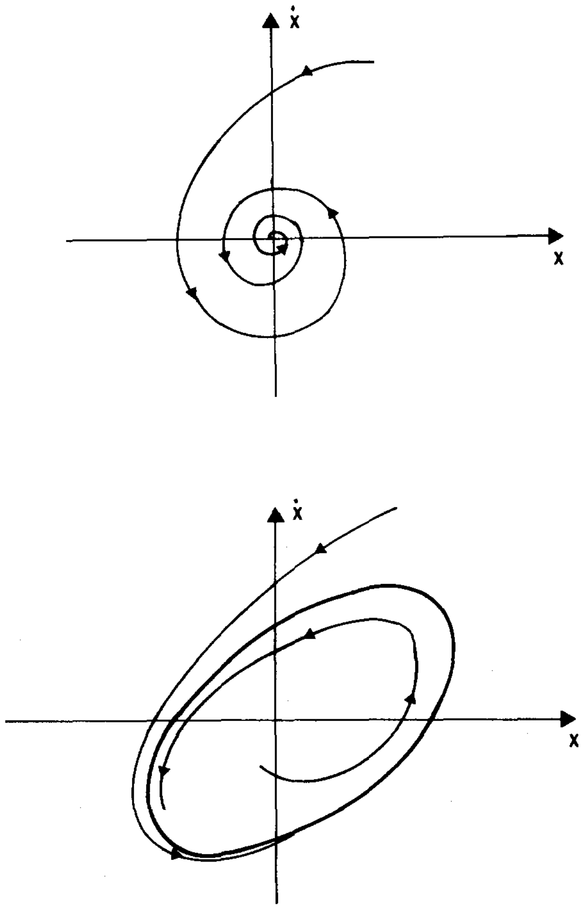

Fig. 1. Example of a stable fixed point (a) and of a stable cycle (b).

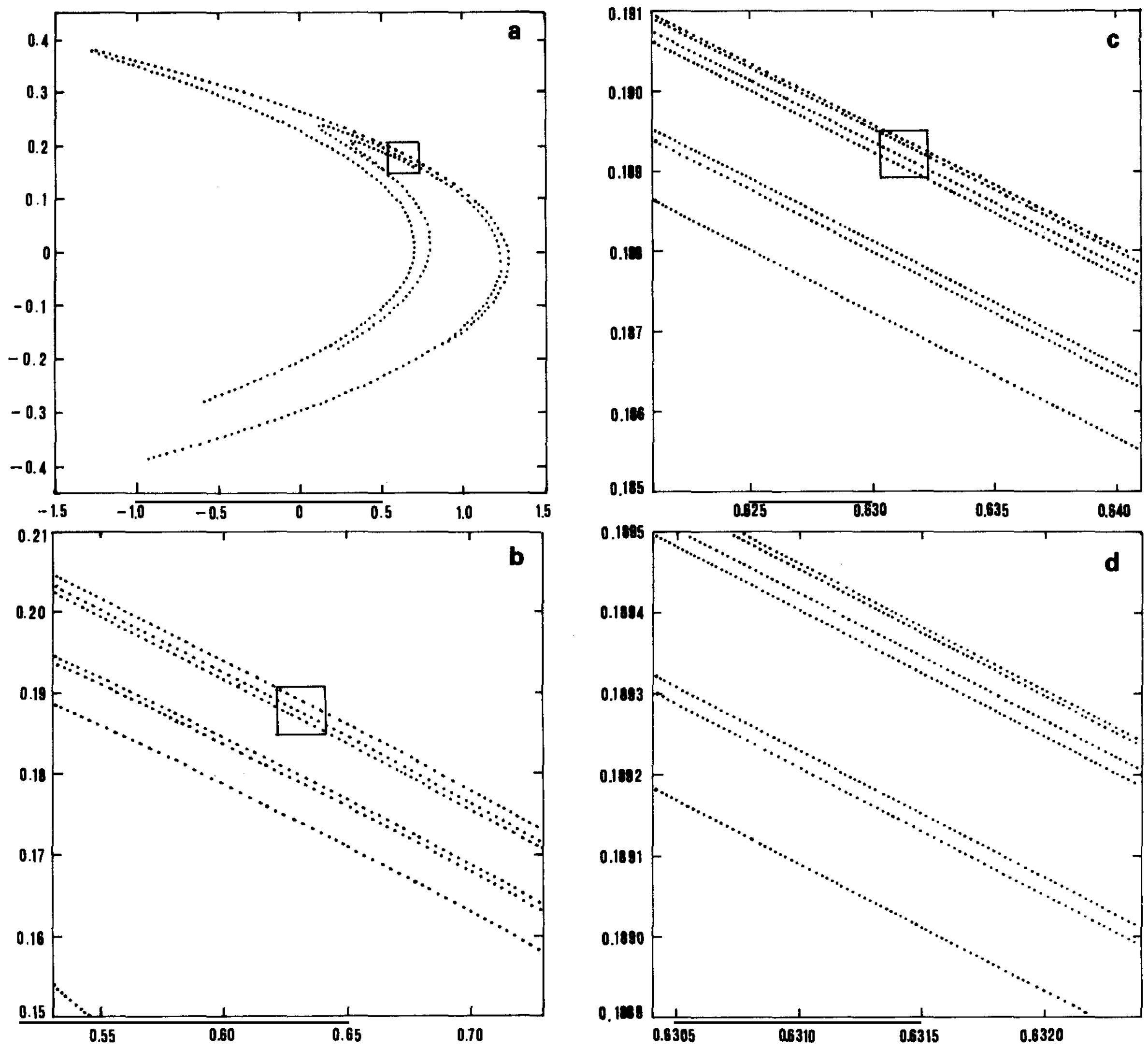

Fig. 2. Structure of the attractor of the Henon map [H76], with $a = 1.4$ and $b = 0.3$. (b), (c) and (d) show successive blow-ups of the regions inside the box of the previous figure.

We have decided here and in the following, to use the term fractal dimension in the sense of the definition (0.1) instead of the term Hausdorff dimension D_H [H19] which involves the evaluation of the extremum among different non-uniform partitions into hypercubes of edge $\leq l$. One has $D_H \leq D_F$, but in typical cases the equality holds.

Let us briefly discuss the procedure to compute D_F. The analysis of the practical difficulties will in fact lead to introduce, in a quite natural way, an infinite set of generalized dimensions.

Remark that we are assuming that the dynamical system is ergodic and mixing so that it is possible to extract the statistical properties by a time average on a single trajectory.

Following the definition of D_F (0.1) the most direct computational method is the box-counting [RHO80]. One generates the series $x_i = x(i\tau)$, $i = 1, \ldots M \gg 1$ and divides the region of R^F where the motion evolves in hypercubes of edge l.

Namely let us consider the number $N(l)$ of F-dimensional hypercubes of size l necessary to cover the attractor (i.e., containing at least one point $\boldsymbol{x}_i$), in the limit $M \to \infty$:

$$N(l) \propto l^{-D_F} \qquad \text{for } l \to 0 . \tag{1.1.3}$$

With this definition, D_F of a regular attractor is the usual (topological) dimension.

The box-counting method is rather difficult to be applied whenever $F \gtrsim 3$ because of the consuming of computer memory. Other methods have therefore been introduced for estimating the fractal dimension of chaotic attractors.

Grassberger and Procaccia [GP83a] have defined a correlation dimension ν by considering the scaling of the correlation integral:

$$C(l) = \lim_{M \to \infty} \frac{1}{M^2} \sum_i \sum_{j \neq i} \theta(l - |\boldsymbol{x}_i - \boldsymbol{x}_j|) \tag{1.1.4}$$

where θ is the Heaviside step function. $C(l)$ is the percentage of pairs $(\boldsymbol{x}_i, \boldsymbol{x}_j)$ with distance $|\boldsymbol{x}_i - \boldsymbol{x}_j| \leq l$ and in the limit $l \to 0$ one has:

$$C(l) \propto l^{\nu} . \tag{1.1.5}$$

If each box has the same density of points, ν is equal to D_F. Typical chaotic attractors however are inhomogeneous fractal and one can show that in general

$$\nu \leq D_F . \tag{1.1.6}$$

ν is a more 'relevant' scaling index than D_F since it is related to the point probability distribution generated by (1.1.1) on the attractor ('natural' or 'physical' measure, see [ER85]) while D_F cannot take into account an eventual inhomogeneity in the visit frequencies. The correlation dimension ν in fact measures the scaling properties of the average density of points even if it gives no information on the density fluctuations.

Indeed, let us define the number of points in an F-dimensional ball of radius l and centre $\boldsymbol{x}_i$

$$n_i(l) = \lim_{M \to \infty} \frac{1}{(M-1)} \sum_{j \neq i} \theta(l - |\boldsymbol{x}_i - \boldsymbol{x}_j|) \tag{1.1.7}$$

in order to write down the correlation integral as the average number of points in the ball:

$$C(l) = \langle n(l) \rangle \tag{1.1.8}$$

where $\langle f \rangle = \lim_{M \to \infty} M^{-1} \sum_{i=1}^{M} f(\boldsymbol{x}_i)$, by ergodicity. On the other hand the fluctuations of n_i are ruled by a probability distribution which can be reconstructed by knowledge of the moments $\langle n(l)^q \rangle$ [PV84].

We must therefore introduce a whole set of generalized scaling exponents [BR56, G83, PV84]:

$$\langle n(l)^q \rangle = \lim_{M \to \infty} \frac{1}{M} \sum_{i=1}^{M} n_i(l)^q \propto l^{\phi(q)} \tag{1.1.9}$$

where $\phi(1) = \nu$. In a homogeneous fractal $\phi(q) = D_F \cdot q$ and the deviations from this linear law are a measure of the degree of inhomogeneity. In fig. 3 $\phi(q)$ vs. q is shown for the Henon map [H76].

If we consider a uniform partition of the phase space into boxes of size l it is convenient to introduce the probability $p_k(l)$ that a point $\boldsymbol{x}_i$ falls into the kth box.

In this case the moments of p_k can be estimated by summing up the boxes:

$$\langle p(l)^q \rangle = \sum_{k=1}^{N(l)} p_k(l)^{q+1} \propto l^{q \cdot d_{q+1}} . \tag{1.1.10}$$

The exponents d_q are called Renyi dimensions and a moment of reflection shows that

$$\phi(q)/q = d_{q+1} \tag{1.1.11}$$

because of the ergodicity $n_i(l) \sim p_k(l)$ if $\boldsymbol{x}_i$ belongs to the kth box and since one can use either an "ensemble" average (weighted sum over the boxes) or a 'temporal' average (sum of the time evolution $\boldsymbol{x}(l)$).

The fractal dimension is obtained by relation (1.1.11) for $q = -1$:

$$D_F = d_0 = -\phi(-1) \tag{1.1.12a}$$

while the correlation dimension is:

$$\nu = d_2 = \phi(1) . \tag{1.1.12b}$$

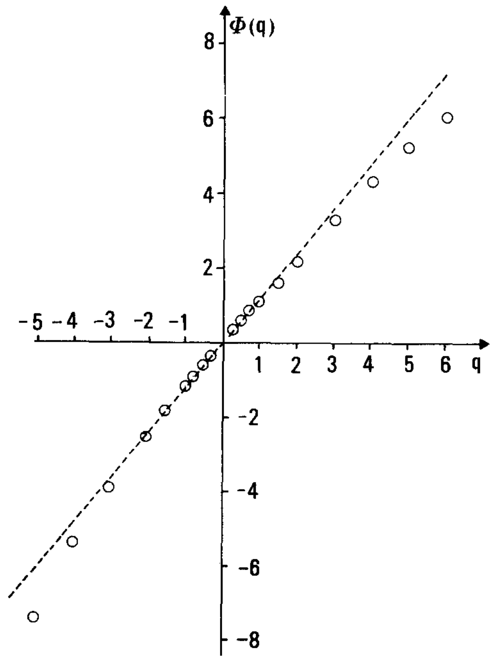

Fig. 3. $\phi(q)$ vs. q for the Henon map [H76] with $a = 1.2$ and $b = 0.3$. The dashed line indicates $\phi(q) = D_1 q$.

One can also show that $\phi(q)$ is a convex function of q by a general theorem of probability theory [F71] and d_q consequently decreases as q increases.

A numerical computation of the Renyi dimensions has to be performed following the definition (1.1.9) instead of (1.1.10). It is in fact easy to generalize the Grassberger–Procaccia method for computing the moments of $n_i(l)$ with a time-consuming of the same order as that for computing ν, while the memory-consuming problems of the box-counting method are escaped [PV84].

Other ways exist for introducing an infinite set of dimensions; Hentschel and Procaccia [HP83a], e.g., proposed the following characterization of the attractors. Let us define:

$$\begin{aligned}&C_n(l) = \lim_{M\to\infty} [\text{number of } n\text{-tuplets of points } (\boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \ldots \boldsymbol{x}_{i_n})\\ &\text{whose distances } |\boldsymbol{x}_{i_\alpha} - \boldsymbol{x}_{i_\beta}| \text{ are less than } l \text{ for all } i_\alpha,\, i_\beta]\end{aligned} \tag{1.1.13}$$

which should scale as:

$$C_n(l) \propto l^{\nu_n} \qquad \text{for small } l \tag{1.1.14}$$

with $\nu_2 = \nu$. Note that the direct computation of ν_n following the definition (1.1.13) requires a CPU time increasing as M^n. However it is easy to recognize that

$$\nu_n = (n-1)\cdot d_n\,, \qquad n = 2, 3, 4, \ldots\,. \tag{1.1.15}$$

The relation (1.1.15) is obtained by the fact that in the ith box the distribution of points is essentially uniform implying

$$\begin{aligned}&C_n^{(i)}(l) = [\text{number of } n\text{-tuplets of points } (\boldsymbol{x}_{i_\alpha}, \ldots \boldsymbol{x}_{i_\beta}) \text{ contained}\\ &\text{in the } i\text{th box with all } |\boldsymbol{x}_{i_\alpha} - \boldsymbol{x}_{i_\beta}| \le l] \propto p_i(l)^n \cdot M^n\,.\end{aligned} \tag{1.1.16}$$

(1.1.15) follows from (1.1.10), (1.1.13) and (1.1.16) since $C_n(l) = \lim_{M\to\infty} (1/M) \sum_{i=1}^{M} C_n^{(i)}(l)$.

1.2. Characterization of chaotic attractors as multifractal objects

It is intuitively reasonable that the dimensions d_q give a measure of inhomogeneity in the distribution of points on the attractor. Let us now show how this inhomogeneity can be related to the existence of a spectrum of singularities of the natural measure.

Let us cover the attractor with $N(l)$ boxes of edge l and define the probability $p_i(l)$ that a point belongs to the ith box $\Lambda_i(l)$

$$p_i(l) = \int_{\Lambda_i(l)} \mathrm{d}\mu(\boldsymbol{x})\,. \tag{1.2.1}$$

$\mathrm{d}\mu(\boldsymbol{x})$ is the natural measure given by the dynamics. In practice, in a numerical experiment $p_i(l)$ is given by $n_k(l)$ defined in (1.1.7) with $\boldsymbol{x}_k$ centred in $\Lambda_i(l)$. In the limit of a homogeneous attractor, $p_i(l) \propto l^{D_F}$, while this does not happen in a generic case. Let us therefore group the boxes with the singularity $\bar{\alpha} \in [\alpha, \alpha + \mathrm{d}\alpha]$:

$$p_i(l) \propto l^{\alpha} \qquad \text{for small } l \tag{1.2.2}$$

into a subset $S(\alpha)$ of the attractor. Roughly speaking α is a 'local mass dimension'.

The number of boxes $dN_{\alpha}(l)$ needed to cover $S(\alpha)$ should behave in the scaling hypothesis like:

$$dN_{\alpha}(l) = d\rho(\alpha)\, l^{-f(\alpha)} \tag{1.2.3}$$

where $f(\alpha)$ are the different fractal dimensions of the sets $S(\alpha)$ upon which the singularities are of kind α. Essentially we are describing the measure on the attractor by interwoven sets each with singularity of kind α and fractal dimension $f(\alpha)$.

Now we can relate $f(\alpha)$ to d_q by computing the quantities (1.1.10) as an integral on α. Following equations (1.2.2, 1.2.3) one has:

$$\sum_{i=1}^{N(l)} p_i(l)^q \propto \int d\rho(\alpha)\, l^{\alpha q - f(\alpha)} . \tag{1.2.4}$$

The integral can be computed for small l by the saddle point method:

$$d_q = \frac{1}{(q-1)} \min_{\alpha} (\alpha q - f(\alpha)) . \tag{1.2.5}$$

If we know $f(\alpha)$, then we can find d_q and, alternatively, given d_q, we obtain $f(\alpha)$ inverting (1.2.5). In the limit case of homogeneous attractors $f(\alpha)$ is defined only for $\alpha = D_F$ implying $f(D_F) = D_F$ and $d_q = D_F$, $\forall q$.

The meaning of (1.2.5) is quite obvious: d_q is detected by a particular value of $\bar{\alpha}(q)$ determined by the extremum conditions of (1.2.5):

$$df/d\alpha|_{\alpha=\bar{\alpha}} = q(\bar{\alpha}) \tag{1.2.6}$$

and

$$d_q = \frac{1}{(q-1)} [q\bar{\alpha} - f(\bar{\alpha})] . \tag{1.2.7}$$

One has the obvious inequality $f(\alpha) \le D_F$ because $N_{\alpha}(l) \le N(l)$, while the information dimension $D_I = d_1$ satisfies the relation

$$D_I = f(D_I) . \tag{1.2.8}$$

We can repeat the above computations for the exponents (1.1.9) by a summation over the points instead of over the boxes. This description is exactly equivalent but allows to emphasize certain physical aspects. Let us introduce the percentage of points $\mathcal{N}_{\alpha}(l)$ which belongs to the boxes of size l such that (1.2.2) holds:

$$d\mathcal{N}_{\alpha}(l) \propto dN_{\alpha}(l) \cdot l^{\alpha} \propto d\rho(\alpha)\, l^{H(\alpha)} \tag{1.2.9}$$

with

$$H(\alpha) = \alpha - f(\alpha) \geq 0\,. \tag{1.2.10}$$

By (1.1.9) and (1.2.9) it follows

$$\langle n(l)^q \rangle \propto \int \mathrm{d}\rho(\alpha)\, l^{\alpha q + H(\alpha)}$$

and one obtains

$$\phi(q) = q d_{q+1} = \min_{\alpha} \left[\alpha q + H(\alpha)\right]. \tag{1.2.11}$$

For $q = 0$, we select the information dimension $d_1 = \mathrm{d}\phi/\mathrm{d}q|_{q=0}$ corresponding to the singularity $\alpha = D_\mathrm{I}$ for which $H = 0$. In the limit $l \to 0$ all the points with $\alpha \neq D_\mathrm{I}$ cannot be detected since $\mathcal{N}_\alpha$ vanishes with l for positive $H(\alpha)$. This situation is well known in the framework of information theory. Indeed, we gain an information $I(l) = -\Sigma_{i=1}^{N(l)} p_i(l) \ln p_i(l)$ in a measurement of the system state with precision l. The Shannon–McMillan theorem [K57] assures that the number $N_\mathrm{R}(l)$ of boxes which give the leading contribution to $I(l)$ should scale like l^{-D_I}. D_I is therefore the most probable 'local mass dimension', if we pick up the points according to the natural measure. This fact has practical consequences in the analysis of experimental signals where one usually computes the correlation dimensions on a small set of points. let us define $\tilde{\nu}$ as

$$\tilde{\nu} = \ln\left(\frac{1}{\tilde{M}} \sum_{k=1}^{\tilde{M}} n_k(l)\right) \Big/ \ln l$$

where one chooses $\tilde{M}$ points $\boldsymbol{x}_k$ at random among the M points $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots \boldsymbol{x}_M$ of the temporal sequence [ER85]. If $\tilde{M}$ is much smaller than M, then $\tilde{\nu}$ is closer to D_I rather than to ν. We think that experimentalists should take into account this warning for the estimation of the dimension of an attractor.

The Legendre transform becomes simple also in the limit $q \to +\infty$ where the minimum condition picks up the extreme values of the singularities:

$$\alpha_{\max} = \lim_{q \to -\infty} d_q\,; \qquad \alpha_{\min} = \lim_{q \to +\infty} d_q\,. \tag{1.2.12a}$$

For q large enough (say $q > q_1 > 0$, $q < q_2 < 0$), the function $\phi(q)$ reaches its asymptotic behaviour:

$$\begin{aligned} \phi(q) &\simeq \alpha_{\min} q + H(\alpha_{\min}) \text{ for } q > q_1 \\ \phi(q) &\simeq \alpha_{\max} q + H(\alpha_{\max}) \text{ for } q < q_2\,. \end{aligned} \tag{1.2.12b}$$

Let us note that $f(\alpha)$ can be negative. In this case the fractal dimension of the corresponding set $\mathrm{S}(\alpha)$ is zero, of course and $\mathrm{S}(\alpha)$ is called 'volatile' fractal [M82]. Negative $f(\alpha)$'s indicate how fast the number of boxes necessary to cover $\mathrm{S}(\alpha)$ converge to zero in the limit of vanishing size l.

Analytical calculations of some features of $f(\alpha)$ have been performed just on particular scale-invariant structures (such as the 2-cycle of period doubling, mode-locking structures, quasi-periodic trajectories for circle maps) [HJKPS86, K86].

In numerical (or real) experiments it is possible to compute the d_q's, e.g., via the generalized correlation integrals (1.1.9). One can then obtain $f(\alpha)$ by means of the Legendre transformation (1.2.5) as shown in fig. 4. $f(\alpha)$ usually approximates the typical parabolic shape, given by a lognormal distribution:

$$\begin{aligned} f(\alpha) &= \alpha - \frac{1}{2\mu}(\alpha - D_{\mathrm{I}})^2 \\ H(\alpha) &= \frac{1}{2\mu}(\alpha - D_{\mathrm{I}})^2 \end{aligned} \tag{1.2.13}$$

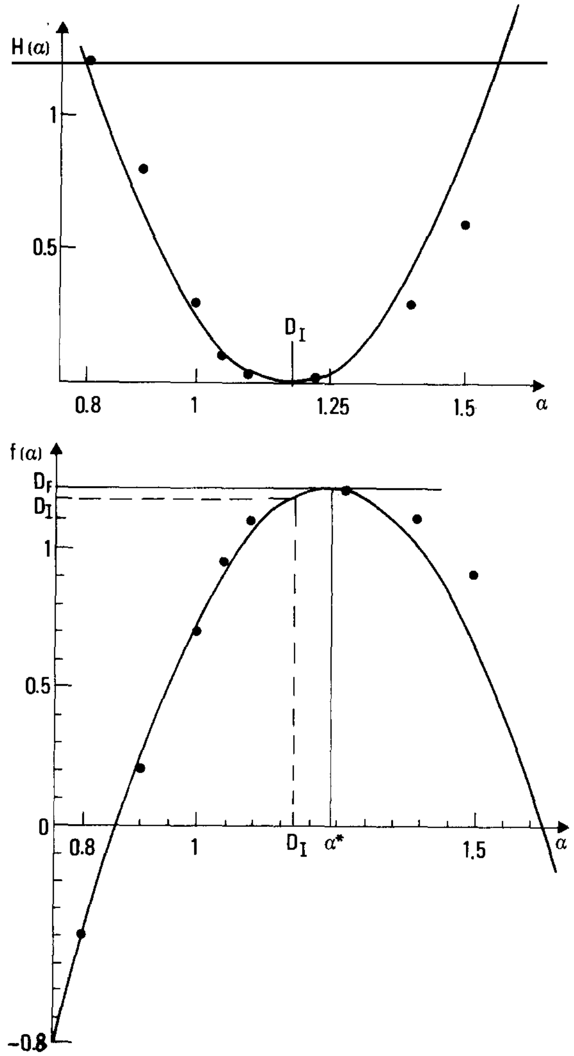

Fig. 4. $H(\alpha)$ (a) and $f(\alpha)$ (b) vs. α for the Henon map [H76] with $a = 1.2$ and $b = 0.3$. The full lines indicate the parabolic approximations given by eq. (1.2.13).

and μ is given by

$$\mu = \lim_{l\to 0} \frac{\langle \ln^2(n(l)) - \langle \ln(n(l))\rangle^2 \rangle}{|\ln l|} . \tag{1.2.14}$$

We shall discuss in the appendix B the limits of this approximation. However, for small q one has in general:

$$d_{q+1} \simeq d_1 - \tfrac{1}{2}\mu q . \tag{1.2.15}$$

1.3. Other characterizations of attractors and experimental problems

The methods developed in sections 1.1 and 1.2 are related to the statistical properties of $p_i(l)$ in the limit of infinite number of points $(M \to \infty)$. There are other possible characterizations of the attractor, of course. We briefly dicuss one of these approaches, due to Badii and Politi [BP84, BP85], which can be easily used in a numerical analysis and allows to give a good description of the inhomogeneity of the attractor.

The method is based on the statistical analysis of the minimal distance $\delta_i(M)$ between $\boldsymbol{x}_i$ and the other $M-1$ points,

$$\delta_i(M) = \min_{j\neq i} |\boldsymbol{x}_i - \boldsymbol{x}_j| .$$

In the homogeneous case one has for each i

$$\delta_i(M) \propto M^{-1/D_{\mathrm{F}}} \tag{1.3.1}$$

but for a generic attractor we could expect that (1.3.1) does not hold. In order to compute D_{F} (or D_{I} and the other d_q) one therefore needs to perform suitable averages of $\delta_i(M)$. Let us introduce the moments of $\delta_i(M)$ and the 'dimension function' $D(\gamma)$:

$$\overline{\delta(M)^\gamma} = \frac{1}{M}\sum_{i=1}^{M} \delta_i(M)^\gamma \propto M^{-\gamma/D(\gamma)} . \tag{1.3.2}$$

It is clear that $D(\gamma)$ is a monotonic non-decreasing function. Moreover the scaling behaviour of the distribution of points on the attractor can be obtained by varying γ.

Badii and Politi [BP85] have in fact proved that the Renyi dimensions are related to the $D(\gamma)$ by:

$$D(\gamma = (1-q)d_q) = d_q . \tag{1.3.3}$$

We want finally to point out the problems which arise in the computation of the fractal dimension (or in general of the d_q's) from the analysis of the chaotic signals. Some practical difficulties are common to computer or real experiments where one always handles with finite time series, noise and so on. A typical puzzle is the choice of the 'meaningful range' (l_0, l_1) which has to be considered in order to fit the data. The noise level and the finite number of points play of course a role in the question, and the choice of a good scaling interval essentially follows by practical 'good sense'.

Moreover, the evolution equations are explicitly known only in computer experiments. On the contrary, in real experiments, one works with just few signals (usually one). The most relevant problem, at least from a conceptual point of view, is thus the construction of the points in phase space (which is in general infinite dimensional) by an experimental time series $u(1), u(2), \ldots$, corresponding to measurements regularly spaced in time ($t = \tau, 2\tau, \ldots$). One therefore has to introduce the points $y_i^{(m)}$ in R^m:

$$y_i^{(m)} = (u(i), u(i+1), \ldots u(i+m-1)) \tag{1.3.4}$$

for computing $\nu(m)$ defined by

$$\langle n_i^{(m)}(l) \rangle \propto l^{\nu(m)} \tag{1.3.5}$$

where $n_i^{(m)}(l)$ is given by (1.1.7) replacing $\boldsymbol{x}$ by $y^{(m)}$.

One expects to obtain ν in the limit of large values of m: $\nu(m) \to \nu$. The above method is not entirely justified from a mathematical point of view but it seems rather reasonable. It however gives the possibility of characterizing an experimental chaotic signal at least on heuristic grounds [MABD83, C85].

We shall see in section 3 that similar problems arise in the estimation of the Kolmogorov entropy from experimental data.

2. Intermittency in fully-developed turbulence

2.1. Basic concepts on fully-developed turbulence

It is well known that at low Reynolds numbers (R_e) an incompressible fluid behaves in a laminar way (i.e. roughly speaking the evolution is regular and stable). On the contrary, at very large Reynolds numbers there is a highly chaotic and irregular behaviour. The regime $R_e \gg R_{crit}$ is called fully developed turbulence where R_{crit} is the value of R_e at which the onset of turbulence appears, i.e. there is a transition from laminar to chaotic flow. We underline that for $R_e \gtrsim R_{crit}$ there are often only temporal chaos and highly spatial coherent structures. In the limit $R_e \gg R_{crit}$ the chaotic behaviour involves fluctuations on a so small scale of space and time that it seems possible only a description in terms of the statical properties of the flow. An idea of the increasing chaos with R_e is given by fig. 5.

Turbulent flows are very common in nature and they are of great interest in applications for their ability to transfer momentum or heat. Other relevant peculiarities of turbulent flows are unstability and unpredictability: a small perturbation at a certain time t_0 may rapidly lead to a strong distortion of the (unperturbed) flow pattern [MY75].

In principle one could build up the statistical mechanics of turbulence on the basis of the Navier–Stokes (N–S) equations:

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\dfrac{1}{\rho} \boldsymbol{\nabla} p + \nu\, \Delta \boldsymbol{u} + \boldsymbol{f} \\ \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \end{cases} \tag{2.1.1}$$
$$+ \text{ initial and boundary conditions}$$

where $\boldsymbol{u}$ is the velocity field, ρ the density, p the pressure, ν the kinetic viscosity and $\boldsymbol{f}$ an external force. Unfortunately, each analytical theory of turbulence, i.e. an approach which makes use of eq. (2.1.1),

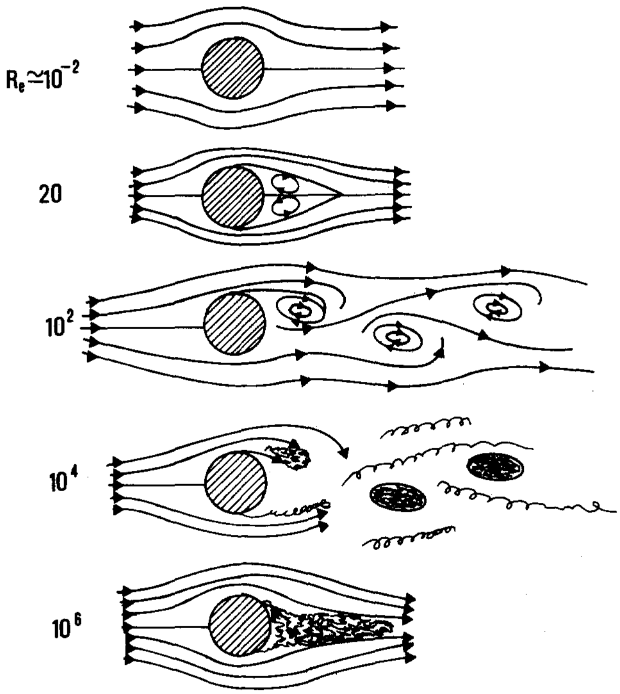

Fig. 5. Schematic pictures of the behaviour of a fluid around a cylinder at different Reynolds numbers.

unavoidably encounters closure problems. If, e.g., one tries to write down an equation for correlation of second order ($\langle u_i(x)\, u_j(y)\rangle$), then third-order correlation terms appear, and so on. These difficulties are typical of all non-linear equations. An analogous situation is present in the gas kinetic theory with the B-B-G-K-Y hierarchy [U59]. We do not consider in this review these approaches (for a general reference see [L73]).

However, an approximate understanding of a large number of statistical properties (at least on small scales) does not require a direct use of the N–S equations since the identification of the fundamental physical mechanism is sufficient.

Richardson [R22, R26] was the first who put forward some penetrating ideas on the physical mechanism acting at large R_e. In his assumption the fully developed turbulence consists essentially in a hierarchy of 'eddies' (i.e. turbulent structures or disturbances) on different scales. 'Eddies' of a certain scale are the result of the unstability of larger 'eddies' at a larger scale, and in their own turn, they generate smaller 'eddies' by their unstability. One has, in this scenario, a *cascade process* of eddy breaking-down in which there is a transmission of energy of the overall flow to motions of smaller and smaller eddies up to the smallest scale η where the fragmentation process is stopped by dissipation.

This physical picture is nicely expressed in the following rhyme ([R22], p. 66):

Big whorls have little whorls;
Which feed on their velocity;
And little whorls have lesser whorls;
And so on to viscosity
(In the molecular sense).

Let us recall that these considerations are related to the three-dimensional case since in the bidimensional situations a quite different phenomenology appears [PV86C] as discussed in section 2.5.

2.2. Kolmogorov theory and the intermittency problem

The qualitative and general ideas of Richardson have been further developed and formulated in a more precise language by Kolmogorov [K41].

Kolmogorov made an addition to the assumptions on the cascade process by noting that, because of the chaotic nature of the energy transfer among the eddies, the orienting effect of the mean flow must be weakened with each breaking down. Consequently it is natural to expect that at spatial scales much smaller than the external length L (i.e. the typical length of the mean flow) and time scales much smaller than the typical time of the mean flow, the velocity fluctuations are homogeneous, isotropic and quasi-steady. At sufficiently small-scale the turbulence is thus characterized by the mean flux of energy $\bar{\varepsilon}$ (from the overall flow to the smallest eddy) and by the dissipation. Moreover, if the scale length is not too small it is natural to assume that the viscosity plays no role, because the dissipation term in the N–S equations is negligible.

We can summarize all the above considerations in fig. 6. In a more quantitative way Kolmogorov formulated the two following hypotheses:

1) The n-variate probability distributions of the velocity difference $\Delta V(r) = \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})$ are universal isotropic function only of r, ν and $\bar{\varepsilon}$, in the case $r \ll L$.

2) If $L \gg r \gg \eta$ (the so-called inertial range) the probability distributions are independent of ν.

The two hypotheses lead immediately, by dimensional analysis, to an explicit form for the moments of $|\Delta V(r)|$ for r in the inertial range:

$$\langle |\Delta V(r)|^p \rangle \propto (\bar{\varepsilon} r)^{p/3} \tag{2.2.2}$$

where $\langle \cdot \rangle$ now denotes a spatial average. Moreover the dissipation length (i.e. the scale at which the dissipation is able to compete with the non-linear transfer) is

$$\eta = (\nu^3/\bar{\varepsilon})^{1/4} \propto R_e^{-3/4} L \,. \tag{2.2.3}$$

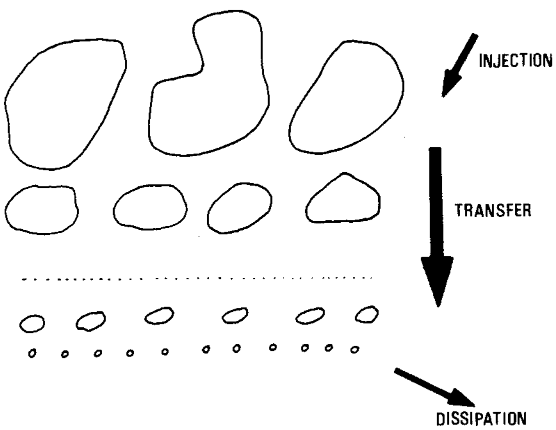

Fig. 6. Scheme of the energy cascade. The reader must think that successive eddies are embedded one within each other.

Independently also Onsager [O45, O49] obtained the same results as Kolmogorov.

Let us stress that the basic assumption in the K41 theory is that $\bar{\varepsilon}$ should be the only relevant parameter in the cascade process. This hypothesis is reasonable only if the energy transfer (or the energy dissipation density) $\varepsilon(\boldsymbol{x})$ does not strongly fluctuate with varying $\boldsymbol{x}$ and over a scale r which one is looking at. The assumption on the smooth behaviour of $\varepsilon(x)$ seems to be not satisfied since experiments [BT49] evidence strong intermittent bursts, both in space and time. For details on experimental studies see refs. [GM72, KC71, KC72, VP72]. Figure 7 obtained by a direct simulation of the N–S equations [S81] shows that $\varepsilon(x)$ is concentrated in a tiny region of the space.

The presence of intermittency, as first pointed out by Landau [LL71], leads to a contradiction in the K41; the statistical laws at small scales have to depend not only on $\bar{\varepsilon}$ but also on the fluctuations of $\varepsilon(\boldsymbol{x})$.

Experiments at large Reynolds number [GM72, VP72, AGHA84] show scaling laws in the inertial range:

$$\langle |\Delta V(\boldsymbol{r})|^p \rangle \propto r^{\zeta_p} \tag{2.2.4}$$

but with $\zeta_p \neq p/3$. The disagreement between the experimental values of ζ_p's and the K41 predictions is small for not too large $p(\lesssim 4\text{–}5)$ and increases with p. In the following we shall compare the experimental data with the estimates obtained by fractal and multifractal models.

These considerations on small-scale intermittency led Kolmogorov and Obukhov [K62, O62, Y66] to modify the K41. In their approach (called lognormal model) the fluctuations of energy dissipation are distributed according to a lognormal distribution. We discuss it in appendix A, since it is not directly relevant to our purpose and because of the peculiarity of the lognormal distribution [M72].

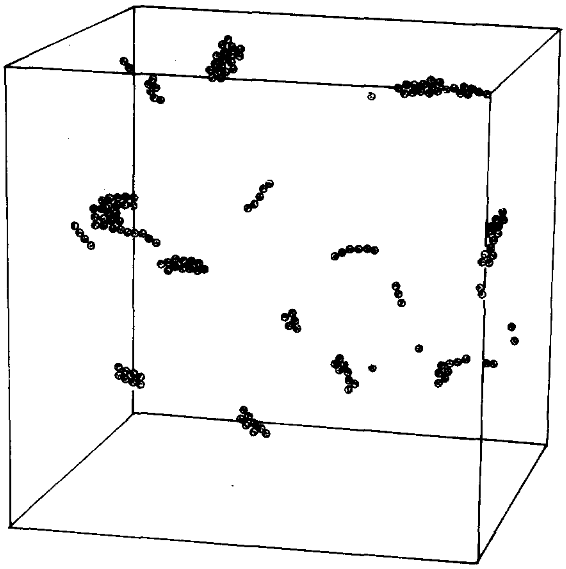

Fig. 7. Numerical simulation [S81] shows the structure of the zones containing the energy dissipation. 95% of the energy dissipation is concentrated in the dark regions.

2.3. Fractal and multifractal models for intermittency

The K41 theory assumes that each point $\boldsymbol{x}$ of the fluid has the same 'singularity' structure:

$$\Delta V_x(r) \propto r^h , \qquad h = \tfrac{1}{3} . \tag{2.3.1}$$

It is easy to see that (2.3.1) is equivalent to assume that $\varepsilon(\boldsymbol{x})$ is smoothly distributed in a region of R^3. Let us define the eddy turn-over time and the kinetic energy per unit mass at scale r:

$$t(r) \sim r/\Delta V(r) \tag{2.3.2}$$

$$E(r) \sim \Delta V(r)^2 . \tag{2.3.3}$$

The transfer rate of energy per unit mass from the eddy at scale r to smaller eddies is then given by

$$\tilde{\varepsilon}(r) = E(r)/t(r) \sim \Delta V(r)^3/r . \tag{2.3.4}$$

Since $\varepsilon_x(r) = (1/r^3) \int_{\Lambda_x(r)} \varepsilon(y)\, \mathrm{d}^3 y$ ($\Lambda_x(r)$ is a cube of edge r around $\boldsymbol{x}$), one has by (2.3.4) and (2.3.1)

$$\int_{\Lambda_x(r)} \varepsilon(y)\, \mathrm{d}^3 y \sim r^3 . \tag{2.3.5a}$$

A simple way to modify the K41 consists in assuming that the active turbulent structures cover a homogeneous fractal S (with $D_\mathrm{F} < 3$) on which $\varepsilon(x)$ is uniformly distributed.

Let us remark that in turbulence we do not use the word fractal in the exact mathematical sense. Indeed in the 'true' limit $r \to 0$ because of the dissipation one probably finds no singular structures. $r \to 0$ means r in the inertial range and the regions containing a large part of $\varepsilon(\boldsymbol{x})$ are a 'physical' approximation of a fractal structure. In this approach (called absolute curdling [M74] or β-model [FSN78]) one replaces (2.3.5a) with

$$\int_{\Lambda_x(r)} \varepsilon(y)\, \mathrm{d}^3 y \propto \begin{cases} r^{D_\mathrm{F}} & \text{if } \boldsymbol{x} \in \mathrm{S} \\ 0 & \text{if } \boldsymbol{x} \notin \mathrm{S} \end{cases} \tag{2.3.5b}$$

or in an equivalent way:

$$\Delta V_x(r) \propto \begin{cases} r^h & \text{if } \boldsymbol{x} \in \mathrm{S} \\ 0 & \text{if } \boldsymbol{x} \notin \mathrm{S} \end{cases} \tag{2.3.6}$$

with $h = (D_\mathrm{F} - 2)/3$. Since at scale r there is only a fraction

$$r^{3-D_\mathrm{F}} = \frac{\text{number of cubes of edge } r \text{ in S}}{\text{number of cubes of edge } r \text{ in the fluid}} \propto \frac{r^{-D_\mathrm{F}}}{r^{-3}}$$

occupied by the 'active' eddies, we have in the inertial range the following behaviour for the structure functions

$$\langle |\Delta V(r)|^p \rangle \propto r^{hp} r^{3-D_F} = r^{\zeta_p}$$
$$\zeta_p = \left(\frac{D_F - 2}{3}\right) \cdot p + (3 - D_F) . \tag{2.3.7}$$

In the limit $D_F = 3$, we recover the K41 results. In this approach D_F is a free parameter and cannot be obtained with simple arguments. In this approach $\varepsilon(\boldsymbol{x})$ is not distributed over all the fluid as in K41, since by (2.3.5b) one sees that $\varepsilon(\boldsymbol{x})$ has a singular structure. We shall discuss again this model in the following. In fig. 8 we report the experimental data [AGHA84] of ζ_p vs. p and the implications of the homogeneous fractal model (eq. (2.3.7)). A linear fit leads to a good agreement with the experimental data for $p \lesssim 7$, while for larger values of p one observes a non-linear behaviour.

The hypothesis of the existence of a whole spectrum of singularities allows to justify the non-linearity of ζ_p. The model can be improved by considering the set on which the energy dissipation is concentrated as a multifractal set, following the lines introduced in section 1 for chaotic attractors. Let us namely define S(h) as the set of points for which in the inertial range

$$\Delta V_x(r) \propto r^h$$

indicating with $d(h)$ the fractal dimension of S(h). It is easy to compute ζ_p noting that the fraction of cubes with edge r in the set S(h) is proportional to $r^{-d(h)}/r^{-3} = r^{3-d(h)}$. Therefore one gets

$$\langle |\Delta V(r)|^p \rangle \propto \int \mathrm{d}\rho(h)\, r^{ph}\, r^{3-d(h)} \propto r^{\zeta_p} \tag{2.3.8}$$

and by the steepest descent method:

$$\zeta_p = \min_h \{ph + 3 - d(h)\} . \tag{2.3.9}$$

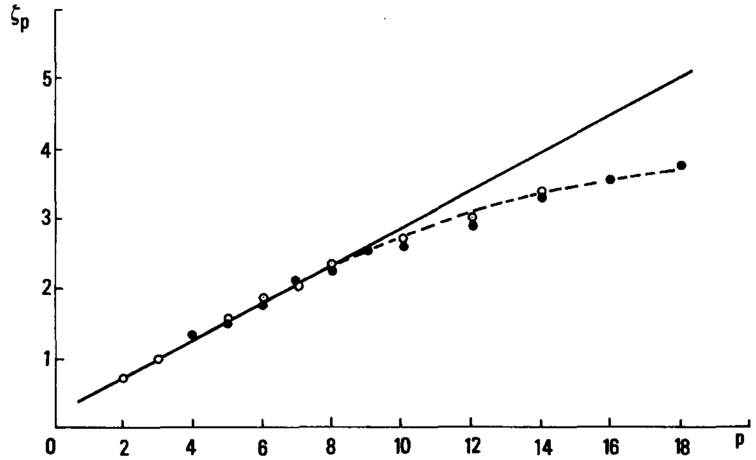

Fig. 8. ζ_p vs. p. Dots and circles are experimental data [AGHA84]; the full line is the β-model result (2.3.7) with $D_F = 2.83$ and the dashed line is the random β-model result (2.3.16) using the distribution (2.3.22) with $x = 0.125$.

Equation (2.3.9) shows that at a given value of p, ζ_p depends on a particular value of h. Hence the kind of instabilities needed to set up the sets S(h) are picked up by different moments.

We have worked out the multifractal approach in terms of singularities of $\Delta V_x(r)$ only because in the literature of turbulence the moments $\langle|\Delta V(r)|^p\rangle$ are usually involved. However it is simple to emphasize the explicit correspondence with chaotic attractors. Let us note that

$$\int_{\Lambda_x(r)} \varepsilon(y)\,\mathrm{d}^3y \propto r^3\,\tilde{\varepsilon}_x(r) \propto r^{3h+2} \tag{2.3.10}$$

so that the analogues of the exponent α and $f(\alpha)$ are $3h+2$ and $d(3h+2)$.

The homogeneous fractal case is the limit of the multifractal one when $d(h)$ is defined only for $h=(D_\mathrm{F}-2)/3$ and $d(h)=D_\mathrm{F}$.

Up to now we have remained at a rather descriptive level in the problem of intermittency. $d(h)$ contains all the relevant features, but cannot be obtained by a first principle calculation based on the N–S equation singularities.

2.3.1. *The β-model (absolute curdling)*

A pictorial scenario of the energy cascade can be given in terms of multiplicative processes (absolute and weighted curdling) [M74], see fig. 9. We think that the details of the model [FNS78] and of the random model [BPPV84] are useful to give a more direct idea of the fragmentation processes in turbulence. We want here to remind that all 'modern' fractal and multifractal cascade approaches have been originated by ad hoc models involving particular structures, as vortex sheets or vortex tubes, for the region containing the energy dissipation [C62, T68, S70]. The explicit idea of the β-model is due to Novikov and Stewart [NS64, N69, N70] and to Kraichnan [K74].

Let us consider the scales $l_n=2^{-n}l_0$, where $l_0=L$ is the scale at which energy is injected and the scaling factor 2 between l_n and l_{n+1} is conventional. Let us call $v_n=\Delta V(l_n)$ the typical velocity

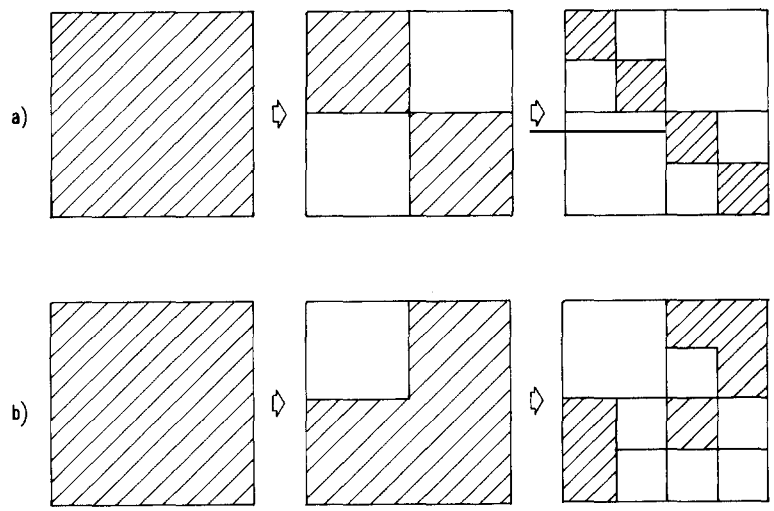

Fig. 9. Schematic view of the β-model (case a) and of the random β-model (case b). The dashed areas are the active zones during the fragmentation process.

difference across a distance l_n in an active eddy. In order to take into account the intermittency Frisch et al. [FSN78] introduced the coefficient $\beta = 2^{D_F-3}$ equal to the fraction between the volume of the daughter eddies at scale l_{n+1} and the volume of the mother eddy at scale l_n. The transfer energy from the eddy at scale l_n to that at scale l_{n+1} is

$$\varepsilon_n \propto v_n^3/l_n .$$

Since the energy transfer rate is constant in the cascade process one has

$$\varepsilon_n = \beta\varepsilon_{n+1} , \qquad v_n^3/l_n = \beta\, v_{n+1}^3/l_{n+1} . \tag{2.3.11}$$

Iterating (2.3.11) one then obtains

$$v_n \propto l_n^{1/3}(l_n/l_0)^{(D_F-3)/3} \tag{2.3.12}$$

i.e. eq. (2.3.6) with $h = (D_F - 2)/3$.

In the previous model one essentially has a two-valued multiplicative process: each eddy at scale l_n is divided into eddies of scale l_{n+1}, in such a way that the energy transfer for a fraction β of eddies increases by a factor $1/\beta$ while it becomes zero for the other ones.

2.3.2. *The random β-model* (*weighted curdling*)

Let us generalize the β-model. We namely assume that at scale l_n there are N_n active eddies, each eddy $l_n(k)$ (k labels the 'mother' eddy and $k = 1, \dots N_n$) generates active eddies covering a fraction of volume $\beta_{n+1}(k)$. Since the rate of energy transfer is constant among mother and daughters, we get:

$$v_n(k)^3/l_n = \beta_{n+1}(k)v_{n+1}(k)^3/l_{n+1} . \tag{2.3.13}$$

The iteration (2.3.13) of v_n gives an eddy generated by a particular 'history' of fragmentations $[\beta_1, \dots \beta_n]$:

$$v_n \propto l_n^{1/3}\Big(\prod_{i=1}^{n} \beta_i\Big)^{-1/3} . \tag{2.3.14}$$

Let us remark that the fraction of volume occupied by an eddy generated by $[\beta_1, \dots \beta_n]$ is $\Pi_{i=1}^n \beta_i$, so from eq. (2.3.14) it follows

$$\langle|\Delta V(l_n)|^p\rangle \propto l_n^{p/3} \int \prod_{i=1}^{n} \mathrm{d}\beta_i\, \beta_i^{(1-p/3)} P(\beta_1, \dots \beta_n) . \tag{2.3.15}$$

Assuming no correlation among different steps of the fragmentation, i.e. $P(\beta_1, \dots \beta_n) = \Pi_{i=1}^n P(\beta_i)$, one obtains for ζ_p

$$\zeta_p = p/3 - \ln_2\{\beta^{(1-p/3)}\} \tag{2.3.16}$$

where $\{\cdot\}$ stands for the average over the distribution $P(\beta)$. If β_i is a constant ($=2^{(D_F-3)}$) one recovers the results of the β-model. The knowledge of the probability distribution $P(\beta)$ is related to the understanding of the nature of the N–S equation singularities. However this problem is far to be solved at present. The fractal object generated by the random β-model (or more general a multifractal object) has no more global dilatation invariant properties; even if, one can still compute the fractal dimension D_F as:

$$\overline{N_n} \propto l_n^{-D_F} \tag{2.3.17}$$

where N_n is the number of active eddies at the nth step of the fragmentation and $\overline{(\cdot)}$ is an average over an ensemble of cascades. It is easy to show that

$$D_F = 3 - \zeta_0 \tag{2.3.18}$$

which in the random β-model is given by:

$$D_F = 3 + \ln_2\{\beta\} . \tag{2.3.19}$$

Moreover the analogue of d_2 is D^* defined by:

$$D^* = 1 + \zeta_6 \tag{2.3.20}$$

and in terms of the random β-model

$$D^* = 3 - \ln_2\{\beta^{-1}\} . \tag{2.3.21}$$

Let us recall that D^* is related to the scaling of energy dissipation density correlation [FSN78],

$$\langle \varepsilon(\boldsymbol{x}+\boldsymbol{r})\, \varepsilon(\boldsymbol{x}) \rangle \propto r^{3-D^*}$$

and that in the experimental literature the fractal dimension is usually estimated by D^* even if in general $D_F > D^*$. $D_F = D^*$ actually holds only for homogeneous fractals.

We have performed a simple fit of the experimental data choosing for $P(\beta)$ the form

$$P(\beta) = x\,\delta(\beta - 0.5) + (1-x)\,\delta(\beta - 1) ; \tag{2.3.22}$$

the value $x = 0.125$ gives good agreement with the experimental data (see fig. 8). There is no deep reason to choose the form (2.3.22); we have simply followed some phenomenological ideas considering two possible kinds of fragmentation: an active eddy can generate either vorticity sheets ($\beta = 0.5$) or space filling Kolmogorov-like eddies ($\beta = 1$). With the fit (2.3.22) we get $D_F = 2.91$ and $D^* = 2.83$, with a small but significant difference. There exist some physical arguments to give a bound to the values of h. In the set $S(h)$ the local Reynolds number at scale r is $R_e(r, h) = r \cdot r^h/\nu \propto r^{1+h}$. Therefore in order to stop the turbulent cascade one must require that $R_e(r, h)$ decreases with r implying $h > -1$. Let us note however, that it is reasonable to assume $h \geq 0$ because for negative h one could have points for which $|\Delta V_x(r)|$ increases when r decreases and this seems quite unrealistic. Note that in the distribution (2.3.22) we have taken $h_{\min} = 0$ (corresponding to $\beta_{\min} = \frac{1}{2}$).

Mandelbrot [M76b] conjectured, using Taylor's frozen turbulence assumption [T38], that fractal structure with a fractal dimension <2 cannot be detected in a real experiment.

However, Mandelbrot's remark is valid only in the (mathematical) limit $\eta = 0$ while in real turbulence, because of the existence of a finite minimal scale for the active region, one has just an approximation of the fractal object. The shape of $d(h)$ obtained with the fit (2.3.22) shown in fig. 10 is therefore not selfcontradictory.

The approaches to the statistical description of turbulence (K41, fractal, multifractal, lognormal) do not directly use the dynamics given by the Navier–Stokes (and Euler) equations. However, the previous phenomenological arguments as well as some closure approximations [AL77] imply that the N–S equations, in the limit of zero viscosity, lead to a singularity in a finite time (see, e.g., [F76]). For finite small ν we expect that the singularity is smoothed by the dissipative term. In our phenomenological model the analogue of the singularity is the hierarchy of eddies down to the dissipative scale. There are some numerical evidences for these singularities [MOF80, BMONMF84]. Perhaps the most relevant point is the 'fractal' spatial nature of these singularities. Their mathematical existence is unproven and controversial, but if they exist, then they must be enormously 'sparser' [P74, S76, K76] and their spatial 'fractality' is a plausible conjecture [M76a, M76b].

Recently some other non-fractal models for the description of intermittency have been proposed [SL84, NN85, N86]. They are alternative to the scenario of this section. We do not discuss them, but we want just to note that they imply $\zeta_p = p/3$ for $p < p_c = 5$–6.

We think that this is in strong disagreement with all experimental results. Even if one could think that for large values of $p(\gtrsim 8)$ the experimental data for ζ_p are not enough accurate in order to prefer

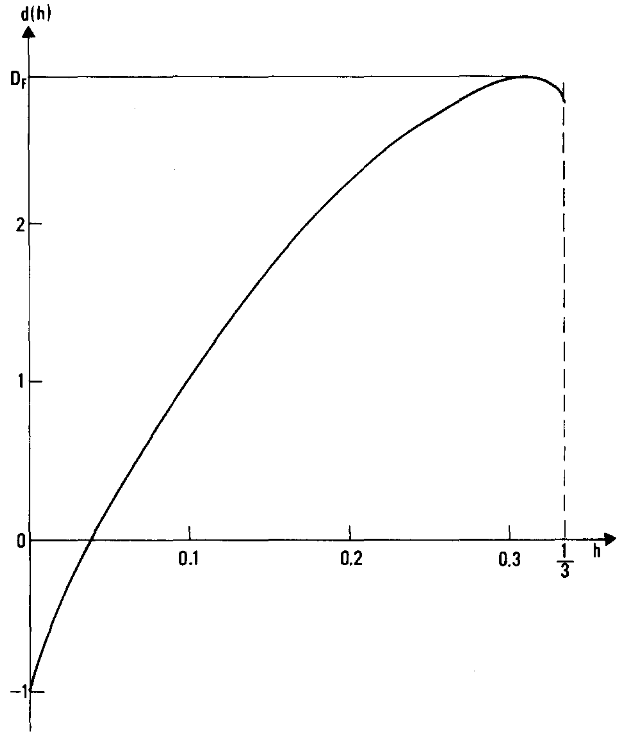

Fig. 10. $d(h)$ vs. h given by the Legendre transform of (2.3.16) using the fit (2.3.22) with $x = 0.125$.

definitively a model or another, all the recent experiments nevertheless give values for ζ_2 and ζ_4 respectively larger and smaller than 2/3 and 4/3 obtained by the K41 (e.g. $\zeta_2 = 0.70 \pm 0.01$).

2.4. Remarks and consequences of multifractality in turbulence

The concept of 'multifractal' is related to the properties of the distribution of 'mass' (in turbulence the density of energy dissipation, in a chaotic system the density of points on the attractor) and not to 'geometrical' properties [M86, BPPV86].

For example there is no anomalous scaling law for the number of active eddies N_n:

$$\overline{N_n^q} \sim (\overline{N_n})^q \propto l_n^{-D_F q} \tag{2.4.1}$$

where $\overline{(\cdot)}$ means the average of an ensemble of cascades. Equation (2.4.1) is a quite trivial consequence of the law of large numbers. For sake of completeness we consider the random β-model of section 2.3.2 where the number of active eddies at scale l_n is

$$N_n = 2^{3n} \prod_{i=1}^{n} B_i \tag{2.4.2}$$

with

$$B_i = \frac{1}{N_i} \sum_{k=1}^{N_i} \beta_i(k)\,. \tag{2.4.3}$$

In the limit of large n (i.e. $N_i \gg 1$) one has (large number theorem):

$$B_i = \{\beta\} \cdot (1 + \mathrm{O}(N_i^{-1/2}))$$

the probability distribution of N_n has therefore a very narrow peak around $2^{3n}\{\beta\}^n = l_n^{-D_F}$ so that (2.4.1) holds. Equation (2.4.1) can easily be proved for positive integer values of q by direct computation [BPPV86].

2.4.1. Irrelevance of multifractality for relative diffusion

Let us now show why the multifractal structure is not relevant for the relative diffusion of particle pairs [CPV87].

Indeed, the growth of the moments of the relative distance R between a pair of particles scales with time as:

$$\overline{R^{2q}} \propto t^{2q\,\nu(q)} \tag{2.4.4}$$

($\overline{(\cdot)}$ is now an average over a large number of pairs). We found that for each q, $\nu(q) = \nu$ and ν is only related to the exponent ζ_1:

$$\nu = 1/(1 - \zeta_1)\,. \tag{2.4.5}$$

Let us consider $M \gg 1$ pairs of particles at positions $\boldsymbol{r}_i^{(1)}$ and $\boldsymbol{r}_i^{(2)}$ $(i = 1, \ldots M)$. The interparticle distance is $\boldsymbol{R}_i = (\boldsymbol{r}_i^{(1)} - \boldsymbol{r}_i^{(2)})$ and the relative velocity $\delta \boldsymbol{V}_i = \boldsymbol{u}(\boldsymbol{r}_i^{(1)}) - \boldsymbol{u}(\boldsymbol{r}_i^{(2)}) = (\mathrm{d}/\mathrm{d}t)\boldsymbol{R}_i$. Let us compute the time derivative of $\overline{R^2}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{R^2} = 2 \lim_{M\to\infty} \frac{1}{M} \sum_{i=1}^{M} \delta \boldsymbol{V}_i(\boldsymbol{R}_i) \cdot \boldsymbol{R}_i$$

$$= 2 \lim_{M\to\infty} \frac{1}{M} \sum_{i=1}^{M} \delta V_i(\boldsymbol{R}_i)\, R_i \cos\theta_i \tag{2.4.6}$$

where θ_i is the angle between $\boldsymbol{R}_i$ and $\delta \boldsymbol{V}_i(\boldsymbol{R}_i)$. In the case of isotropic turbulence the relative positions and velocities are uncorrelated [MY75 section 24, C69, O70a], moreover $\cos\theta_i$ is positive and does not depend on R [C69, O70]. Therefore this one has

$$\mathrm{d}\overline{R^2}/\mathrm{d}t = 2\,(\overline{\cos\theta})\,\overline{\delta V(\boldsymbol{R})\,R}\,. \tag{2.4.7}$$

In order to compute $M^{-1} \cdot \Sigma_{i=1}^{M} \delta V_i(\boldsymbol{R}_i) R_i = \overline{\delta V(\boldsymbol{R})\, R}$ we group together all the $n(\alpha)$ pairs with the same value $R(\alpha)$ of the interparticle distance in such a way that one obtains:

$$\sum_{i=1}^{M} \delta V_i(R_i)\, R_i = \sum_{\alpha} \left[\frac{1}{n(\alpha)} \sum_k \delta V_k\right] R(\alpha)\,. \tag{2.4.8}$$

The sum Σ_α is over the set of different values of $R(\alpha)$ and Σ_k is over the $n(\alpha)$ pairs with $R_k = R(\alpha)$. Note that for large M (and $n(\alpha)$)

$$\frac{1}{n(\alpha)} \sum_k \delta V_k \simeq \langle |\Delta V(R(\alpha))| \rangle \propto R(\alpha)^{\zeta_1} \tag{2.4.9}$$

therefore

$$\mathrm{d}\overline{R^2}/\mathrm{d}t \propto \overline{R^{1+\zeta_1}}\,. \tag{2.4.10}$$

It is trivial to repeat the same computations for $\overline{R^{2q}}$:

$$\mathrm{d}\overline{R^{2q}}/\mathrm{d}t \propto \overline{R^{2q+\zeta_1-1}}\,. \tag{2.4.11}$$

Now it is easy to see that $\nu(q)$, as defined in (2.4.4), must be constant. In fact by (2.4.11) and (2.4.1) one has

$$2q\,\nu(q) - 1 = (2q - 1 + \zeta_1) \cdot \nu\left(\frac{2q - 1 + \zeta_1}{2}\right);$$

moreover $\nu(q)$ does not increase when q increases [F71], therefore one obtains:

$$\nu(q) = \nu = 1/(1 - \zeta_1)\,, \qquad \forall q\,.$$

Let us remark that the multifractal structure induces only a (slight) correction to the Richardson law

$\nu = \frac{3}{2}$ [R26] and in the diffusion process there is no anomalous scaling. The scaling laws (2.4.4, 2.4.5) are parametrized by a single exponent ζ_1; the same result holds for a homogeneous fractal [HP83b] (with $\zeta_1 = (7 - 2D_F)/3$) or a multifractal.

2.4.2. The problem of the number of degrees of freedom

It is clear that a satisfactory description of turbulent fluids needs a resolution up to scale of the same order of the dissipative Kolmogorov length η at which the molecular friction is able to compete with the non-linear transfer. One has:

$$\eta = (\nu^3/\varepsilon)^{1/4} \tag{2.4.12}$$

where ε is the rate of the energy dissipation for unit mass and time (assumed to be constant in K41).

If L is the system characteristic length at which the external energy input is pumped then the adimensional ratio $R_e = (\varepsilon L^4)^{1/3}/\nu$ is the Reynolds number. The number of grid points for unit volume necessary to obtain a resolution up to η is thus

$$N(R_e) \sim (L/\eta)^3 \propto R_e^{9/4}\,. \tag{2.4.13}$$

This argument (due to Landau and Lifschitz [LL71]) hides the central assumption that all the fluid is 'active', i.e. that the energy dissipation density field is smoothly distributed on a three-dimensional region.

Kraichnan [K85] has repeated the Landau–Lifschitz argument by making the hypothesis that the energy dissipation $\varepsilon(\boldsymbol{x})$ is concentrated on a homogeneous fractal with non-integer dimension $D_F < 3$. The dissipation scale η can be now determined by imposing that the Reynolds number related to an eddy of length scale l is of order one:

$$\eta\, \Delta V(\eta)/\nu \sim O(1)\,. \tag{2.4.14}$$

This is equivalent to require that the dissipative (linear) term of the N–S equation is able to compete with the non-linear transfer term.

Inserting eq. (2.3.6) in eq. (2.4.14) we obtain:

$$\eta \propto L/R_e^{1/(1+h)}\,. \tag{2.4.15}$$

It follows that:

$$N(R_e) \sim (L/\eta)^{D_F} \propto R_e^{3D_F/(1+D_F)}\,. \tag{2.4.16}$$

Let us remark that some other variables are also necessary for describing the non-active regions of the fluid but their number does not depend on R_e. If the β-model assumptions were correct, eq. (2.4.16) would give (in principle) the scaling law for $N(R_e)$.

The number of degrees of freedom necessary for describing the multifractal of turbulence must be defined with much more carefulness [PV87a]. In fact, for each singularity h a different dissipative length $\eta(h)$ is picked up by condition (2.4.15): $\eta(h) \propto R_e^{-1/(1+h)}$.

Since the number of eddies at scale l with singularity h is proportional to $l^{-d(h)}$, one deduces that the number of grid points which have to be considered for resolving the set $S(h)$ is:

$$N_h(R_e) \sim (L/\eta(h))^{d(h)} \propto R_e^{d(h)/(1+h)} . \tag{2.4.17}$$

We thus get the total number of degrees of freedom by integrating (2.4.17) over h:

$$N(R_e) = \int d\rho(h)\, N_h(R_e) \propto R_e^{\delta} \tag{2.4.18}$$

where δ can be estimated by the steepest descent method in the limit of large R_e

$$\delta = \max_h [d(h)/(1+h)] . \tag{2.4.19}$$

A fit of the experimental data [AGHA84] gives the value $\delta \simeq 2.2$ which is close to the value given by eq. (2.4.16).

The results (2.4.16) and (2.4.18, 19) are nevertheless quite different from a conceptual point of view.

We must stress that the estimate (2.4.18, 19) has just a theoretical relevance since it is rather difficult in a computer simulation to locate the grid points on the sets S(h) (which also evolves in time). Indeed one usually works with a fixed grid or with a pseudo-spectral method [PO71]. It follows that the only relevant parameter is the minimal scale $l_{\min}$ considered which is bounded from below by the dissipative length related to the strongest singularity:

$$l_{\min} \sim \eta(h_{\min}) \propto R_e^{-1/(1+h_{\min})} . \tag{2.4.20}$$

The estimate $l_{\min} = \eta(h_{\min})$ assures that all the sets S(h) (i.e. even very improbable events) are taken into account.

The number of equations which allows us to get such a fully accurate description is thus:

$$N_T^* \sim (L/l_{\min})^3 \propto R_e^{3/(1+h_{\min})} . \tag{2.4.21}$$

Equation (2.4.21) is in agreement with rigorous bounds [R82, CFMT85]. On the other hand, if one decides to neglect the rare events a resolution $\tilde{l} \gg l_{\min}$ is sufficient; just the relevant features of turbulence are reproduced loosing some details. In this case the number of equations is reduced to

$$\tilde{N}^* \sim (L/\tilde{l})^3 .$$

This scale $\tilde{l}$ can be estimated by the dissipative length $\eta(\tilde{h})$ related to an effective singularity $\tilde{h}$.

Let us define an 'effective' mass dimension $\tilde{D}$ of the object on which the energy dissipation is concentrated, by:

$$\tilde{h} = (\tilde{D} - 2)/3 .$$

Mandelbrot [M76b] has, e.g., assumed $\tilde{D} = D_1$, the information dimension, and from the data [AGHA84] one has $D_1 \simeq 2.87$. This assumption corresponds to select a $\tilde{h} = d\zeta_p/dp|_{p=3} \simeq 0.29$ and, roughly speaking, $\tilde{l}$ is thus the smallest scale on which in average active eddies are still present.

On the other hand, some heuristic arguments (see section 2.3) as well as the fit of experimental data shown in fig. 10 indicate $h_{\min} = 0$. It follows that:

$$N_{\mathrm{T}}^{*} \propto R_{\mathrm{e}}^{3} \qquad \text{and} \qquad \tilde{N}^{*} \propto R_{\mathrm{e}}^{3/(1+\tilde{h})} \propto R_{\mathrm{e}}^{2.3} .$$

Let us emphasize that N_{T}^{*} is much greater than $\tilde{N}^{*}$ which is close to the estimate of the number of degrees of freedom obtained respectively in K41, in the β-model and in the framework of the multifractal approach.

2.5. Two-dimensional turbulence

Two-dimensional fluids have a relevant interest, apart from a mathematical point of view, because they idealize geophysical phenomena in the atmosphere, oceans and magnetosphere and are a starting point for understanding these phenomena (see for a general review [KM80]).

The motion of fluids in two dimensions has many remarkable regularity properties which are essentially due to the fact that the vorticity of each fluid element is constant if viscosity and external forcing are absent. For this reason $\bar{\varepsilon}$ depends on viscosity and there is no forward energy cascade [O77].

Nevertheless, quite reasonable arguments [K67, B69] allow to repeat a K41-like approach by assuming the mean dissipation enstrophy $\bar{\eta} = -(\mathrm{d}/\mathrm{d}t)\langle(\mathrm{rot}\, \boldsymbol{u})^2\rangle$ as the relevant parameter (instead of $\bar{\varepsilon}$) and thus a forward cascade of enstrophy (instead of energy). One obtains by dimensional analysis, in the inertial range:

$$\langle |\Delta V(r)|^{p} \rangle \propto r^{\zeta_p} , \qquad \zeta_p = p \tag{2.5.1}$$

and for the energy spectrum

$$E(k) \propto k^{-3} . \tag{2.5.2}$$

There exist numerical evidences for an energy spectrum in strong disagreement with (2.5.2), i.e. $E(k) \propto k^{-\alpha}$, $\alpha \simeq -4$ to -6 [BLSB81, M84]. Some authors argued that this difference is related to intermittency [BLSB81].

Let us briefly show that in two-dimensional turbulence a K41-like theory gives the exact scaling for ζ_p (i.e. $\zeta_p = p$) which is not affected by the 'intermittency'.

One could naively think that either the fractal or multifractal approach can be repeated for describing the enstrophy cascade. This is not true because in the two-dimensional Euler equation, as consequence of the vorticity conservation for each fluid particle, one can prove [RS78]:

$$|\Delta V(r)| < \text{const.}\ r\, |\ln r| . \tag{2.5.3}$$

The inequality (2.5.3) also holds for the N–S equations so that

$$\zeta_p \geq p . \tag{2.5.4}$$

Moreover ζ_p must be convex [F71] and $\zeta_3 = 3$, in order to have a constant forward enstrophy cascade. All these constraints compel us to conclude that:

$$\zeta_p = p \tag{2.5.5}$$

which implies the apparently surprising result that the K41-like theory prediction $\zeta_p = p$ also holds in presence of intermittency.

It is easy to see that a two-dimensional β-model (or random β-model), whenever $\beta_n \neq 1$ gives a wrong result for the enstrophy cascade. Indeed if we repeat the considerations of section 2.3 for the three-dimensional case imposing a constant enstrophy transfer rate we get, instead of eq. (2.3.13):

$$v_n(k)^3/l_n^3 = \beta_{n+1}(k)\, v_{n+1}^3/l_{n+1}^3 \,. \tag{2.5.6}$$

This equation gives a convex function ζ_p (if $\beta_n(k) \neq 1$). The apparition of singularities in the velocity field follows by the Bernoullian nature of the fragmentation process (β_n is independent by β_{n-1}).

However, the random β-model can be modified assuming that

$$\beta_{n+1} = 1 \qquad \text{if } (v_n/l_n)^3 > \eta_{\max} \,. \tag{2.5.7}$$

Equation (2.5.7) corresponds in some sense to a "Markovian" assumption in the fragmentation model because the steps of the cascade are no longer independent. The constraint (2.5.3) is now satisfied in this Markovian random β-model, but the enstrophy is concentrated in regions with fractal dimension equal to 2, which cover a non-decreasing area for decreasing scale length.

Let us note that unlike the three-dimensional case, we can reach no conclusions on the shape of $E(k)$ from ζ_2. The naive dimensional counting gives $E(k) \propto k^{-\alpha}$ with $\alpha = 1 + \zeta_2$.

This is wrong if $\zeta_2 \geq 2$ [BBS84] and one can just derive from the bound (2.5.3):

$$\alpha \geq 3 \,. \tag{2.5.8}$$

Therefore, neither the α value nor the structure functions (since $\zeta_p = p$) give us information about the 'intermittency' in two-dimensional turbulence. Numerical experiments [BPPSV86] confirm that the fragmentation is space-filling on the small scales (but larger than the viscosity dissipation ones). The intermittency should be regarded as a somewhat "macroscopic" phenomena related to coherent structures.

The enstrophy cascade indeed seems to be inhibited in some highly organized structures which dominate the energy spectrum. Moreover the turbulent field seems to be decomposed in two parts: a background with an energy spectrum k^{-3} in the inertial range and a finite number of vortices (coherent structures) which advect the background field.

3. Temporal intermittency in chaotic dynamical systems

3.1. General remarks

One of the relevant features of chaotic systems consists in their unpredictability. Essentially one observes that nearby trajectories diverge exponentially in time. However, there exist time variations of the 'chaoticity' which appear in all generical situations and consequently the mean exponential growth of the uncertainty on the initial state does not exhaust all the typical behaviours.

One can, e.g., observe a regular motion in phase space for long times, interrupted by randomly distributed bursts of strong chaoticity. This phenomenon, called temporal intermittency, plays an important role. For example intermittency has been shown to be [PM80] one of the fundamental mechanisms for the transition to turbulence.

A quite simple example is the one-dimensional map $x_n = g_\varepsilon(x_{n-1})$ where $g_\varepsilon(x)$ has for $\varepsilon = 0$ a tangent contact with the line $x_n = x_{n-1}$. For small ε the state x_n spends a large number of iterations ($\sim\varepsilon^{-1/2}$) in the 'laminar' phase, i.e. near the bisectrix. Nevertheless x_n is finally expelled out and a chaotic burst destroys correlations, see fig. 11. This behaviour is displayed also by non-trivial dynamical systems as the Lorenz model at $r \gtrsim r_c \simeq 166.07$, where there are regular oscillations interrupted by randomly-distributed bursts which become more and more frequent as r increases.

We shall use the word 'intermittency' in a broader sense than that used in the transition to stochasticity via tangent bifurcation. Even weak variations of the chaoticity degree are included in the class of intermittent behaviours.

Let us remark that quantities as the Lyapunov exponents and the Kolmogorov entropy cannot give a characterization of the intermittency. Indeed they give 'global' indications on the mean exponential divergence of nearby trajectories and cannot measure the variations of the 'chaoticity degree' along a given trajectory.

Our purpose is to characterize the intermittency degree in the same way the Lyapunov exponents and the Kolmogorov entropy do for the 'global' chaoticity giving a quantitative criterion which allows to discriminate between weak and strong intermittency.

We want to show how to reach these goals by means of a generalization of the Lyapunov exponents and of the Kolmogorov entropy which leads to introduce sets of exponents quite analogous to the Renyi dimensions.

The intermittency will then appear as a manifestation of the multifractality with regard to the time dilations in the trajectory space. This approach stresses the link between chaotic dynamic systems and the equilibrium statistical mechanics, relating the new sets of exponents to a kind of free energy. We can thus pick out the analogue of the thermodynamical state functions extending to a first rough level the rigorous results of Bowen, Ruelle, Sinai and Walters.

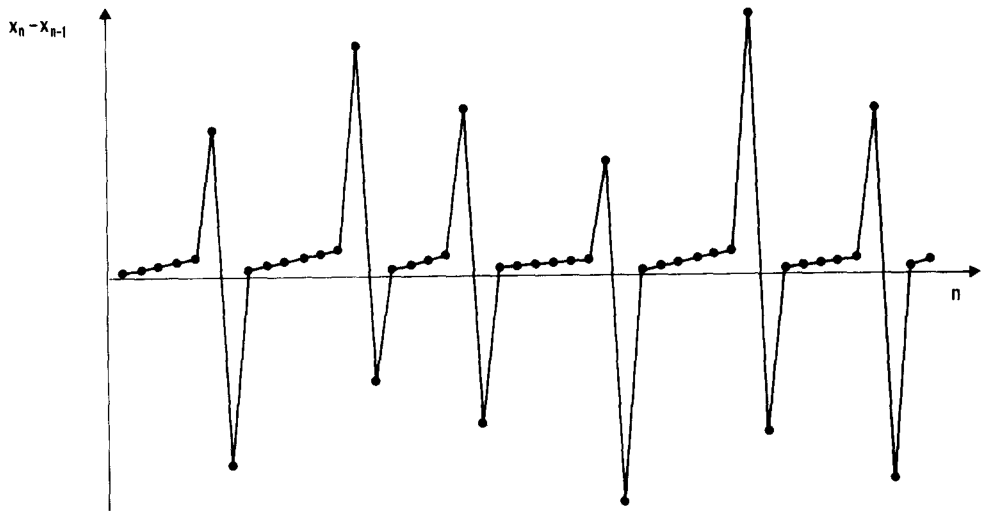

Fig. 11. $x_n - x_{n-1}$ vs. n for a map near the tangent bifurcation. The values close to zero are related to iterations inside the 'channel' and the peaks are related to chaotic bursts when x_n goes out of the 'channel'.

3.2. Generalized Lyapunov exponents

A practical tool for characterizing the global degree of chaos in dynamical systems consists in considering the set of Lyapunov exponents because they can be computed by means of simple algorithms [BGS76, BGGS80]. Moreover they are linked to other stochasticity indicators like the Kolmogorov entropy [P77] and the information dimension [KY78].

Let us define the spectrum of the Lyapunov exponents $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_F$ of the flow $T: \mathrm{R}^F \to \mathrm{R}^F$ generated by the set of differential equations $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ by considering the linear evolution of the tangent vector $\boldsymbol{\zeta}$:

$$\dot{\zeta}_i = \sum_{j=1}^{F} J_{ij} \zeta_j \tag{3.2.1}$$

where J_{ij} is the matrix $\partial f_i / \partial x_j|_{\boldsymbol{x}(t)}$.

One can introduce the Lyapunov exponents also for a discrete map $\boldsymbol{x}(n) = \boldsymbol{g}(\boldsymbol{x}(n-1))$ by considering the corresponding evolution for $\boldsymbol{\zeta}(n)$:

$$\zeta_i(n) = \sum_{j=1}^{F} A_{ij}(n-1)\, \zeta_j(n-1)$$

with $A_{ij}(n) = \partial g_i / \partial x_j|_{\boldsymbol{x}(n)}$.

Oseledec [O68] proved that for almost all initial conditions $\boldsymbol{x}(0)$ there is a basis $\{\hat{\boldsymbol{e}}_i\}$ in R^F such that:

$$\boldsymbol{\zeta}(t) = \sum_{i=1}^{F} |\boldsymbol{\zeta}(0)|\, c_i\, \hat{\boldsymbol{e}}_i \exp(\lambda_i t) \tag{3.2.2}$$

for large enough times.

Roughly speaking, (3.2.2) tells us that in the phase space a sphere of radius ε and centre $\boldsymbol{x}(0)$ is deformed with time into an 'ellipsoid' of semi-axes $\varepsilon_i(t) = \varepsilon \exp(\lambda_i t)$ directed along the $\hat{\boldsymbol{e}}_i$ vectors.

In the next section we shall show that the Oseledec theorem corresponds to the existence of a thermodynamic limit of infinite volume in a statistical mechanics language.

The positive maximal Lyapunov exponent λ_1 measures the growing of an error on the initial condition knowledge. A small incertitude $\delta\boldsymbol{x}(0)$ is exponentially amplified along $\hat{\boldsymbol{e}}_1$ with characteristic time λ_1^{-1}. To be more specific eq. (3.2.2) implies that for $\lambda_1 > \lambda_2$:

$$\delta\boldsymbol{x}(t) \sim |\delta\boldsymbol{x}(0)|\, \hat{\boldsymbol{e}}_1 \exp(\lambda_1 t)\, [1 + \mathrm{O}(\exp\{-(\lambda_1 - \lambda_2)t\})]\,.$$

This relation leads us to introduce the response R to a perturbation in $\boldsymbol{x}(\tau)$ after a time t by the error growth rate:

$$R_\tau(t) \equiv \|\boldsymbol{\zeta}(t+\tau)\| / \|\boldsymbol{\zeta}(\tau)\| \approx |\delta\boldsymbol{x}(t+\tau)| / |\delta\boldsymbol{x}(\tau)|\,. \tag{3.2.3}$$

The maximal Lyapunov exponent can be defined by averaging the logarithm of the response over the possible initial conditions along the trajectory:

$$\lambda_1 = \lim_{t\to\infty} \frac{1}{t} \langle \ln R(t) \rangle \tag{3.2.4}$$

where $\langle \cdot \rangle$ denotes $\lim_{T\to\infty} (1/T) \int_\tau^{\tau+T} \cdot \, \mathrm{d}t$.

In the same way we can define the other Lyapunov exponents by the divergence of a small n-dimensional point volume in phase space.

Let us namely consider n different tangent vectors $\zeta^{(1)}, \zeta^{(2)}, \ldots \zeta^{(n)}$. The rate of an n-dimensional volume is then measured by an n order response $R^{(n)}$ defined as:

$$R_\tau^{(n)}(t) = \frac{\|\zeta^{(1)}(t+\tau) \wedge \zeta^{(2)}(t+\tau) \wedge \cdots \wedge \zeta^{(n)}(t+\tau)\|}{\|\zeta^{(1)}(\tau) \wedge \zeta^{(2)}(\tau) \wedge \cdots \wedge \zeta^{(n)}(\tau)\|} \tag{3.2.5}$$

where $\wedge$ denotes the vectorial product.

The sum of the first n Lyapunov exponents is [BGGS80]:

$$\sum_{i=1}^{n} \lambda_i = \lim_{t\to\infty} \frac{1}{t} \langle \ln R^{(n)}(t) \rangle \ . \tag{3.2.6}$$

Let us also remark that, in the case of continuous flow, at least one of the Lyapunov exponents has to be zero since $\zeta(t)$ cannot grow exponentially in time along the direction tangent to the flow.

3.2.1. *Characterization of intermittency*

The Lyapunov exponents do not describe the degree of intermittency because of their global character. Equation (3.2.4), e.g., defines the average of the characteristic time scale on which correlations are lost but it does not give any further information about the fluctuations around this average. Indeed, we must still consider a non-uniform distribution in time of the 'chaoticity degree', i.e. of the response R.

The reconstruction of the probability distribution of R can be achieved by the analysis of the moments $\langle R^q \rangle$.

Let us therefore introduce the function [F83, BPPV85]:

$$L(q) = \lim_{t\to\infty} \frac{1}{t} \ln\langle R(t)^q \rangle \ . \tag{3.2.7}$$

$L(q)$ is called generalized Lyapunov exponent (of order q) since:

$$\lambda_1 = \mathrm{d}L/\mathrm{d}q|_{q=0} \tag{3.2.8}$$

and in the absence of fluctuations

$$L(q) = \lambda_1 q \tag{3.2.9}$$

while in the general case $L(q)$ is concave in q [F71].

The deviations from this linear law give a first rough indication on the intermittency degree in the same way that the set of $\phi(q)$'s is used to characterize the spatial inhomogeneity with regard to the point density (see section 1.1).

Equation (3.2.9) corresponds to long range correlations which inhibit fluctuations. However if the correlations are weak enough the response at time $N\,\Delta t$ can be considered as the product of independent random variables $R_i = R_{\tau+i\cdot\Delta t}(\Delta t)$:

$$R_\tau(t = N\,\Delta t) = \prod_{i=0}^{N-1} R_i(\Delta t)\,. \tag{3.2.10}$$

As consequence of the central limit theorem, the probability distribution of R_τ is therefore well approximated by the lognormal:

$$\mathcal{P}(R) = \frac{1}{(2\pi\mu t)^{1/2}}\,\frac{1}{R}\exp\left\{-\frac{(\ln R - \lambda_1 t)^2}{2\mu t}\right\} \tag{3.2.11}$$

where $\mu = \lim_{t\to\infty} t^{-1}[\langle \ln R(t)^2\rangle - (\lambda_1 t)^2]$.

If (3.2.11) is exact then the generalized Lyapunov exponents are:

$$L(q) = \lambda_1 q + \tfrac{1}{2}\mu q^2\,. \tag{3.2.12}$$

It is thus natural to conjecture that, in the general case, $L(q)$ is bounded between the linear form (3.2.9) (strong correlations) and the parabolic one (3.2.12) (weak correlations).

Let us however recall that at large q the moments deviate from (3.2.12) even if the lognormal is a good approximation because of the pathologies of this distribution (see appendix B).

In (3.2.11) the fluctuations are fully characterized by the second cumulant μ and it is easy to show that the value $\mu/\lambda_1 = 1$ delimits the borderline between weak and strong intermittency. To be more specific let us remark that $\mathcal{P}(R)$ reaches its maximum for

$$\tilde{R}(t) = \exp(\lambda_1 t(1 - \mu/\lambda_1))\,. \tag{3.2.13}$$

It follows that for large times:

$$\begin{aligned} &\tilde{R}\to 0 \quad \text{if } \mu/\lambda_1 > 1 \\ &\tilde{R}\to \infty \quad \text{if } \mu/\lambda_1 < 1\,. \end{aligned} \tag{3.2.14}$$

The equations (3.2.14) give, in the phase transition jargon, a mean field result and it is interesting to analyse the effects of the corrections. For $\mu/\lambda_1 < 1$ the fluctuations can be neglected at a first level since they just slightly modify the characteristic time on which the 'average' response diverges from $\lambda_1^{-1}(1 - \mu/\lambda_1)^{-1}$ to λ_1^{-1}.

On the contrary, the mean field picture fully breaks down for $\mu/\lambda_1 > 1$ where it predicts a 'laminar' stable phase ($\tilde{R}\to 0$) instead of the 'turbulent' chaotic one characterized by a positive maximal Lyapunov exponent.

We have performed a series of numerical computations of $L(q)$ for the Henon–Heiles model [HH64], the Henon map [H76] and the Lorenz system [L63] (see fig. 12). The Henon–Heiles model is consistent with $L(q)$ as done by eq. (3.2.12) with μ/λ_1 near 1. On the contrary, the Henon map and the Lorenz model for r near 166.07 (the critical value of intermittency transition to turbulence, see [PM80]) show strong deviations from the lognormal prediction.

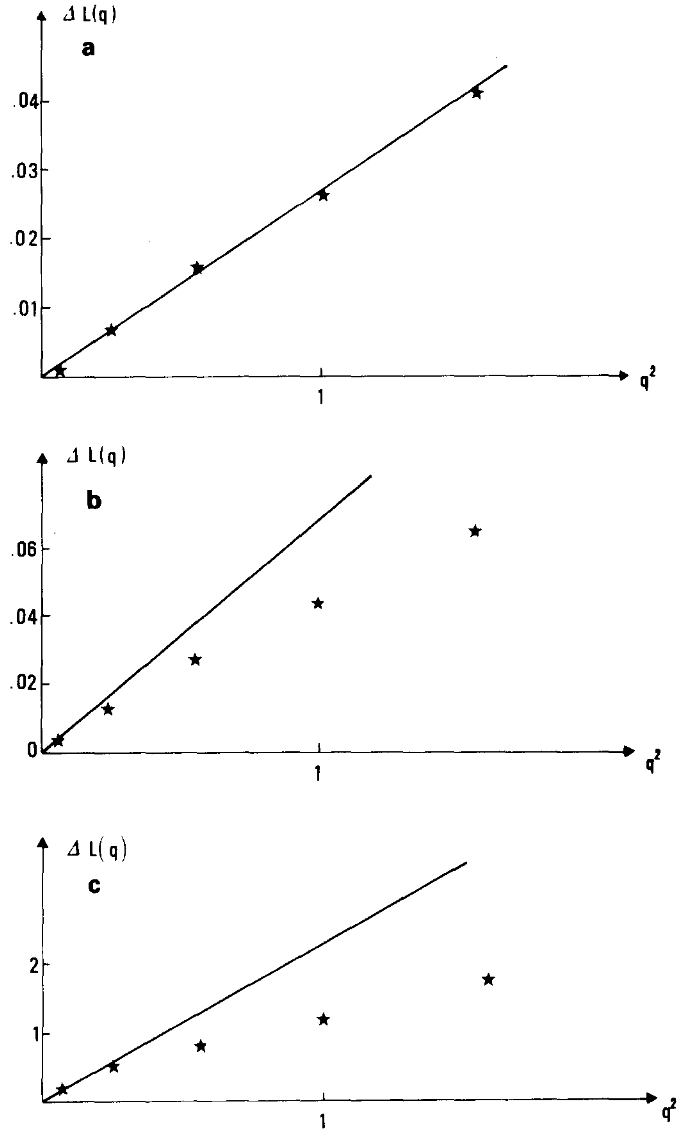

Fig. 12. $\Delta L(q) = L(q) - \lambda_1 q$ vs. q^2; (a) for the Henon–Heiles model [HH64] with initial condition on the chaotic region at the energy surface $E = 0.125$; (b) Henon map [H76] with $a = 1.2$ and $b = 0.3$; (c) Lorenz model [L63] with $r = 166.3$. The full line indicates the parabolic approximation $\Delta L(q) = (\mu/2)q^2$.

3.2.2. Conceptual relevance of the intermittency in the definition of chaos

The introduction of the set of the generalized Lyapunov exponents is relevant from a conceptual point of view since they allow to clear out when a system is chaotic. Indeed, it is commonly stated that a system with negative λ_1 is stable under small perturbations. The relation (3.2.7) evidences that even in this case small perturbations might cause large responses with finite probability: if $L(q)$ are positive for $q > q_c$. It is clear that the smaller is q_c, the larger is the probability of having chaotic bursts in the framework of a regular behaviour. A simple example is useful to stress this important point.

Let us compute $L(q)$ in the case of the Langevin equation:

$$\dot{x} = -\mathrm{d}V/\mathrm{d}x + \eta \tag{3.2.15}$$

where η is a white noise, i.e. a random Gaussian process with zero average and covariance:

$$\langle \eta(t)\, \eta(t') \rangle = \delta(t - t') . \tag{3.2.16}$$

We consider two trajectories $x(t)$ and $\hat{x}(t) = x(t) + \varepsilon(t)$, both satisfying eq. (3.2.15) with the same realization of noise. We define:

$$R(x(0), \tau|\eta) = \lim_{\varepsilon(0)\to 0} \frac{|\varepsilon(\tau)|}{|\varepsilon(0)|} \tag{3.2.17}$$

which is of course a functional of η. The function $L(q)$ is then:

$$L(q) = \lim_{\tau\to\infty} \frac{1}{\tau} \ln \int \mathrm{d}[\eta]\, P[\eta]\, R^q(x(0), \tau|\eta) \tag{3.2.18}$$

where $P[\eta]$ is the Gaussian probability distribution functional. Let us remark that the definition of the Lyapunov exponents for stochastic equations is a subtle mathematical point; we have here followed the naive standard procedure (see e.g. [CFU82]). Usual arguments imply that $L(q)$ does not depend on $x(0)$. Our aim is to evaluate $L(q)$. We first notice that:

$$R(x(0), \tau|\eta) = \exp\left[-\int_0^\tau V''(x(t))\, \mathrm{d}t\right] \tag{3.2.19}$$

where the dependence on η comes through the dependence of $x(t)$ on η. Let us evaluate:

$$\langle R^q(\tau) \rangle = \int \mathrm{d}\mu[x]^\tau \exp\left(-q \int_0^\tau V''(x(t))\, \mathrm{d}t\right) \tag{3.2.20}$$

where $\mathrm{d}\mu[x]^\tau$ is the measure induced on the trajectories by the stochastic differential equation (3.2.15). It is well known that (see for example [G78]):

$$\begin{aligned} \mathrm{d}\mu[x]^\tau &= \exp\left(-\int_0^\tau \mathrm{d}t\, \{\tfrac{1}{2}\dot{x}(t)^2 + \mathcal{U}(x(t))\}\right) \mathrm{d}[x]^\tau \\ &= \exp\left(-\int_0^\tau \mathcal{U}(x(t))\, \mathrm{d}t\right) \mathrm{d}P[x]^\tau \end{aligned} \tag{3.2.21}$$

where $\mathrm{d}P[x]^\tau$ is the usual Wiener measure on the trajectories and we have neglected boundary terms (i.e. terms depending on $x(0)$ and $x(\tau)$). The function $\mathcal{U}$ is given by:

$$\mathcal{U}(x) = \tfrac{1}{2}(\mathrm{d}V/\mathrm{d}x)^2 - \tfrac{1}{2}\,\mathrm{d}^2V/\mathrm{d}x^2\,. \tag{3.2.22}$$

The Feynman path integral representation for quantum mechanics at imaginary time implies that [FH63]:

$$\int \mathrm{d}\mu[x]^\tau \xrightarrow[\tau\to\infty]{} \exp(-\tau\, E_0(\mathcal{U})) \tag{3.2.23}$$

where we have neglected the prefactor and $E_0(\mathcal{U})$ is the ground state of the Hamiltonian

$$\hat{H} = -\tfrac{1}{2}\,\mathrm{d}^2/\mathrm{d}x^2 + \mathcal{U}(x)\,. \tag{3.2.24}$$

Consistency requires that $E_0(\mathcal{U}) = 0$, which is indeed trivial to check. We can now write:

$$\begin{aligned}\langle R^q(\tau)\rangle &= \int \mathrm{d}[x]^\tau \exp\Big(-\int_0^\tau \mathrm{d}t\,[\tfrac{1}{2}\dot{x}(t)^2 + \mathcal{U}(x) + q\,V''(x)]\Big)\\ &\propto \exp(-\tau E_0(q))\end{aligned} \tag{3.2.25}$$

where $E_0(q)$ is the ground state of the Hamiltonian

$$\hat{H}(q) = -\tfrac{1}{2}\,\mathrm{d}^2/\mathrm{d}x^2 + \mathcal{U}(x) + q\,V''(x)\,. \tag{3.2.26}$$

It is clear that

$$L(q) = -E_0(q)\,. \tag{3.2.27}$$

Equation (3.2.27) gives a method to compute $L(q)$ in an approximate way. A few exact solutions are available: for example if $V = x^2$ we have $L(q) = -2q$. In the general case we remark that $\lim_{q\to\infty} L(q)/q = -L^*$ is easy to compute and it is given by $L^* = \min_x V''(x)$. This means that, although $L(q)$ can be negative for small q and the system is stable in the usual sense, the system can be unstable under a small perturbation as soon as $V''(x)$ is negative somewhere as previously discussed.

We remark that this situation is somehow similar to what happens in the case of a multifractal set in both fully developed turbulence and chaotic attractors: the strongest singularity dominates the behaviour of the high moments for the structure functions [BPPV84]; for a similar case in different context see [B77, B82].

3.2.3. Generalized Lyapunov exponents of higher order

A more accurate description of the chaoticity in a dynamical system can be achieved by taking into account the fluctuations in the divergence of volumes in phase space under the dynamics. Toward this goal, we can define a set of generalized Lyapunov exponents of order n which generalize the relation (3.2.7) [PV86b]:

$$L^{(n)}(q) = \lim_{t\to\infty} \frac{1}{t} \ln\langle R^{(n)}(t)^q\rangle \tag{3.2.28}$$

where $L^{(1)}(q) = L(q)$. It is then easy to verify that:

$$\sum_{i=1}^{n} \lambda_i = \left.\frac{\mathrm{d}L^{(n)}}{\mathrm{d}q}\right|_{q=0} . \tag{3.2.29}$$

The whole spectrum of the Lyapunov exponent $\{\lambda_1, \lambda_2, \ldots \lambda_F\}$ has now found its counterpart but it seems rather artificial to distinguish the contribution of each eigendirection $\hat{e}_i$ in the fluctuations of the exponential divergence of an n-dimensional volume. One is of course tempted to define:

$$l_i(q) = \frac{L^{(i)}(q) - L^{(i-1)}(q)}{q} \tag{3.2.30}$$

with $F \geq i \geq 2$ and $l_1(q) = L(q)/q$.

This definition is however justified only by the trivial fact that $l_i(0) = \lambda_i$. Moreover some preliminary results [G86a] indicate that it cannot be correct for systems which are not hyperbolic where the stable and unstable manifolds can mix each other under the dynamics. The measurable quantities are just the $L^{(n)}$ and we can repeat for them all the considerations done for $L(q)$.

We want to stress that $L^{(n)}(q)$ as well as $L(q)$ cannot be extracted by an experimental signal while they can be easily computed from the numerical analysis of a dynamical system evolution using the methods developed by Benettin et al. [BGGS80] even if some precautions must be taken in order to avoid computer overflows due to the fast divergence of R^q.

3.3. The Renyi entropies

The Kolmogorov entropy is related to the sum of the positive Lyapunov exponents which measure the divergence rate along the expanding directions. Indeed Pesin [P77] proved that for an ergodic measure with a compact support

$$K_1 \leq \sum_{i=1}^{p} \lambda_i \tag{3.3.1}$$

where p is the number of exponents $\lambda_i > 0$. In many interesting situations, for example the Hamiltonian systems, (3.3.1) becomes the identity [ER85]:

$$K_1 = \sum_{i=1}^{p} \lambda_i = \left.\frac{\mathrm{d}L^{(p)}}{\mathrm{d}q}\right|_{q=0} \tag{3.3.2}$$

which we shall assume valid in the following.

On the other hand, temporal intermittency corresponds to fluctuations in the predictability time which can be described by a set of generalized entropies called in the information theory Renyi entropies [R70].

Let us give their definition in a simple constructive way which can be made more rigorous without excessive difficulties, see e.g. [CP85]. Take a record of measures of a signal $x(t)$ at uniform spacing τ:

$$x_i = x(i\tau) \qquad \text{with } i = 1, 2, \ldots M \gg 1 \tag{3.3.3}$$

considering, as usual, a partition of the phase space R^F into a grid of boxes of size l. A trajectory is thus specified up to a time τd with resolution l and τ by a sequence of boxes $i_1, i_2, \ldots i_d$ (see fig. 13). The joint probability $P_l(i_1, i_2, \ldots i_d)$ that $\boldsymbol{x}_1$ falls into i_1, $\boldsymbol{x}_2$ into i_2, and so on, defines a 'mass density' in the trajectory space R^{Fd}, whose moments give a possible characterization of intermittency. The Renyi entropy of order q is:

$$K_{q+1} = -\lim_{\tau\to 0} \lim_{l\to 0} \lim_{d\to\infty} \frac{1}{\tau d q} \ln\langle P_l^q \rangle \tag{3.3.4}$$

where the average is now taken over the different histories of the system:

$$\langle \cdot \rangle = \sum_{\{i_1, i_2, \ldots i_d\}} (\cdot) P_l(i_1, i_2, \ldots i_d) . \tag{3.3.5}$$

It is simple to show [F71] that $K_q \leq K_{q'}$ if $q > q'$ and that K_q is constant in absence of intermittency. The Kolmogorov entropy is obtained in the limit case of vanishing q:

$$K = -\lim_{\tau\to 0} \lim_{l\to 0} \lim_{d\to\infty} \frac{1}{\tau d} \langle \ln P_l \rangle = \lim_{q\to 1} K_q . \tag{3.3.6}$$

K gives the average loss of information for unit time which is independent of l. On the other hand, K_0 is the topological entropy h_T:

$$h_T = \lim_{\tau\to 0} \lim_{l\to 0} \lim_{d\to\infty} \frac{\ln \mathcal{N}_d}{d\tau} = K_0 \tag{3.3.7}$$

where $\mathcal{N}_d$ is the number of different sequences $\{i_1, \ldots i_d\}$ which have to be considered up to a time τd knowing the initial state of the system with resolution l and τ. h_T is a topological invariant [ER85] which does not require the introduction either of a metric or of a measure in the trajectory space at difference with the other entropies K_q.

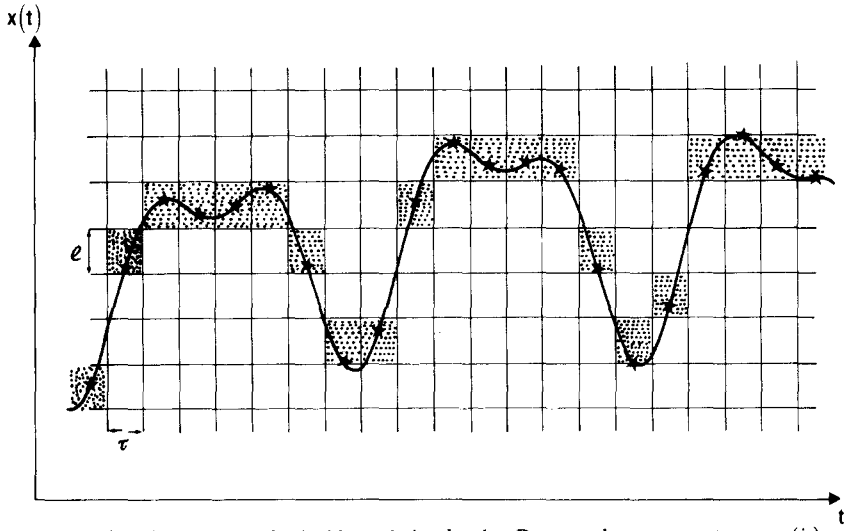

Fig. 13. Partition of 'space-time' with resolution l and τ. Dots are the system states $\boldsymbol{x}_i = \boldsymbol{x}(i\tau)$.

It is evident that the topological entropy is the analogue of the fractal dimension as well as the Kolmogorov entropy of the information dimension. The probability $P_l(i_1, \dots i_d)$ is deduced from the (chaotic) temporal evolution of the system and, in this sense, the entropies refer to the natural measure. A more refined definition requires an extremum over the possible partitions of space-time but it goes further than the purposes of this paper [W86, CP85].

Grassberger and Procaccia [GP83b] have firstly proposed a method for the computation of K_2 (a lower bound of K) which is quite analogous to that for the computation of d_2. This numerical algorithm has been extended to the calculation of K [CP85] as well as of the whole set of the K_q's [PV86b].

It is worth noting that the Renyi entropies can be extracted by an experimental signal while the generalized Lyapunov exponents cannot. Indeed, let us estimate the average (3.3.4) by a record of M measures at uniform spacing τ. The joint probability $P_l(i_1, \dots i_d)$ is then given for M large enough by:

$$n_i^{(d)}(l) = \frac{1}{(M-1)} \sum_{j \neq i} \theta\Big(l - \sum_{n=1}^{d} |\boldsymbol{x}_{i+n} - \boldsymbol{x}_{j+n}|\Big) . \tag{3.3.8}$$

In the limit $d = 1$, $n^{(d)}(l)$ becomes the density $n_i(l)$ of points in a ball of centre $\boldsymbol{x}_i$ and radius l defined in eq. (1.1.7).

Increasing the time this density decreases since many points which are initially in the ball of centre $\boldsymbol{x}_i$ do not remain in the ball around $\boldsymbol{x}_{i+d}$ (the d-iterated point of $\boldsymbol{x}_i$). The Kolmogorov entropy measures the inverse of the average characteristic time of this exponential decay. We can however define for each moment the quantity:

$$C^{(d)}(l, q) = \frac{1}{M} \sum_{i=1}^{M} (n_i^{(d)}(l))^q \simeq \langle P_l^q \rangle . \tag{3.3.9}$$

The corresponding entropy K_{q+1} is thus given by the d-asymptotic value of the ratio:

$$\lim_{\tau \to 0} \lim_{l \to 0} \lim_{d \to \infty} \frac{1}{\tau q} \ln\left(\frac{C^{(d)}(l, q)}{C^{(d+1)}(l, q)} \right) . \tag{3.3.10}$$

3.4. Relation between the generalized Lyapunov exponents and the Renyi entropies

We can find a simple relation between entropies and generalized exponents under the hypothesis that the Pesin identity $K = \Sigma_{i=1}^{p} \lambda_i$ holds.

As a starting point it is convenient to define the concept of 'mass' M in the trajectory space. Following Grassberger [G86b], let us focus on a given trajectory $\boldsymbol{x}(t)$ and the trajectories which stay within a distance $\leq l$ from $\boldsymbol{x}(t)$ for all the time in the interval $[t^*, t^* + \Delta t]$. One defines a domain B:

$$\mathrm{B} \equiv \{ y(t) \colon |y(t) - x(t)| \leq l, \forall t \in [t^*, t^* + \Delta t] \} . \tag{3.4.1}$$

Let us denote by $M_l(x(t^*), \Delta t)$ the measure of B. Then, this mass decays with Δt since there are $\tilde{p}$ expanding directions (at least 'in average') corresponding to the $\tilde{p}$ positive Lyapunov exponents. Let us assume that the number of contracting directions does not change, i.e. defining $p \geq \tilde{p} + 1$ as the number of non-negative Lyapunov exponents:

$$L^{(p+1)}(q) < L^{(p)}(q)\,, \qquad \forall q > 0; \qquad L^{(p+1)}(q) > L^{(p)}(q)\,, \qquad \forall q < 0\,. \tag{3.4.2}$$

This inequality plays a fundamental role when relating K_{q+1} to $L^{(p)}(q)$. It is probably satisfied in a hyperbolic system and some arguments indicate that in general it remains valid only up to a certain value q_c above which p may change with a unity [BP87b]. If (3.4.2) holds, we see that for each point $\boldsymbol{x}(t)$ the mass should decay as:

$$M_l(\boldsymbol{x}(t), \Delta t) \approx n_x(l) / R_t^{(p)}(\Delta t) \tag{3.4.3}$$

as schematically illustrated in fig. 14. The scaling of the mass is thus due to that of the response $R^{(p)}$ and eq. (3.2.28) implies for the moments:

$$\begin{aligned} \langle M_l^q \rangle &\propto \langle n(l)^q \rangle \exp\{\Delta t \cdot L^{(p)}(-q)\} \\ &\propto l^{\phi(q)} \exp\{\Delta t \cdot L^{(p)}(-q)\}\,. \end{aligned} \tag{3.4.4}$$

On the other hand, the joint probability $P_l(i_1, i_2, \ldots i_d)$ is given by the mass part factors of order one:

$$M_l(x(t), \Delta t = \tau d) \simeq P_l(i_1, i_2, \ldots i_d)\,. \tag{3.4.5}$$

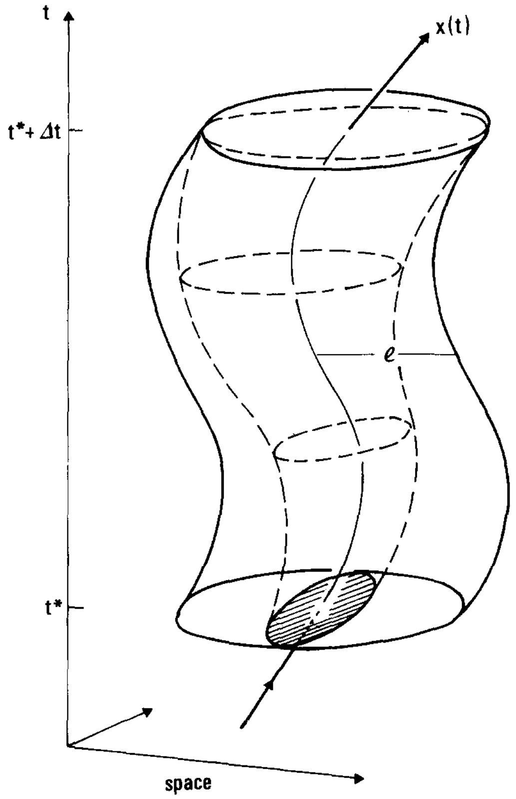

Fig. 14. Tube of trajectories $y(t)$ which stay at a distance $\leq l$ from the trajectory $x(t)$ between t^* and $t^* + \Delta t$. The shadowed region B defined by eq. (3.4.1) decays exponentially in t.

We can therefore estimate the average in (3.4.4) by a weighted sum over the possible sequences of boxes $\{i_1, i_2, \ldots i_d\}$:

$$\langle M_l^q \rangle = \sum_{\{i_1, i_2, \ldots i_d\}} P_l(i_1, i_2, \ldots i_d)^{q+1} \propto \exp(-\tau d q K_{q+1}) . \tag{3.4.6}$$

It follows, by comparing (3.4.6) and (3.4.4), the relation [PV86b]

$$K_{q+1} = \frac{L^{(p)}(-q)}{-q} . \tag{3.4.7}$$

Equation (3.4.7) becomes the Pesin identity in the limit $q \to 0$ while h_{T}, the topological entropy K_0, is equal to $L^{(p)}(1)$, the exponent which rules the exponential divergence, in average, of a p-dimensional volume under the chaotic dynamics.

3.5. Multifractality in the trajectory space

We can regard the set of trajectories (box sequences) of an intermittent system as a multifractal object with regard to the mass defined in eq. (3.4.1).

Let us namely call history $\boldsymbol{X}^{(d)}$ the finite sequence of measurements of the signal $\boldsymbol{x}(t)$:

$$\boldsymbol{X}_k^{(d)} \equiv (\boldsymbol{x}_k, \boldsymbol{x}_{k+1}, \ldots \boldsymbol{x}_{k+d-1}) \tag{3.5.1}$$

where $\boldsymbol{x}_k = \boldsymbol{x}(k\tau)$.

$\boldsymbol{X}_k^{(d)}$ represents a point in the history space $\mathrm{R}^F \otimes \mathrm{R}^F \cdots \otimes \mathrm{R}^F$ (d times). We also introduce a metric by defining the distance between two trajectories as:

$$\mathscr{D}(\boldsymbol{X}_k^{(d)} - \boldsymbol{Y}_n^{(d)}) = \sup_{0 \le j \le d-1} |\boldsymbol{x}_{k+j} - \boldsymbol{y}_{n+j}| . \tag{3.5.2}$$

Around each history $\boldsymbol{X}_k^{(d)}$ there is a tube T of trajectories $\boldsymbol{Y}_k^{(d)}$ such that:

$$\mathrm{T}_l(\boldsymbol{X}_k^{(d)}, d) = \{\boldsymbol{Y}_k : \mathscr{D}(\boldsymbol{Y}_k^{(d)} - \boldsymbol{X}_k^{(d)}) \le l\} . \tag{3.5.3}$$

A natural measure in the space of histories is introduced if the initial state $\boldsymbol{x}_k$ is distributed according to the invariant natural measure on the attractor. We can thus define the probability $P_l(\boldsymbol{X}_k^{(d)})$ that a given history falls within the tube T_l around $\boldsymbol{X}_k^{(d)}$. Such a probability can be estimated [GP83a, GP83b] if we know a great number $\mathcal{N}$ of histories $\boldsymbol{X}_j^{(N)}$ (or a very long record of consecutive states of the system, which is then broken up into histories of length N). One has:

$$P_l(\boldsymbol{X}_k^{(d)}) \simeq \frac{1}{\mathcal{N}} \sum_{j=1}^{\mathcal{N}} \theta(l - \mathscr{D}(\boldsymbol{X}_k^{(d)} - \boldsymbol{X}_j^{(d)})) . \tag{3.5.4}$$

This probability decreases exponentially with a characteristic time γ^{-1} which depends on the particular initial condition $\boldsymbol{x}_k$:

$$P_l(X_k^{(d)}) \propto \mathrm{e}^{-\tau d\gamma} \quad \text{for large times}. \tag{3.5.5}$$

The temporal intermittency is thus characterized by the fluctuations of these local expansion parameters, LEP, γ. It is not difficult to apply the multifractal approach [EP86, PPV86, SSS86] to the distribution of γ in the same way as the distribution of α for the phase space has been characterized.

It is convenient at this point to divide the space of histories into boxes of linear size l. The 'history' boxes are then the cartesian product (for $1<j<d$) of the 'phase space' boxes. The time evolution of the system is thus given by a particular history box $\mathrm{I}=(i_1, \ldots i_d)$ as we have seen in the section 3.3. The joint probability $P_l(i_1, \ldots i_d)$ is the probability $P_l(\mathrm{I})$ that a history $X^{(d)}$ belongs to the box I.

We now consider the set $\Omega(\gamma)$ of histories whose LEP belongs to the interval $[\gamma, \gamma+\mathrm{d}\gamma]$. The number $\mathcal{N}_\gamma$ of boxes necessary to cover this set will increase with d according to the law:

$$\mathcal{N}_\gamma(d) \propto \exp(\tau\, d\, h(\gamma)) \tag{3.5.6}$$

where $h(\gamma)$ is the topological entropy of the set $\Omega(\gamma)$, with the warning that if $h(\gamma) \le 0$, then the topological entropy is zero. We have by definition $h(\gamma) \le h_{\mathrm{T}}$.

Let us now show that $h(\gamma)$ and K_q are related to each other by a Legendre transform analogous to (1.2.5).

Indeed, the partition of the (multifractal) set of the histories into the sets $\Omega(\gamma)$ allows to compute the averages (3.3.4) as weighted sums over the boxes I, by grouping together all the boxes which belong to the same $\Omega(\gamma)$:

$$\begin{aligned} \langle P_l(\mathrm{I})^{q-1}\rangle &= \sum_{\{\mathrm{I}\}} P_l(\mathrm{I})^q \\ &\propto \int \mathrm{d}\rho(\gamma)\exp(-d\tau\,(q\gamma - h(\gamma))) \end{aligned} \tag{3.5.7}$$

where $\mathrm{d}\rho(\gamma)$ is a smooth measure.

A saddle point estimation of the integral yields in the limit of large τd:

$$K_q(q-1) = \min_\gamma\,(q\gamma - h(\gamma)). \tag{3.5.8}$$

The term $\exp(-d\tau)$ is here the scaling parameter in the history space like l in the phase space.

Let us emphasize that the probability that a trajectory (i.e. a box I) belongs to a set $\Omega(\gamma)$ scales as:

$$\mathcal{P}(X \in \Omega(\gamma)) \propto \exp(-S(\gamma)\,\tau d) \tag{3.5.9}$$

with $S(\gamma) \ge 0$.

Arguments quite similar to those which lead to eq. (1.2.11) show that:

$$\langle P_l^q\rangle = \int \mathcal{P}(X \in \Omega(\gamma))\,\mathrm{e}^{-\tau d\gamma q}\,\mathrm{d}\gamma \tag{3.5.10}$$

and so:

$$K_{q+1}q = \min_{\gamma} (q\gamma + S(\gamma)) . \tag{3.5.11}$$

It follows by comparing (3.5.8) and (3.5.11) that:

$$S(\gamma) = -h(\gamma) + \gamma . \tag{3.5.12}$$

The equality $S = 0$ holds only for $\gamma = K_1$ implying that in the asymptotic limit $t \to \infty$ one has vanishing probability of finding a trajectory with LEP different from the Kolmogorov entropy. Each q peaks up a particular value of the LEP by minimizing (3.5.8) or (3.5.11):

$$q = \mathrm{d}h/\mathrm{d}\gamma|_{\tilde{\gamma}(q)} \qquad \text{and} \qquad q = -\mathrm{d}S/\mathrm{d}\gamma|_{\bar{\gamma}(q)} . \tag{3.5.13}$$

In the limit $q \to \pm\infty$ the Legendre transform becomes trivial and one finds the extrema between which γ is bounded:

$$\gamma_{\min} = K_{\infty} \qquad \text{and} \qquad \gamma_{\max} = K_{-\infty} . \tag{3.5.14}$$

At $q = 0$, $\tilde{\gamma} = K^*$ (and $\bar{\gamma} = K_1$) whereas h attains its maximum:

$$h(\gamma = K^*) = K_0 . \tag{3.5.15}$$

S has its minimum at K_1: $S(K_1) = 0$. It is easy to show that h (and $-S$) are convex functions defined on the interval $[K_{\infty}, K_{-\infty}]$; S has a quadratic minimum implying for $\gamma \simeq K_1$ the parabolic shape:

$$S(\gamma) = (\gamma - K_1)^2/2b . \tag{3.5.16}$$

Let us note that $h(\gamma)$ is also approximated by a parabola around K_1 and not around its vertex K^*. For small enough q we then have:

$$K_{q+1} = K_1 - \tfrac{1}{2}bq \tag{3.5.17}$$

where b is the second-order cumulant:

$$b = \lim_{\tau \to 0} \lim_{l \to 0} \lim_{d \to \infty} \frac{1}{\tau d} [\langle \ln^2 P_l(\mathrm{I}) \rangle - \langle \ln P_l(\mathrm{I}) \rangle^2] . \tag{3.5.18}$$

For a lognormal distribution of the 'mass' in the trajectory space the relations (3.5.16) and (3.5.17) remain valid for all the q values.

The relation between $-S(\gamma)$ and K_{q+1} is reminiscent of that linking the entropy to the free energy in thermodynamics $\beta\, F(\beta) = U(\beta) - \mathcal{S}(\beta)$. The role of the inverse of the temperature β is played by the moment index q, $F(\beta)$ by K_{q+1}, U by $\bar{\gamma}$ and $\mathcal{S}(\beta)$ by $-S(\bar{\gamma}(q))$.

Let us limit ourselves to consider the branch of positive temperatures (i.e. $q > 0$). As $q \to \infty$ (i.e. vanishing temperature) the system is found in the state of minimal energy $\gamma_{\min}$ and minimal entropy $-S(\gamma_{\min})$.

A typical phase diagram of K_q vs. q is shown in fig. 15.

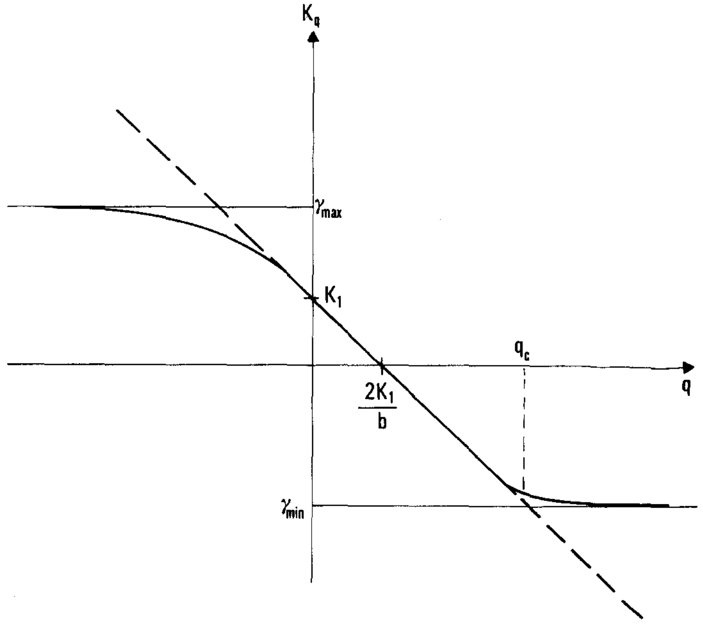

Fig. 15. Typical shape of K_{q+1} vs. q (full line). The dashed line indicates the lognormal approximation $K_{q+1} = K_1 - (b/2)q$.

One may also speculate on the possible existence of 'phase transitions' which would appear as edges in the K_q vs. q curve at a critical value q_c. Let us, e.g., suppose that the linking of the large q behaviour $K_q = \gamma_{\min} q$ with the 'high temperature' expansion ($q \ll q_c$) $K_q = \Sigma_i c_i q^i$ is not smooth enough. In this case one should have a discontinuity in the n-order derivative of K_q at q_c implying a continuous transition between the intermittent high temperature phase and the non-intermittent ('ordered') low temperature phase.

3.6. A thermodynamical formalism for unidimensional maps

We want to discuss here how the statistical mechanics for spin systems on a lattice can be applied to chaotic one-dimensional maps g [R78, B75, R76a,b],

$$x_{k+1} = g(x_k) .$$

For a system of n spins the partition function Z_n is given by a sum over all configurations of the Boltzmann weight $\exp(-\beta H_n)$ where H_n is the configuration energy and β^{-1} the temperature. The free energy per spin is then given in the thermodynamic limit by:

$$F(\beta) = -\lim_{n\to\infty} \frac{1}{n\beta} \ln Z_n .$$

Such a thermodynamical approach can be rigorously used only for a particular class of dynamical systems (see e.g. [R78]) but its generalization is possible at least on heuristic grounds.

For "typical" unidimensional maps before chaos each initial point is attracted into a periodic stable orbit such that $x_k = g^{(n)}(x_k)$ where $g^{(n)}$ is the nth iterate of g. The evolution becomes chaotic whereas all the orbits become unstable. Nevertheless, the statistical properties can still be characterized by these unstable periodic orbits [B79].

A global degree of instability is measured by the Lyapunov exponents [MR82]:

$$\lambda = \lim_{n\to\infty} \int \mathrm{d}\mu_n(x) \ln|g'(x)| \tag{3.6.1}$$

where $g' = \mathrm{d}g/\mathrm{d}x$ and $\mathrm{d}\mu_n$ is an invariant measure which converges with n to the invariant ergodic measure $\mathrm{d}\mu(x)$.

Kai and Tomita [KT82] suggested the following form for μ_n:

$$\mu_n(x) = \sum_{i=1}^{\mathcal{N}(n)} a_i\, \delta(x - z_i) \tag{3.6.2}$$

where z_i are the $\mathcal{N}(n)$ unstable fixed points of period n (i.e. $z_i = g^n(z_i)$), the set of which will be denoted by $\mathrm{J}^{(n)}$. Its cardinality $\mathcal{N}(n)$ is proportional to $\exp(h_\mathrm{T} n)$ by definition.

The weight a_i for measures which are absolutely continuous with respect to the Lebesgue measure [L81] is:

$$a_i = c_n / |g^{(n)}(z_i)'| \qquad \text{with } \sum_i a_i = 1\,. \tag{3.6.3}$$

We can thus define a partition function following Takahashi and Oono [TO84]:

$$Z_n(g, \beta) = \sum_{z_i \in \mathrm{J}^{(n)}} \exp\{-\beta \ln|g^{(n)}(z_i)'|\}\,. \tag{3.6.4}$$

The fluctuation of the degree of chaos can be characterized varying β by the generalized Lyapunov exponents. Let us in fact call $R_n(z) = |(g^{(n)}(z))'|$ the response after n iterations of the map to a perturbation in z, unstable periodic solution of order n.

The partition function (3.6.4) becomes:

$$Z_n(g, \beta) = \sum_{z_i \in \mathrm{J}^{(n)}} \exp\{-\beta \ln R_n(z_i)\} \tag{3.6.5}$$

which can be estimated by a time average in the chaotic phase. Moreover, let us recall the Bowen–Ruelle relation [B75]:

$$F(g, \beta = D_\mathrm{H}) = 0 \tag{3.6.6}$$

where D_H is the Hausdorff dimension. The definition of Hausdorff dimension requires a minimum over the possible partitions into boxes of size $\le l$ for $l \to 0$. We have escaped these troubles considering uniform partitions into boxes of size l. If these procedures are equivalent (which is often true), then the Hausdorff dimension is the fractal dimension.

For measures whose support is fractal, the ansatz (3.6.3) suggests to assume the weight $a_i \propto |g^{(n)}(z_i)'|^{-D_\mathrm{F}}$, which corresponds to a "uniform" measure for which $\phi(q) = D_\mathrm{F} \cdot q$ [BPPV85].

The partition function (3.6.5) can be estimated by an ensemble average $\overline{(\cdot)} = \sum_{z_i \in J^{(n)}} (\cdot) |g^{(n)}(z)'|^{-D_F}$ for large n:

$$Z_n(g, \beta) = \langle R_n^{-\beta + D_F} \rangle = \overline{R_n^{-\beta + D_F}} \tag{3.6.7}$$

where $\langle \cdot \rangle$ is a time average.

The generalized Lyapunov exponents computed with respect to the uniform measure therefore allow one to determine the free energy $F(\beta)$ by the relation:

$$P(\beta) = -\beta F(g, \beta) = L(q = (D_F - \beta)) , \tag{3.6.8}$$

where we have introduced the so-called topological pressure $P(\beta)$.

The Bowen–Ruelle formula (3.6.6) now simply follows from the trivial identity $L(q = 0) = 0$.

It is useful to note that the temperature parametrizes a whole class of probability measures (called Gibbs equilibrium measures) on the invariant set J given by the closure of $\lim_{n\to\infty} J^{(n)}$:

$$\mathrm{d}\mu_\beta(x) = \lim_{n\to\infty} \sum_{z_i \in J^{(n)}} \mathrm{d}x \, \delta(z_i - x) \, |g^{(n)}(z_i)'|^{-\beta} , \tag{3.6.9}$$

where $\beta = D_F$ picks up the uniform measure. A moment of reflection shows that the characteristic Lyapunov exponent $\lambda^{(\beta)}$ computed by an average over the measure $\mathrm{d}\mu_\beta$ is:

$$\lambda(\mu_\beta) = \langle \ln|g'(x)| \rangle_\beta \equiv \int \mathrm{d}\mu_\beta(x) \ln|g'(x)| . \tag{3.6.10}$$

It corresponds to an internal energy $U(\beta)$ defined as:

$$U(\beta) = -\lim_{n\to\infty} \frac{1}{n} \frac{\mathrm{d} \ln Z_n(\beta)}{\mathrm{d}\beta} = \frac{\mathrm{d}\rho}{\mathrm{d}\beta} .$$

Let us note that the Pesin equality is not satisfied if $D_F < 1$ since one has [LM85, G86b]:

$$K_1(\mu) = \lambda(\mu) \cdot D_1(\mu) , \tag{3.6.11}$$

and $D_1(\mu) \le D_F$ for any measure μ. Nevertheless, for the uniform measure where $D_1 = D_F$ we can extend relation (3.4.7) [BPTV87], which becomes

$$K_{q+1} = -L(-D_F \cdot q)/q . \tag{3.6.12}$$

Let us also discuss the limit of infinite temperature $\beta \to 0$, which picks up the maximum entropy measure (3.6.5) such that $K_1 = h_T = \max_\mu K_1(\mu)$ and $K_q = h_T$ for all q. Relation (3.6.5) now gives the number $\mathcal{N}(n)$ of unstable fixed points of period n. In the limit $n \to \infty$ we find that:

$$h_T = \lim_{n\to\infty} \frac{\ln \mathcal{N}(n)}{n} = -\lim_{\beta\to 0} \beta F(\beta) = P(0) . \tag{3.6.13}$$

We finally note that (3.6.6) can be extended to a generical q value in the case of the uniform measure [BPTV87] and of the maximum entropy measure [LPC87, T86, F87] to $P(D_{\mathrm{H}}q) = (1-q)K_q$ and $P(-\phi(q)) = (q+1)h_{\mathrm{T}}$, respectively.

3.7. Partial dimensions and entropies

The introduction of dimensions and entropies related to the different eigendirections of the linearized flow allows to find some relations linking Renyi dimensions to Renyi entropies and to generalized Lyapunov exponents, even if its applicability seems rather questionable as noted in section 3.2.

Let us assume that a small box centred around a point $\boldsymbol{x}_i$ of the trajectory is stretched along the kth eigendirection by a factor

$$m_k(\boldsymbol{x}_i, t) \propto \exp\{\gamma_k(\boldsymbol{x}_i)\cdot t\} \qquad \text{for large } t \tag{3.7.1}$$

where one can easily recognize from the definition (3.2.30) that

$$\langle m_k^q \rangle \propto \exp\{l_k(q)\cdot t\}\,, \qquad \langle \gamma_k \rangle = \lambda_k \qquad \text{for large } t\,. \tag{3.7.2}$$

In the same spirit, Grassberger [G86] proposed to assign a 'partial dimension' $D_q(k)$ to each eigendirection. The Renyi dimensions are given by the sum of all the partial dimensions:

$$d_q = \sum_{k=1}^{F} D_q(k)\,. \tag{3.7.3}$$

The attractor is thus decomposed in the direct product of continua (directions with $D_1 = 1$), discrete points ($D_1 = 0$) and Cantor like sets ($D_1 < 0$).

Following ref. [BP87a] with slight modifications, let us define a (generalized) volume contraction rate of order q around a point $\boldsymbol{x}_i$:

$$V_i(q, t) = \prod_{k=1}^{F} m_k^{D_{q+1}(k)}\,. \tag{3.7.4}$$

The average volume conservation then implies the relation:

$$\lim_{t\to\infty} \langle V(q, t)^{-q} \rangle = 1\,. \tag{3.7.5}$$

This is a constraint on the choice of the proper dimension list of the D_{q+1}'s. In the limit $q \to 0$, condition (3.7.5) becomes:

$$\sum_{k=1}^{F} D_1(k)\,\lambda_k = 0 \tag{3.7.6}$$

and it is possible to show [G86a] that the Kolmogorov entropy is given by:

$$K_1 = \sum_k^{(+)} D_1(k)\, \lambda_k \tag{3.7.7}$$

where the sum $\Sigma^{(+)}$ runs over the directions with positive λ_k. The Pesin relation (3.3.2) is recovered if all the expanding directions have dimensions $D_1 = 1$ as it is the case of axiom A systems [ER85].

Furthermore, for $q \geq 0$ the partial dimension must satisfy the constraint

$$0 \leq D_q \leq 1 . \tag{3.7.8}$$

The simplest choice compatible with (3.7.5) and (3.7.8) is given by the assumption that the attractor is Cantor-like only along the $(j+1)$th direction, where j is the largest integer for which $\Sigma_{k=1}^{j} \lambda_k \geq 0$. We have $D_q(k) = 1$ for $k \leq j$, $D_q(k) = 0$ for $k > j+2$ and $D_q(j+1) \neq 0$ as well as $\neq 1$. This choice maximizes the estimation of the information entropy and gives the Kaplan–Yorke formula [KY78]:

$$d_1 = j + \sum_{k=1}^{j} \lambda_k \Big/ |\lambda_{j+1}| . \tag{3.7.9}$$

Equation (3.7.9) is in general an upper bound but it has been proved to be exact for two-dimensional diffeomorphisms [Y82]. Analogous upper bounds can be obtained for d_{q+1} with $q > 0$. However, in the case of maps with more than two degrees of freedom, one needs to assign several trial values of D_{q+1} in order to determine the right ones by interpolation. It was proposed [BP87a] to decrease the dimensions $D_q(k)$ starting from the largest possible list ($D_q = 1$ for $k = 1, 2, \ldots F$) until the requirement (3.7.5) is satisfied. In this procedure one always imposes that only one direction, say j_q, is Cantor-like where j_q is the largest integer for which:

$$\lim_{t\to\infty} \frac{1}{t} \ln\left\langle \prod_{k=1}^{j_q} m_k(t)^{-q} \right\rangle^{-1/q} \geq 0 . \tag{3.7.10}$$

In this way, we obtain $D_q(j_q + 1)$ from (3.7.5) and thus an upper bound for d_q. Only in the case of two-dimensional maps with constant Jacobian, the equality $d_q = j_q + D_q(j_q + 1)$ can be explicitly solved [BP87a].

Nevertheless, it is possible to perform a perturbative expansion of eq. (3.7.10) defining the generating function g:

$$\left\langle \prod_k m_k^{z_k} \right\rangle \propto \exp\{ g(z_1, z_2, \ldots z_F) t \} , \tag{3.7.11}$$

for large t.

The average conservation of the volume (3.7.5) implies that [G86b]:

$$g(\{ z_k = D_{q+1}(k) \cdot q \}) = 0 \tag{3.7.12}$$

with:

$$g(z_1, z_2, \ldots z_F) = \sum_i z_i \lambda_i + \frac{1}{2} \sum_{i,k} z_i Q_{ik} z_k + \cdots \tag{3.7.13}$$

noting $\lambda_i = \langle \gamma_i \rangle$ and $Q_{ik} = \langle (\gamma_i - \lambda_i)(\gamma_k - \lambda_k) \rangle$.

The generalization of the Kaplan–Yorke formula (3.7.9) is then given by inserting eq. (3.7.13) into eq. (3.7.12) so that:

$$\sum_k D_{q+1}(k)\,\lambda_k = \frac{q}{2}\sum_{i\neq k} D_{q+1}(i)\,Q_{ik} D_{q+1}(k) + \cdots \tag{3.7.14}$$

It is also possible to find the analogue of relation (3.7.7):

$$K_{q+1} = \sum^{(+)} \lambda_k - \frac{q}{2}\sum_{k,j}^{(+)} D_{q+1}(k)\,Q_{k,j}\,D_{q+1}(j) \tag{3.7.15}$$

which in the limit of axiom A system reduces itself to a Pesin-like equality:

$$K_{q+1} = \sum^{(+)} \lambda_k - \frac{q}{2}\sum_{k,j}^{(+)} Q_{k,j} + \mathrm{o}(q^2) = \frac{L^{(j)}(-q)}{-q}\,. \tag{3.7.16}$$

We must finally discuss the connections with the thermodynamic formalism. As the conservation laws (3.7.5) involve both Renyi dimensions and Renyi entropies, Badii and Politi [BP87b] argue that the function $H(\alpha)$ is related to the analogous $S(\gamma)$, respectively defined in eqs. (1.2.9) and (3.5.9). Indeed, under the hypothesis that to each singularity $\bar{\alpha}$ corresponds a given local expanding parameter $\bar{\gamma}$, we can identify the factor $\mathrm{e}^{-\tau\gamma}$ around a point x_i of the trajectory with the scaling of the measure l^{α}. It follows that:

$$\exp(-\tau\,\gamma(x_i)) \propto l^{\alpha(x_i)} \tag{3.7.17}$$

and the probability of finding a singularity α in the range $[\bar{\alpha}, \bar{\alpha} + \mathrm{d}\alpha]$

$$\mathcal{P}(\bar{\alpha}) \propto l^{H(\bar{\alpha})} \propto \exp(-\tau\bar{\gamma}\,H(\bar{\alpha})/\bar{\alpha})$$

should scale as the probability $P(\bar{\gamma}) \propto \exp\{-S(\bar{\gamma})\,\tau\}$ of finding a LEP $\in [\bar{\gamma}, \bar{\gamma} + \mathrm{d}\gamma]$.

This implies that

$$H(\alpha)/\alpha = S(\gamma)/\gamma\,. \tag{3.7.18}$$

The appearance of discontinuities in the Renyi functions can be analysed in terms of partial dimensions.

If the index j_q, at which $D_{q+1}(j_q) \neq 0$ and $\neq 1$, is not fixed but jumps by one unit at q_c, we should expect an edge in the Renyi dimensions. In this case, the partial dimension associated with the Cantor-like direction for $q > q_c$ decreases with increasing q and vanishes for $q \geq q_c$. The preceding dimension $D_{q+1}(j_q - 1)$ which is equal to 1 for $q \leq q_c$ begins to decrease with q, at the same time.

This involves a kind of first-order phase transition in the function d_{q+1} (regarded as a free energy at the inverse temperature $\beta = q$) corresponding to vanishing of the contribution of the multiplier m_j to the generalized volume (3.7.4) at q_c. We can correctly speak of transition if the 'Kaplan–Yorke' choice for the list of D_q is not only an upper bound but coincides with the real D_q's. Otherwise the transition can be related to the change in the rule used to define the partial dimensions.

We have described a mechanism which can explain the appearance of 'phase transition' just for systems where each eigendirection has its own individuality. On the other hand, in generic systems, like the Henon map, an expanding direction can coincide with a contracting one when it is computed at different points. For q larger than q_c a transition can occur because of an exchange of two multipliers.

This second mechanism is not yet fully understood and we refer the reader to [BGP87] for more details.

4. Replicas of disordered systems as multifractal objects

4.1. Introduction

The techniques developed for the characterization of the temporal intermittency are quite useful for the study of disordered systems. Indeed, in some systems, the sample to sample fluctuations of the physical observables can be described in terms of the generalized Lyapunov exponents related to the product of appropriate transfer matrices whose elements are random variables. We have here to note that the random matrix product is a rather interesting problem also in dynamical systems where it is often used as a model of the chaotic behaviour of quasi-integrable Hamiltonian systems [B84, PV86a].

In this section we want to show how the different realizations of the system with respect to disorder correspond to trajectories generated by iterating the transfer matrices.

It is thus possible to define a multifractal object in the 'replica ensemble': in this sense the fluctuations induced by the disorder are an intermittency phenomenon.

We must however stress the peculiarity of this sort of intermittency. In a time evolution the characterization of the degree of chaos variations has a great practical relevance. One is interested in the knowledge of the (chaotic) signal as function of time and not only in the 'global' behaviour. On the other hand, the fluctuations around the average value in a disordered system are rather less important since they decay exponentially with volume in the thermodynamic limit which is always reached in a real sample.

The multifractal formalism becomes in this case just a fluctuation theory which is quite analogous to that introduced by Einstein in the canonical ensemble.

Let us in fact consider a large number of different samples of a usual equilibrium system with N particles. A sample can of course be regarded just as a 'small' part of a larger system. Each sample is then characterized by its energy per particle, say x, and we can group all the realizations with energy per particle between x and $x + \mathrm{d}x$ in the same set $\Omega(x)$. The number H_x of realizations belonging to $\Omega(x)$ is assumed to increase exponentially with N like:

$$H_x \sim \exp(s(x)N)\,, \tag{4.1.1}$$

where $s(x)$ is the microcanonical entropy per particle corresponding to the energy x.

We can now compute the partition function of the Gibbs distribution integrating over x:

$$Z_N(\beta) = \int H_x\, \mathrm{e}^{-\beta x N}\, \mathrm{d}x \sim \int \mathrm{e}^{N(s(x)-\beta x)}\, \mathrm{d}x\,. \tag{4.1.2}$$

In the limit $N \to \infty$ only one set gives a relevant contribution to Z_N. By means of the steepest descent method one in fact gets:

$$Z_N(\beta) \sim \exp(-\beta\, f(\beta)\, N) \tag{4.1.3}$$

with

$$\beta\, f(\beta) = \min_x\, [\beta x - s(x)] = \beta u - s(u)$$

where the minimal condition in the Legendre transformation (4.1.3) implies:

$$\left.\frac{\mathrm{d}s}{\mathrm{d}x}\right|_{x=u} = \beta(u)\,. \tag{4.1.4}$$

The probability of finding a sample with $x \neq u$ vanishes for large N like:

$$P_N(x) = Z_N^{-1}\, \mathrm{e}^{-\beta x N} \cdot H_x \sim \exp(-\beta N[\psi_\beta(x) - f(\beta)]) \tag{4.1.5}$$

where $\psi_\beta = \beta x - s(x) \geq f$ and f is the free energy per particle of the system. We have derived these well-known results in order to point out that the hierarchy of sets $\Omega(x)$ can be investigated by varying the temperature. The same approach can be used for characterizing the finite volume fluctuations in disordered systems by means of the replica formalism.

We shall explicitly consider two models: the Schroedinger equation with a random potential and the Ising model with random couplings, analysing respectively the localization length and the free energy. A short discussion will finally deal with the possibility that there exist 'phase' transitions in the diagram $L(q)$ versus q.

4.2. The one-dimensional Anderson model

A first simple model of the conductivity in a disordered medium is given by the Schroedinger equation with a randomly distributed potential V_i. We shall consider a discrete version of the Anderson model [A58] defined on a one-dimensional lattice. One can write the finite difference equation:

$$\psi_{i+1} + \psi_{i-1} + wV_i\psi_i = E\psi_i \tag{4.2.1}$$

where i labels the site of the lattice, ψ_i is the amplitude of the eigenfunction at the site i, V_i a random potential and $E+2$ the energy.

In the following each V_i is uniformly distributed on the range $[-1/2, 1/2]$ and is uncorrelated from one site to another:

$$\langle V_i \rangle = 0\,, \qquad \langle V_i V_j \rangle = A^2 \delta_{ij} \tag{4.2.2}$$

with $A^2 = 1/12$.

It is well known that whenever $w \neq 0$ the eigenfunction is exponentially localized [DLS85]. This means that if we have assumed periodic boundary conditions (i.e. $\psi_N = \psi_{-N}$) an eigenfunction $\{\psi_n | -N \leq n \leq N\}$ exponentially decreases (with probability one) at large distances around its maximum (say for $n = 0$):

$$|\psi_n| \le |\psi_0| \exp(-\text{const.}\, |n|)\,. \tag{4.2.3}$$

The localization length is then usually defined as

$$\xi_0^{-1} = \lim_{|n|\to\infty} -\frac{1}{|n|} \langle \ln |\psi_n| \rangle \tag{4.2.4}$$

where the average is taken over disorder.

On the other hand Mielke and Wegner [MW85] have recently considered:

$$\xi^{-1} = \lim_{|n|\to\infty} -\frac{1}{|n|} \ln \langle |\psi_n| \rangle\,. \tag{4.2.5}$$

Let us generalize (4.2.5) introducing ξ_q [PV87b]:

$$\xi_q^{-1} = \lim_{|n|\to\infty} \frac{1}{|n|\, q} \ln\langle |\psi_n|^{-q} \rangle\,. \tag{4.2.6}$$

A direct numerical computation of ξ_q is a hard task since it is rather difficult to calculate the exact eigenvalue and eigenvector of (4.2.1) for large chains.

Let us show that the problem can be reduced to the calculation of the generalized Lyapunov exponents associated with a product of random matrices. Equation (4.2.1) can in fact be written in the recursive form:

$$Z(i+1) = A_w(i)\, Z(i) \tag{4.2.7}$$

where

$$A_w(i) = \begin{pmatrix} E - wV_i & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{Z}(i) = \begin{pmatrix} \psi_{i+1} \\ \psi_i \end{pmatrix}. \tag{4.2.8}$$

One has a product $\Pi_{i=1}^{N} A_w(i)$ of random symplectic matrices which relates $\mathbf{Z}(N)$ to $\mathbf{Z}(0)$.

It is useful to introduce the 'response' after n sites whose value depends on the starting site m:

$$R_m(n) = |\mathbf{Z}(m+n)| / |\mathbf{Z}(m)|\,. \tag{4.2.9}$$

The generalized Lyapunov exponents $L(q)$ are then given by the asymptotic behaviour of the moments:

$$\langle R(N)^q \rangle = \lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} R_m(N)^q \sim \exp\{L(q)\, N\} \qquad \text{for large } N\,. \tag{4.2.10}$$

It is easy to recognize that ξ_0^{-1} is the maximal Lyapunov exponent λ_1:

$$\xi_0^{-1} = \lambda_1 = \lim_{N\to\infty} \frac{1}{N} \langle \ln R(N) \rangle = \left.\frac{\mathrm{d}L}{\mathrm{d}q}\right|_{q=0}\,. \tag{4.2.11}$$

We have seen that a linear behaviour $L(q) = \lambda_1 q$ indicates that there are no fluctuations of R (i.e. no intermittency). By comparing (4.2.10) with (4.2.6) one has, since $R_m(n) \sim |\psi_m|/|\psi_{m+n}|$:

$$\xi_q^{-1} = L(q)/q . \tag{4.2.12}$$

Let us emphasize that the sequence $\{Z(m+i)$ with $0 < i < N\}$ generated by iterating the transfer matrix is not an eigenfunction $\{\psi_i\}$ on the lattice because it does not take into account boundary conditions. Nevertheless, if eigenstates at the chosen energy exist, then the Lyapunov exponent λ_1 coincides with the inverse of the (average) localization length ξ_0. On the other hand, the localization length fluctuates around ξ_0 with varying realizations of disorder. Let us therefore define a local expansion parameter LEP γ by the relation:

$$R_m(N) \sim \exp\{\gamma(m)\, N\} \qquad \text{for } N \text{ large} . \tag{4.2.13}$$

γ is a random variable which depends on the starting site m and with average value λ_1, whose probability distribution $P(\gamma)$ coincides with the distribution of the localization lengths among different realizations of disorder.

At this point we can apply all the multifractal machinery of the previous section by grouping in the same subset all the trajectories $\{Z(m+i),\ 0 < i < N\}$ which have a LEP in the range $[\gamma, \gamma + \mathrm{d}\gamma]$.

It is quite reasonable to assume that after N sites the probability $P(\gamma)$ of having a response with a given LEP has the following form:

$$\mathrm{d}P_N(\gamma) = \mathrm{d}\rho(\gamma) \exp\{-S(\gamma)N\} . \tag{4.2.14}$$

The ansatz (4.2.14) is necessary in order to obtain an exponential behaviour for the moments $\langle R^q(N)\rangle$ and is similar to the hypothesis made for the singularity structure in turbulence and in other strange sets. We can now calculate the average over the disorder by the average over the γ-distribution:

$$\langle R^q(N)\rangle \propto \int \mathrm{d}\rho(\gamma) \exp\{[\gamma q - S(\gamma)]N\} \propto \exp\{L(q)N\} . \tag{4.2.15}$$

A saddle point estimation of (4.2.15) gives:

$$L(q) = \max_\gamma [\gamma q - S(\gamma)] = q\xi_q^{-1} . \tag{4.2.16}$$

From eq. (4.2.13) all the ξ_q's must be positive implying $\gamma > 0$: when all the potentials V_i are equal one recovers the pure system and this corresponds to the minimal value of the LEP $\gamma_{\min}$. More generally one can see from eq. (4.2.16) that:

$$\lim_{q\to\infty} \xi_q^{-1} = \gamma_{\max} , \qquad \lim_{q\to-\infty} \xi_q^{-1} = \gamma_{\min} = 0 \tag{4.2.17a}$$

$$0 \le S(\gamma) \le \max[S(\gamma_{\max}), S(\gamma_{\min})] . \tag{4.2.17b}$$

Let us recall that $S(\gamma = \lambda_1) = 0$ while $S(\gamma \neq \lambda_1)$ is positive in the S-definition range $[0, \gamma_{\max}]$. The fluctuations around λ_1 therefore decay with N as $\exp(-S(\gamma)N)$. The existence of this thermodynamical limit corresponds to the Oseledec theorem [O68] on the existence of the Lyapunov exponent λ_1. The absence of fluctuations is analogous to a non-intermittent chaotic behavior. If the response is obtained as the product of independent random variables, the lognormal distribution is a good approximation and $S(\gamma)$ is a parabola truncated at $\gamma_{\min} = 0$ and at $\gamma_{\max}$. The corresponding form $L(q) = \lambda_1 q + (\mu/2)q^2$ is then valid only for not too large q values (see appendix B).

4.3. Anomalous scaling laws at the band edge

For a pure system ($w = 0$) the energies of the Hamiltonian (4.2.1) are concentrated within a band $-2 \le E \le 2$.

In the sequel, we shall limit ourselves to the case $\langle V_i \rangle = 0$ since it is always possible to include the average value of the potential in the definition of E. Let us concentrate on the band edge $E = 2$ of the pure system where the generalized Lyapunov exponents have a particularly interesting scaling behaviour with respect to the disorder amplitude w [PV87b]. The weak disorder expansion of the Lyapunov exponent is non-analytic and one finds [DG84]:

$$\lambda_1 = C_1 w^{2/3}, \qquad C_1 \simeq 0.2893 A^{2/3}. \tag{4.3.1}$$

On the other hand it is quite simple to extend a perturbative calculation of Parisi and Vulpiani [PVu86] for computing $L(2)$ in our case:

$$L(2) = 2C_2 w^{2/3} \tag{4.3.2}$$

with $C_2 = A^{2/3} 2^{1/3}/2 \simeq 0.6299 A^{2/3}$.

One can therefore estimate the variance μ by assuming that for low disorder $L(q) = \lambda_1 q + (\mu/2)\, q^2$ has a parabolic shape up to $q > 2$:

$$\mu = L(2)/2 - \lambda_1 \simeq 0.3406 A^{2/3} w^{2/3}. \tag{4.3.3}$$

Figure 16 shows the plot of $L(q)/(C_2 w^{2/3} q)$ for different values of w in the case of V_i uniformly distributed in the range $[-1/2, 1/2]$. We observe a linear behaviour $L(q)/q = \lambda_1 + (\mu/2)q$ for $q < \bar{q}(w)$ which breaks down for larger q's and then saturates to the plateau $L(q)/q = \gamma_{\max}$ for $q > \bar{q}(w)$.

The maximal LEP can be estimated by a rather naive (but stringent) argument. The maximal eigenvalue of a random matrix $A_w(i)$ is $l(V_i) = 1 + \sqrt{wV_i} + O(w)$. It follows that the maximal LEP is obtained for a disorder realization such that V_i are typically of the same order of $\max\{V_i\} = \frac{1}{2}$:

$$\gamma_{\max} \propto w^{1/2} \tag{4.3.4}$$

for small w.

This result is in good agreement with the numerical extrapolation of $L(q)/q$ in the limit of large q (see fig. 17).

The value of $\bar{q}(w)$ can easily be obtained by matching the asymptotic behaviour $L(q) \propto w^{1/2}$ with the lognormal one for small q, i.e. $L(q)/q = \lambda_1 + (\mu/2)q$. One thus obtains $\bar{q}(w) \propto w^{-1/6}$. These two kinds of behaviour for $L(q)$ (i.e. $L(q) \propto w^{2/3}$ for small q and $L(q) \propto w^{1/2}$ for large q) implies a crossover for the scaling law of ξ_q when q is very large (see eq. (4.2.12)).

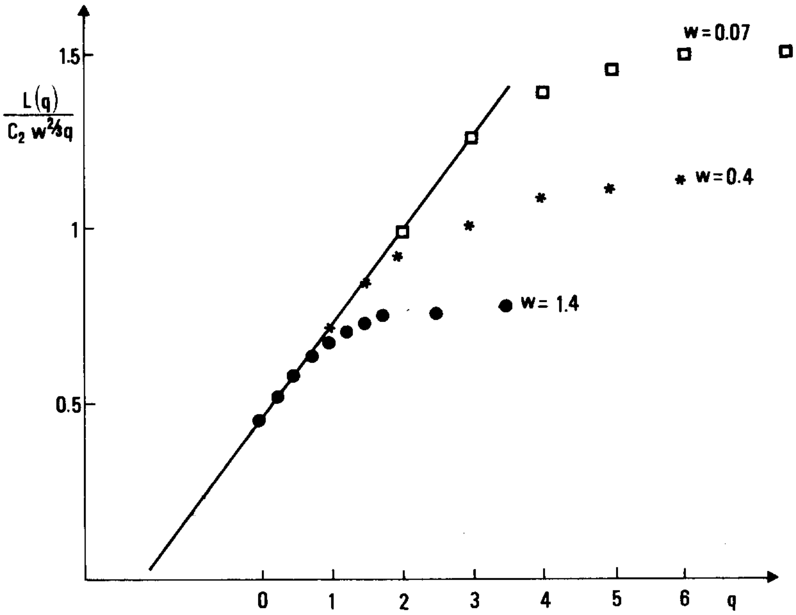

Fig. 16. Numerical calculation of $L(q)/[C_2 w^{2/3} q]$ at different values of the disorder amplitude w. The full line is the parabolic behaviour $L(q)/q = \lambda_1 + (\mu/2)q$.

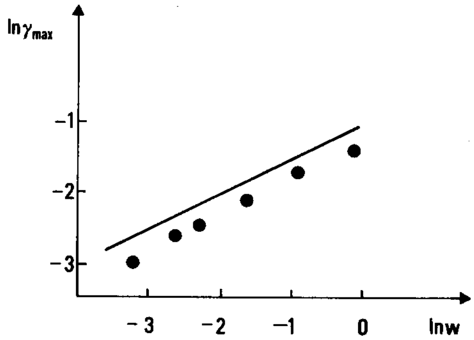

Fig. 17. $\gamma_{\max}$ as function of w; the line has slope 1/2.

In fact one sees that for large q there is a value $\bar{w}(q) \propto q^{-6}$ such that:

$$\xi_q \propto w^{-1/2} \qquad \text{if } w \gtrsim \bar{w}(q) \tag{4.3.5}$$

while the "true" scaling law $\xi_q \propto w^{-2/3}$ holds for very small $w(<\bar{w}(q))$. Note that the value of $\bar{w}(q)$ goes quickly to zero when q goes to infinity.

In fig. 18 we show $S(\gamma)$ vs. γ as obtained via a Legendre transformation from the direct numerical calculation of $L(q)$. Note that for $\gamma < \gamma_{\max}$, $S(\gamma)$ is close to a parabolic form because the probability distribution of $R(N)$ is nearly lognormal for $R(N) < (\gamma_{\max})^N$ and zero otherwise. The maximum is found at $\gamma = \lambda_1$ since λ_1 is the most probable value of the LEP.

Moreover the probability distribution of the response $R(N)$ is close to a lognormal distribution with parameters λ_1 and μ (see appendix B for a detailed discussion).

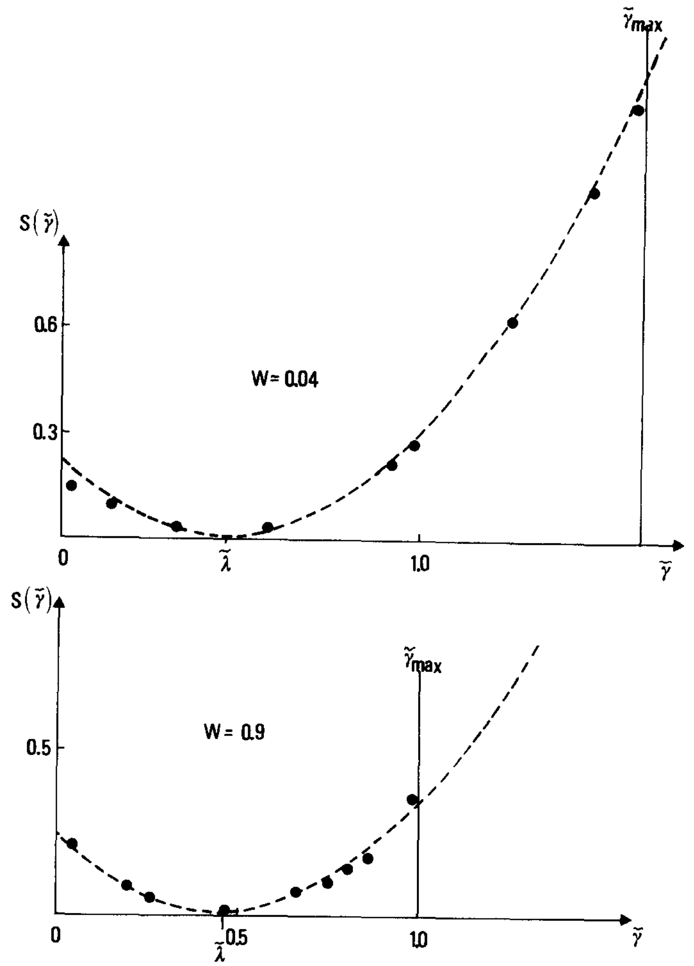

Fig. 18. $\tilde{S}(\tilde{\gamma})$ vs. $\tilde{\gamma}$ ($\tilde{S} = S/(C_2 w^{2/3})$ and $\tilde{\gamma} = \gamma/(C_2 w^{2/3})$) for $w = 0.9$ (a) and $w = 0.04$ (b). The dashed lines indicate the lognormal approximation; the dots are obtained by a Legendre transform from the numerical calculation of $L(q)$ shown in fig. 16.

In our case we obtain $\mu/\lambda_1 \simeq 1.17 > 1$ which implies that the $R(n)$ fluctuations, due to intermittency, are relevant also from a qualitative point of view.

In the case $\mu/\lambda_1 > 1$ the intermittency cannot be neglected in the correction to the 'mean field' which takes into account only the maximum $\tilde{R}$ of the probability distribution, since it gives a quite different qualitative behaviour. In fact for $N \to \infty$, $\tilde{R}(N) \to 0$ while $\langle R(N) \rangle \to \infty$ (see section 3.2).

Let us remark that the above behaviour of $L(q)/q$ (i.e. linear for small q and then a saturation to $\gamma_{max} \propto w^{1/2}$) is not peculiar to the probability distribution of V_n but is typical of all cases with $\langle V \rangle = 0$ and $V_n < V_{max} < \infty$, i.e. it does not depend on the probability distribution of the V_n's considering that the distribution of V_n is concentrated on a finite interval with a zero mean value.

In the case $\langle V \rangle = a \neq 0$ and $E = 2$ (or equivalently $\langle V \rangle = 0$ and $E = 2 - wa$) it is easy to show that $\lambda_1 \propto w^{1/2}$, $L(1) \propto w^{1/2}$, $\gamma_{max} \propto w^{1/2}$. Therefore it follows that the crossover phenomenon is peculiar to the band edge.

Bouchaud et al. [BGHLM86] have calculated the set of the $L(q)$ in the case of the potential wV_i distributed according to a Gaussian distribution of variance σ and mean value v. They define $L(q)$ as:

$$L(q) = \lim_{N\to\infty} \frac{1}{N} \ln\left\langle \left(\mathrm{Tr} \prod_{i=1}^{N} A_w(i) \right)^q \right\rangle \tag{4.3.6}$$

and they use the equality:

$$\left\langle \left(\mathrm{Tr} \prod_{i=1}^{N} A_w(i) \right)^q \right\rangle = \mathrm{Tr}\left\langle \prod_{i=1}^{N} A_w(i)^{\otimes q} \right\rangle = \mathrm{Tr}\langle A_w^{\otimes q} \rangle^N \tag{4.3.7}$$

where $A_w^{\otimes q}$ denotes the direct product of $A_w \otimes A_w \otimes \cdots \otimes A_w$, q times. One has to estimate the highest eigenvalue Λ_q of the $2^q \times 2^q$ matrix $\langle A_w^{\otimes q} \rangle$ which can be reduced to a $(q+1) \times (q+1)$ matrix using symmetry considerations [P82].

The calculation inside the band except for $v = 2$, gives:

$$L(q) = \lambda_1 q + \tfrac{1}{2}\mu q^2 \qquad \text{for } q \text{ even} \tag{4.3.8}$$

with $\lambda_1 = \sigma/[2v(4-v)]$.

At the band edge $v = 0$, no perturbation theory has yet been applied but for $q = 2$ and $q = 4$ one obtains respectively:

$$\frac{\Lambda_q - 1}{q\sigma^{1/3}} = \frac{2^{1/3}}{2}\,;\ \frac{(42)^{1/3}}{4} \tag{4.3.9}$$

where the $L(2)$ value agrees with the [PVu86] result.

Moreover for q integer >2 the $\sigma^{1/3}$ behaviour was shown to always hold in the limit of weak disorder, at difference with the case of a bounded potential where at fixed w a "transition" to the behaviour (4.3.5) appears for $q > \bar{q} \propto w^{-1/6}$.

We finally recall that some features of the multifractal nature of one-dimensional localization can be obtained by considering a simple multiplicative process [PS86, PSTZ87]. The random matrices product problem is reduced to a scalar product (i.e. a random number multiplication, see appendix B) in the framework of a random phase-like approximation.

4.4. Free energy fluctuations in spin glasses

In the statistical mechanics of disordered systems the fluctuations of the free energy among different replicas can be regarded as the analogue of the temporal intermittency in a chaotic signal. Let us, e.g., consider a spin model on a D-dimensional lattice with Hamiltonian:

$$H[\{J_{ij}\}] = -\sum_{(i,j)} J_{ij}\sigma_i\sigma_j \tag{4.4.1}$$

where $\sigma_i = \pm 1$ is the value of the spin on the site i and the coupling J_{ij} is an independent random variable distributed according to a probability distribution $P(J_{ij})$. Given a coupling realization $\{J_{ij}\}$, the partition function of an N spin system is the trace of the Boltzmann weight $\exp(-\beta H_N)$:

$$Z_N(\beta, \{J_{ij}\}) = \sum_{\{\sigma_i\}} \exp\{-\beta H_N[\{J_{ij}\}]\}\,. \tag{4.4.2}$$

The free energy per spin then is in the thermodynamic limit $N \to \infty$:

$$F(\beta) = \lim_{N\to\infty} -\frac{1}{N\beta}\langle \ln Z_N \rangle = \lim_{N\to\infty} -\frac{1}{N\beta}\int P(J_{ij})\,\mathrm{d}J_{ij} \ln Z_N(\beta, \{J_{ij}\})\,. \tag{4.4.3}$$

In order to compute this quenched average one usually applies the so-called replica trick. One namely computes the free energy per spin of q non-interacting replicas:

$$\mathscr{F}(q) = \lim_{N\to\infty} -\frac{1}{N\beta q} \ln\langle Z_N^q \rangle\,. \tag{4.4.4}$$

The results obtained at integer positive q are then extrapolated at $q = 0$:

$$F = \lim_{q\to 0}\lim_{N\to\infty} \frac{1 - \langle Z_N^q \rangle}{N\beta q} = \lim_{q\to 0} \mathscr{F}(q)\,. \tag{4.4.5}$$

Even in the mean field approximation of infinite range couplings, this procedure encountered great difficulties due to the replica symmetry breaking [see e.g. SK75, HP79] which were finally overcome by the famous mean field Parisi solution [P79].

We do not want to enter into this problem but just show how the $\mathscr{F}(q)$ are interesting by their own. Indeed, they allow to reconstruct the probability distribution of the partition function among the different realizations with respect to disorder [DT81, P86].

The multifractal approach can be easily extended to spin glasses. It is sufficient to introduce the free energy per spin of a coupling realization $\{J_{ij}\}$ of an N-spin system:

$$\Xi_N = -\frac{1}{N\beta} \ln Z_N(\beta, \{J_{ij}\})\,. \tag{4.4.6}$$

In the thermodynamical limit almost each sample has to have free energy F and we thus recover the selfaveraging of Ξ_N:

$$F = \lim_{N\to\infty} \Xi_N\,. \tag{4.4.7}$$

This is an ergodicity hypothesis since we are assuming that the ensemble average (4.4.3) and the spatial average (4.4.6) are equivalent.

In a unidimensional system with first neighbour interactions and uniform field h, we can write the partition function (4.4.2) as the trace over 2×2 random transfer matrices product. The Hamiltonian is now $H = -\Sigma_i\,(J_i\sigma_i\sigma_{i+1} + h\sigma_i)$ so that one has:

$$Z_N = \mathrm{Tr}\prod_{i=1}^{N} M_i\,, \qquad M_i = \begin{pmatrix} \exp(\beta J_i + \beta h) & \exp(-\beta J_i + \beta h) \\ \exp(-\beta J_i - \beta h) & \exp(\beta J_i - \beta h) \end{pmatrix}. \tag{4.4.8}$$

The maximal Lyapunov exponent is thus related to the free energy while the generalized Lyapunov exponents to the $\mathscr{F}(q)$:

$$L(q) = -\beta\,\mathscr{F}(q)\,q\,, \qquad \lambda_1 = -\beta F\,. \tag{4.4.9}$$

Derrida et al. [DPV78] have, e.g., found the expression of λ_1 in the limit of vanishing temperature for any value of the uniform field if $P(J_i)$ is the sum of two delta functions and their method can be extended to the calculation of $L(q)$.

The reader can identify $-\beta\Xi_N$ with the LEP γ (see definition (3.5.5)) whose probability distribution is reconstructed by the knowledge of the generalized exponents $L(q)$.

We can however use the multifractal approach without having recourse to the transfer matrix formalism in a generic dimension.

Let us group all the N spin system realizations with free energy in the range $[\Xi, \Xi + \mathrm{d}\Xi]$ into the same subset $\Omega(\Xi)$, as usual. If we want a finite free energy per spin and per replica $\mathcal{F}(q)$, then we must assume that the probability $\Pi(\Xi)$ that a realization belongs to $\Omega(\Xi)$ decays exponentially with N for $\Xi \neq F$:

$$\Pi(\Xi) \propto \exp\{-S(\Xi)N\} \tag{4.4.10}$$

where $S(\Xi) \geq 0$ and $S(F) = 0$. The moments of the partition function can be estimated as an integral over the spectrum of the possible free energies $[\Xi_{\min}, \Xi_{\max}]$:

$$\langle Z_N(\beta)^q \rangle \propto \int \Pi(\Xi)\, \mathrm{d}\Xi \exp(-\beta\Xi q N)\,. \tag{4.4.11}$$

We thus have in the limit $N \to \infty$:

$$q\,\beta\,\mathcal{F}(q) = \min_{\Xi}\,[\beta\Xi q + S(\Xi)]\,. \tag{4.4.12}$$

Let us note that the Oseledec theorem corresponds to the self-averaging (4.4.7) as in the previous section.

Let us conclude with the remark that if the J_{ij} are bounded (i.e. $|J_{ij}| \leq C$), it is possible to give a rough estimate on the minimal value of the free energy $\Xi_{\min}$. It is in fact given by the free energy of the pure system $J_{ij} = C$ for each couple of neighbours i, j.

5. Multifractal structures in condensed matter systems

5.1. General remarks

In this section we briefly review some critical or critical-like phenomena in the study of which the multifractal method appears as a powerful tool. The usual attitude is to assume that the relevant features of a system near the critical point are determined by a finite number of relevant operators.

It is therefore natural to ask whether this idea is in agreement or not with the existence of anomalous scaling and of multifractal structures.

Fourcade et al. [FBT86a, FBT86b] have shown that the anomalous scaling is compatible with a finite number of relevant operators even if an infinite set of irrelevant operators (irrelevant in the usual technical sense used in critical phenomena) are also important. They are related to the Renyi dimensions d_n with $n > 0$ and give a complete characterization of the statistical properties of the system.

Let us, e.g., consider the case of a lattice whose bonds are occupied by a resistor with probability p

or by an insulator with probability $1-p$. In these random resistor networks, close to the percolation threshold p_c, the measurable quantities of interest have the form

$$\sum_k y_k^n$$

where y_k is the power dissipated in the branch k. Now we define the exponent x_n (analogous to d_n) as follows:

$$\left\langle \sum_k y_k^n \right\rangle_{\xi,L} \propto L^{-x_n}, \qquad L \to \infty \tag{5.1.1}$$

where $\langle \cdot \rangle_{\xi,L}$ refers to the average over the sample realizations, L is the system size $L \lesssim \xi$ and $\xi \propto (p-p_c)^{\nu}$ is the correlation length. In analogy with the critical phenomena we introduce the joint probability for $\Sigma_k y_k^0, \Sigma_k y_k^1, \dots$:

$$P\Big(\sum_k y_k^0, \sum_k y_k^1, \dots, \xi, L\Big)$$

and we assume the following finite-size scaling behaviour:

$$P\Big(\sum_k y_k^0, \sum_k y_k^1, \dots, \xi, L\Big) = \lambda^{x_0}\lambda^{x_1} \dots P\Big(\sum_k y_k^0 \Big/ \lambda^{-x_0}, \sum_k y_k^1 \Big/ \lambda^{-x_1}, \dots, \xi/\lambda, L/\lambda\Big) \tag{5.1.2}$$

where λ is a rescaling parameter.

No check of the scaling (5.1.2) has appeared yet. However it has been numerically verified [ARC85] and in some cases proved [SW76] that partial distribution functions

$$P_n\Big(\sum_k y_k^n \Big/ \lambda^{-x_n}, \infty, 1\Big) = \int \prod_{j\neq n} \mathrm{d}\Big(\sum_k y_k^j \Big/ L^{-x_j}\Big)\, P\Big(\sum_k y_k^0 \Big/ L^{-x_0}, \dots \infty, 1\Big) \tag{5.1.3}$$

do scale as predicted by eq. (5.1.2). We remark that eq. (5.1.2) implies

$$\left\langle \Big(\sum_k y_k^n\Big)^m \right\rangle_{\xi,L} \propto L^{-mx_n} \tag{5.1.4}$$

i.e. powers of a given $U_n = \Sigma_k y_k^n$ obey to a gap scaling but different U_n scale in an anomalous way.

One can see [RTT85] that there is a relevant operator, controlled by $p-p_c$, while there is an infinite set of irrelevant operators because $x_0 - x_n < 0$ for $n > 0$.

A similar result has been obtained in a ϕ^4 theory: an infinite set of irrelevant operators associated with $\int \mathrm{d}x\, \phi^n(x)$ is necessary to give a complete characterization of the distribution function of the field ϕ [FBT86a].

5.2. Multifractal wavefunction at the localization threshold

We want here to dicuss further the Anderson model by considering the structure of the wavefunction at the localization transition. The universal properties near the localization threshold can be investigated by an expansion in the difference between the spatial dimension D and the lower critical dimension which is commonly believed to be two [AALR79]. It was in fact argued that even the presence of weak disorder induces a (marginal) localization of all the quantum states of independent electrons in two dimensions. Wegner [W79, W80] showed how the localization problem can be handled by means of ε-expansion techniques applied to the non-linear σ-model.

He determined [W80, W86] the exponents of the moments P_q of the wavefunctions for the Anderson localization defined as:

$$P_q(E) = \left\langle \sum_\lambda |\psi_\lambda(\boldsymbol{r})|^{2q} \delta(E - E_\lambda) \right\rangle \Big/ \hat{\rho}(E) \tag{5.2.1}$$

where ψ_λ is the amplitude of the localized wavefunction $|\lambda\rangle$ with energy E_λ at site $\boldsymbol{r}$ and $\hat{\rho}$ is the density of states.

The average is taken over realizations of the disorder. P_q vanishes near the mobility edge E_c like:

$$P_q(E) \propto (E - E_c)^{\Pi_q} \propto \xi^{-\Pi_q/\nu} . \tag{5.2.2}$$

As before, ξ is the localization length which scales as

$$\xi(E) \propto (E - E_c)^{-\nu} . \tag{5.2.3}$$

The value $\nu = 1/\varepsilon + \mathrm{O}(\varepsilon^3)$ evaluated in the ε-expansion is often considered exact [KS83, H83]. The critical exponents in (5.2.2) have been obtained [W86] up to the third order in ε:

$$\begin{aligned} \Pi_q &= D\nu(q-1) - \nu\zeta_{2q} \\ \zeta_{2q} &= q(q-1)\left[1 - \frac{\zeta(3)}{4}\,\varepsilon^3\,(q^2 - q + 1)\right] + \mathrm{O}(\varepsilon^4) \end{aligned} \tag{5.2.4}$$

with $\zeta(3) \simeq 1.2020569$.

Castellani and Peliti [CP86] related the anomalous scaling of P_q to the multifractal structure of the wavefunction near the localization threshold.

They assumed that just one typical wavefunction φ_E gives a significant contribution to the moments $P_q(E)$ for $E \simeq E_c$.

It follows that it is possible to define a coarse grained probability density, 'a mass', $p_i(l)$ by integrating $|\varphi_E|^2$ over a box Λ_i of size l:

$$p_i(l) = \int_{\Lambda_i} |\varphi_E(\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x} . \tag{5.2.5}$$

Its support is the whole space of dimension $D = d_0$ which is not fractal of course, but can be regarded as multifractal with respect to the scaling of $p_i(l)$.

On the other hand, we can estimate the moments $P_q(E)$ for $E \simeq E_c$ integrating $|\varphi_E|^2$ over a box of size ξ since φ_E rapidly decays for scales larger than ξ:

$$P_q(E) \simeq \sum_{\{\text{boxes } \Lambda_i\}} p_i(l)^q \propto \left(\frac{\xi}{l}\right)^{-\Pi_q/\nu} . \tag{5.2.7}$$

Near the critical point we can however evaluate P_q by means of the coarse grained probability density $p_i(l)$ because of the scaling invariance.

With the standard assumptions, the relevant scaling parameter for the rescaled system made of boxes of size l is ξ/l and one has:

$$P_q(E) \simeq \sum_{\{\text{boxes } \Lambda_i\}} p_i(l)^q \propto \left(\frac{\xi}{l}\right)^{-\Pi_q/\nu} . \tag{5.2.7}$$

A direct comparison of (5.2.7) with (5.2.6) shows that

$$\phi(q) = d_{q+1} q = \Pi_{q+1}/\nu . \tag{5.2.8}$$

The singularities of the measure $|\varphi_E(\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x}$ with respect to the Lebesgue measure can then be characterized by means of the Renyi dimensions computed with field theoretical techniques. One namely finds to fourth order in ε:

$$d_{q+1} = D - \nu^{-1}(q+1) + \frac{\zeta(3)}{4} \, \nu^{-1} \, \varepsilon^3 \, (q^3 + 2q^2 + 2q + 1) . \tag{5.2.9}$$

We can write (5.2.9) using the estimate $\nu^{-1} = \varepsilon + \mathrm{O}(\varepsilon^5)$ [H83] and noting $D = 2 + \varepsilon$:

$$d_{q+1} = D_{\mathrm{I}} - \frac{\mu q}{2} + \frac{\zeta(3)}{4} \, \varepsilon^4 \, (2q^2 + q^3) + \mathrm{O}(\varepsilon^5) \tag{5.2.10}$$

where:

$$D_{\mathrm{I}} = 2 + \frac{\zeta(3)}{4} \, \varepsilon^4 + \mathrm{O}(\varepsilon^5) \tag{5.2.11a}$$

$$\frac{\mu}{2} = \varepsilon\left(1 - \frac{\zeta(3)}{2} \, \varepsilon^3\right) + \mathrm{O}(\varepsilon^5) . \tag{5.2.11b}$$

This result indicates that the wavefunction has a multifractal structure since the information dimension differs from the space dimension $d_0 = D$.

We are interested in the three-dimensional case where we have to extrapolate formula (5.2.10) up to $\varepsilon = 1$ to obtain the information dimension:

$$d_1 \simeq 2.3 . \tag{5.2.12}$$

Let us recall that $\phi(q)$ is a convex function of q (i.e. d_q is non-increasing).

Moreover, $\phi(q)$ is non-decreasing if one excludes the possibility of negative measure singularities α. At $\varepsilon = 1$ we can therefore trust the ε-expansion result (5.2.10) only up to $q \simeq 1.25$ corresponding to $d_{1.25} \simeq 2.24$, see fig. 19.

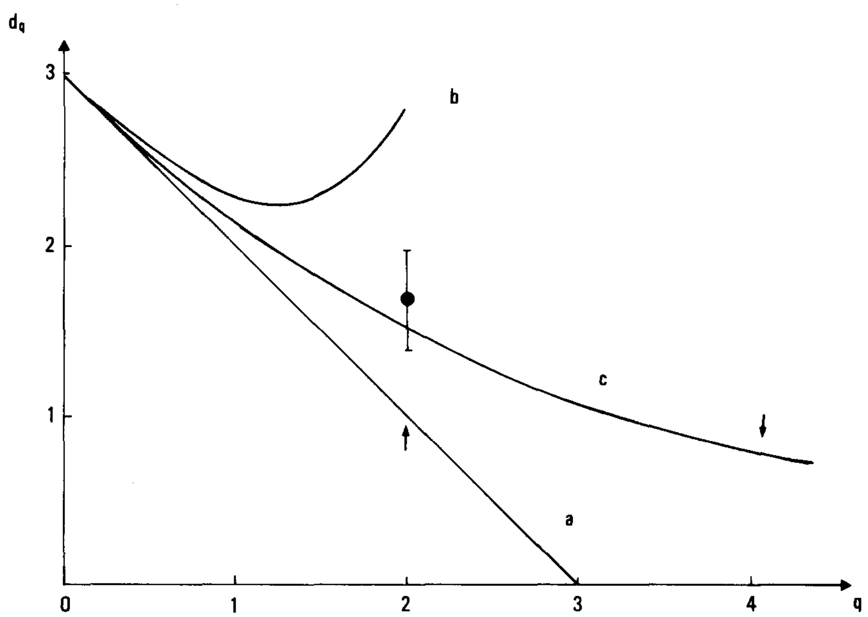

Fig. 19. The Renyi dimensions d_q vs. q for the wavefunction at the localization threshold in three dimensions. Line (a) indicates the ε-expansion result at first order, line (b) the fourth order result (5.2.10) and line (c) the Borel resummation (5.2.15). The dot is the numerical result for d_2 with the error bar [SE84]. The little arrows indicate the q value for which the function $d_q(q-1) = \phi(q-1)$ has zero derivative. For larger q one has negative derivatives which corresponds to unphysical negative singularities.

On the other hand, Soukoulis and Economou [SE84] have numerically calculated the correlation integral at the mobility edge:

$$A(L) = \int \mathrm{d}x\, |\psi(x)|^2 \int_0^L \mathrm{d}x'\, |\psi(x+x')|^2 \tag{5.2.13}$$

which should scale as L^{d_2}.

The correlation dimension d_2, called by them 'fractal dimension' (of the measure), is thus estimated to be

$$d_2 = 1.7 \pm 0.3\,. \tag{5.2.14}$$

A comparison between this value and the result of Wegner $d_2 = 3 - 2\,\varepsilon + \frac{3}{2}\,\zeta(3)\,\varepsilon^4 \simeq 2.8$ makes no sense because the fourth order ε-expansion gives d_q increasing with q for $q \gtrsim 1.25$ at $\varepsilon = 1$.

However, we can try to improve the ε-expansion result by means of a Borel resummation of the Pade's approximant of (5.2.4):

$$\frac{\zeta_{2q}}{q(q-1)} = \int_0^\infty \frac{\mathrm{e}^{-y} y\,\mathrm{d}y}{(1 + B\varepsilon^3 y^3)} \tag{5.2.15}$$

with $B = \zeta(3)\,(q^2 - q + 1)/(4\cdot 4!)$. In three dimensions, we thus get $d_1 = 2.14$ and $d_2 = 1.51$ in good agreement with the numerical estimate (5.2.14). Figure 19 shows the results for the Renyi dimensions obtained to first order in ε, at the fourth order in ε (5.2.10) and by the Borel resummation (5.2.15).

In our discussion we have implicitly extended the results (5.2.4) for Π_q to non-integer q-values. It is then possible to apply the usual Legendre transformation to obtain either $f(\alpha)$ or $H(\alpha) = -f(\alpha) + \alpha$ which characterize the singularities α of the measure.

It is rather useful to write $H(\alpha)$ in powers of $\tilde{D} = D_{\mathrm{I}} - \alpha = 2 - \alpha + (\zeta(3)/4)\varepsilon^4 + \mathrm{O}(\varepsilon^5)$:

$$H(\alpha) = \frac{\tilde{D}^2}{2\mu}\left[1 + \varepsilon\frac{\zeta(3)}{16}\tilde{D}^2 + \varepsilon^2\frac{\zeta(3)}{4}\tilde{D} + \mathrm{O}(\varepsilon^4)\right]. \tag{5.2.16}$$

Let us stress that a first-order calculation [CP86] gives $d_{q+1} = 2 - \varepsilon q$ and the parabolic shape $H(\alpha) = (2-\alpha)^2/(2\mu)$ with $D_{\mathrm{I}} = 2$ and $\mu = 2\varepsilon$. In this case, the 'relevant' (in the information theory sense) part of the critical wavefunction would cover a bidimensional structure in any space dimensionality $D \geq 2$.

The function $H(\alpha)$ is defined in a bounded range $[\alpha_{\min} > 0, \alpha_{\max}]$ as discussed in the first section. Near the critical point, the probability $P(\alpha)$ of finding a box for which $p_i(l) \sim l^{\alpha}$ should scale like:

$$P(\alpha) \propto \xi^{-H(\alpha)}$$

This means that at fixed l, the probability of finding a singularity which differs from the information dimension vanishes as a power of ξ as E tends toward E_{c}, the mobility edge. In the limit $E = E_{\mathrm{c}}$ only the singularity $\alpha = D_{\mathrm{I}}$ survives.

In this sense we have a hierarchy of critical exponents d_q in contrast with usual critical phenomena. The larger is $H(\alpha)$, the less important is the corresponding exponent α. At the top of the hierarchy there is the information dimension for which H reaches its minimum value $H = 0$.

5.3. *Growth probability distribution in kinetic aggregation processes*

The diffusion limited aggregates DLA were introduced by Witten and Sander [WS81] for the description of growth phenomena. In this model the growth on a lattice starts from an initial seed. At each step a diffusive particle is released from infinity. When it strikes the aggregate cluster, it becomes part of the cluster.

It is simple to generate such an aggregate on a two-dimensional lattice by a numerical simulation. One thus obtains a fractal object with $D_{\mathrm{F}} \simeq 1.7$ which has a highly ramified structure since particle deposition is favoured at the tips. This implies practically full screening of the bulk.

The relevant properties of these clusters can be characterized by the site growth probability distribution. We have namely to consider the probability $\rho(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}$ that a random walker lands on the boundary of the cluster $\partial\Gamma$ between the points $\boldsymbol{r}$ and $\boldsymbol{r} + \mathrm{d}\boldsymbol{r}$.

$\rho(\boldsymbol{r})$ is the harmonic measure which can be defined for the boundary $\partial\Gamma$ of the cluster as the normal derivative $\partial_{\mathrm{n}} g$ of the Green's function $g(\boldsymbol{r})$ for the Laplace equation $\Delta g = 0$ with the boundary conditions $g(\infty) = \text{constant}$ and $g(\partial\Gamma) = 0$. The singularities of $\rho(\boldsymbol{r})$ were first characterized by means of the multifractal approach by Halsey et al. [HMP86]. The coarse grained probability over a box Λ_i of size l and centre $\boldsymbol{r}_i$ is:

$$p_i(l) = \int_{\Lambda_i} \rho(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}. \tag{5.3.1}$$

Using the properties of the Green's function [HMP86] one can show that the information dimension of the harmonic measure is:

$$d_1 = 1 \tag{5.3.2}$$

for the DLA as well as for any connected set in two dimensions. The Renyi dimensions are given by the scaling of the moments:

$$\langle p_i(l)^q \rangle = \sum_{i=1}^{N(l)} p_i(l)^{q+1} \propto \exp(q\, d_{q+1})$$

where $N(l)$ is the number of intervals of size l necessary to cover the boundary. The fractal dimension of $\partial\Gamma$ is thus d_0. Moreover heuristic arguments [TS85] indicate that:

$$d_\infty = \alpha_{\min} = d_0 - 1 \simeq 0.7\,. \tag{5.3.3}$$

The Renyi dimensions can be computed by a direct simulation sending many particle probes toward the aggregate to estimate $p_i(l)$. By this method it is however difficult to calculate the negative moments, for which very improbable events are relevant. On the contrary, Amitrano et al. [ACL86] used the electrostatic analogy between DLA and dielectric breakdown [NPW84] to compute the whole set of the Renyi dimensions. Their results are shown in fig. 20 where we plot $\phi(q-1) = d_q(q-1)$ vs. q. They solved the discretized version of the Laplace equation for an electrostatic potential $\psi(x)$ on the DLA considered as a conducting cluster. One thus has

$$\rho(x) = K[\partial_n\, \psi(x)]^\eta \tag{5.3.4}$$

with $\eta = 1$. Other values of η allow to describe different models (e.g. $\eta = 0$ corresponds to the Eden model and $\eta = \infty$ produces one-dimensional clusters).

Let us stress that in the spirit of the scaling approach to critical phenomena it is reasonable (see e.g.

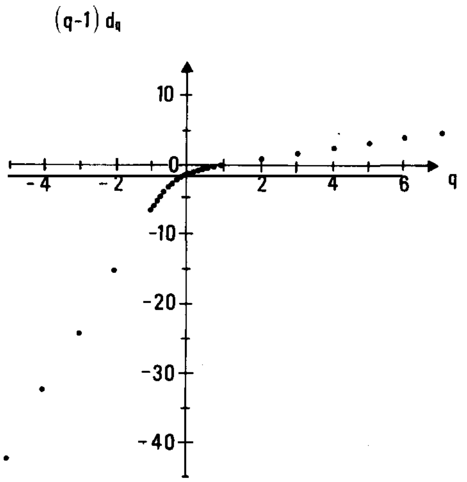

Fig. 20. Renyi dimensions of the harmonic measure on DLA obtained by numerical simulations [HMP86, ACL86].

[C86]) that the probability that a particle lands on an l-size interval of an aggregate of gyration radius L should scale like:

$$p_i^{(L)}(l) \propto (l/L)^{\alpha} . \tag{5.3.5}$$

L then plays the role of the correlation length ξ in critical phenomena. The hypothesis (5.3.5) corresponds to assume that the growth probability is scale invariant for l/L small enough with a hierarchy of scaling indices $\alpha(i)$. The asymptotic limit $L \to \infty$ is a sort of 'critical point'. In the numerical calculations one usually increases L at fixed lattice constant l. There is nevertheless another subtle point worth stressing. Indeed, one studies the moments

$$\sum_{i \in \partial\Gamma(L)} p_i^q \propto L^{-(q-1)\tilde{d}_q} \tag{5.3.6}$$

where p_i is not given by (5.3.1) but by the probability that the site i on the surface of the cluster becomes part of the aggregates. The dimensions $\tilde{d}_q$ and the standard ones d_q are assumed to be equal. This is true only if the aggregate is a homogeneous fractal with respect to the point density, i.e. if each interval Λ_i of the boundary contains approximatively the same number of points $n_i(l)$ so that:

$$\sum_{i=1}^{N(l)} n_i(l)^q \propto l^{D_F(q-1)} . \tag{5.3.7}$$

In this case the aggregate is a multifractal object with respect to the harmonic measure but not to the 'natural measure' given by the point density. With this warning, the numerical calculation [ACL86] shows that $\phi(q)$ is essentially linear for $q > 1$ and $q < -2$ and the result $\alpha_{\min} = 0.7$ agrees with the 'theoretical' prediction $\alpha_{\min} = D_F - 1$. On the other hand, $\phi(q)$ is highly non-linear around the value $q = -1$ corresponding to the fractal dimension $d_0 = -\phi(-1)$. The derivative

$$\mathrm{d}\phi/\mathrm{d}q|_{q=0} = 1 \tag{5.3.8}$$

is in agreement with (5.3.2) while the fractal dimension $d_0 \simeq 1.5$ is slightly different from the independent result $D_F \simeq 1.7$. This fact is attributed by the authors of ref. [ACL86] to the existence of a non-screened part of the perimeter (where $\psi(\boldsymbol{r})$ is neither zero nor exponentially decreasing with L) which should scale with an exponent $d_0 < D_F$.

It is easy to obtain the fractal dimension $f(\alpha)$ of the set $S(\alpha)$ on which is concentrated the measure with singularity α by means of the usual Legendre transform, see fig. 21.

Let us briefly recall the first phenomenological theory on the multifractal structure of the surface layer of DLA proposed by Halsey et al. [HMP86]. They assumed that the measure can be described in terms of two singularity values:

$$\alpha_1 = D_I = 1 , \qquad \alpha_2 = \alpha_{\min} = 0.7 . \tag{5.3.9}$$

It follows that there is one adjustable parameter $f(\alpha_2)$ since $f(\alpha_1) = f(D_I) = 1$ by definition.

Recalling that $\phi(q) = \min_\alpha [\alpha q + H(\alpha)]$ we thus see that the relation (5.3.9) corresponds to a function $\phi(q)$ which is a piecewise linear function of q:

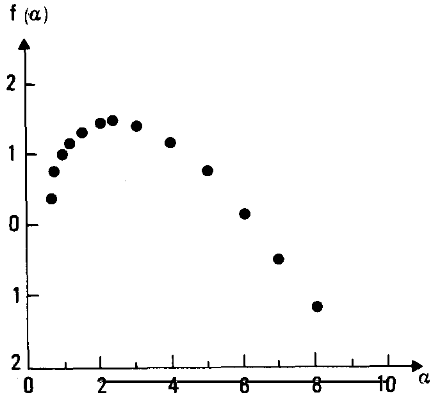

Fig. 21. $f(\alpha)$ vs. α for DLA obtained by a Legendre transform of the d_q's shown in fig. 20.

$$\begin{cases} \phi(q) = \alpha_1\, q + H(\alpha_1) = D_{\mathrm{I}} q = q\,, & \text{for } q < q_c \\ \phi(q) = \alpha_2\, q + H(\alpha_2) = (D_{\mathrm{F}} - 1)(q+1) - f(\alpha_2)\,, & \text{for } q > q_c \end{cases} \tag{5.3.10}$$

where $H(D_{\mathrm{I}}) = 0$ and $H(\alpha_2) = f(\alpha_2) - \alpha_2$.

The knowledge of q_c is equivalent to that of $H(\alpha_2)$ since by matching the two lines one gets:

$$q_c = H(\alpha_2)/(1 - \alpha_2)\,. \tag{5.3.11}$$

These results agree quite well with the direct calculation of the moments for positive $q \geq 1$ using the value $f(\alpha_2 = 0.71) = 0.42$, i.e. $q_c = 0.74$. On the other hand the formula (5.3.10) is inconsistent with the result for the fractal dimension since it implies $D_{\mathrm{F}} = -\phi(-1) = 1$ which is much smaller than the value $D_{\mathrm{F}} = 1.71$ assumed for estimating α_2.

5.4. *Multifractality in percolation*

The critical behaviour of random resistor networks is a percolation problem which can be studied by means of the multifractal approach [ARC85, RTBT85].

Let us consider a lattice of size L where each bond is conducting with probability p or insulating with probability $1 - p$. In the limit of infinite L there is a percolation threshold p_c above which there is an infinite cluster of conducting bonds.

At the critical point p_c one isolates the set of bonds carrying a non-zero current from the rest of the infinite cluster. This set is called the backbone and is a fractal object with dimension:

$$D_{\mathrm{F}}^{(\mathrm{BB})} = \ln n_{\mathrm{BB}}(L)/\ln L \qquad \text{for } L \to \infty \tag{5.4.1}$$

where $n_{\mathrm{BB}}(L)$ is here the number of bonds belonging to the backbone of a system of size L.

Let us stress that this is *not* the box-counting definition (0.1). The backbone is made of singly connected bonds (links) and of multiply connected bonds (blobs) in a selfsimilar way. The fractal dimension of the set of the links is $1/\nu$ where ν is the connectedness length exponent [C81].

Let us now consider the backbone as a random resistor network each bond having a unit resistance. A unit voltage is then applied to the opposite boundaries of a box containing the percolating cluster.

Each bond can thus be characterized by the voltage drop across it. Let us define the strength $\alpha(i)$ of a bond i as:

$$V_i \propto L^{-\alpha(i)} . \tag{5.4.2}$$

The minimum value of α is associated with the links since they carry the total current I which is equal to the conductance of the system R^{-1} and to the maximal value of the voltage $V_{\max}$:

$$V_{\max} \propto L^{-\alpha_{\min}} , \qquad \alpha_{\min} = \zeta_R/\nu$$

where ζ_R is the resistance exponent defined by $R \propto L^{\zeta_R/\nu}$. The number of bonds $N_\alpha(L)$ with a voltage drop $V \sim L^{-\alpha}$ is:

$$N_\alpha(L) \sim L^{D(\alpha)} \tag{5.4.3}$$

where $D(\alpha)$ is a sort of fractal dimension. One can compute the scaling of the moments

$$\langle V^q \rangle = \sum_i V_i^q \propto L^{-(q-1)d_q}$$

in terms of $D(\alpha)$ with the multifractal machinery and the saddle point method:

$$(q-1)\, d_q = \min_\alpha (\alpha q - D(\alpha)) .$$

Note that $d_0 = D_F^{(BB)}$.

Also in this case we must repeat the warning done in section 5.3: $D(\alpha)$ can only be considered as a fractal dimension under certain assumptions.

De Arcangelis et al. [ARC85] computed the exponent d_q on a simple hierarchical model which describes the backbone properties for any spatial dimension. They found a reasonably good agreement with the existing numerical data.

The model is obtained by successive iterations starting from a single bond of unit resistance. As shown in fig. 22 at each step n of the construction the bonds are replaced by two links in series with a blob.

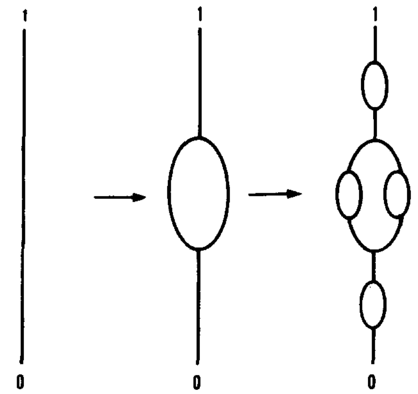

Fig. 22. Hierarchical model for the backbone of the infinite cluster near the percolation threshold.

It is easy to check that there are 4^n bonds and 2^n links, so that one identifies L with $2^{n\nu}$. This model is given by a multiplicative process and d_q can therefore be computed by a simple direct computation. For example, taking in the first step of the construction in fig. 22 the single link potential $V = \frac{2}{5}$ and the blob potential $V = \frac{1}{5}$ (values suggested by phenomenological considerations) one has:

$$\nu(q-1)\, d_q = 1 + \ln_2[(\tfrac{1}{5})^q + (\tfrac{2}{5})^q]$$

in good agreement with the numerical data in two dimensions.

We finish this section by recalling that anomalous scaling laws have been found in many condensed matter systems. It is impossible to analyse all these cases and we must limit ourselves to recall some of them: depletion of a diffusing substance in the vicinity of an absorbing fractal [CW86], random superconducting networks [ARC85], some properties of Laplacian walks [EL87].

6. Conclusions

The existence of continuous spectra of independent exponents arises as a common feature of the scaling laws in many physical systems. Table 1 indicates the main multifractal objects introduced up to now.

The popularity of the multifractal approach is due, in our opinion, to its usefulness in the interpretation and organization of theoretical as well as experimental results in a simple and unique scheme.

Table 1

Phenomenon	Mass density
Fully developed turbulence in three dimensions	Density of energy dissipation
Chaotic attractors	Density of points distributed according to the natural measure
Temporal intermittency	Density of trajectories in 'history' space
Metal–insulator transition in $2+\varepsilon$ dimensions	Square modulus of the wavefunction
Anderson localization in one dimension	Replicas of the systems with the same localization length
Aggregation phenomena	Growing probability
Conduction on percolative clusters	Voltage on the bonds of the cluster
Spin glasses	Replicas of the systems with the same free energy in the realization space

However, one must use proper techniques for each particular situation in order to reach a deeper understanding of a phenomenon. A typical example is given by the Anderson localization in one dimension where the calculation of the generalized exponents have been done following some methods developed in the context of disordered systems [BGHLM86, PS86, PV87b]. The opposite situation is fully-developed turbulence where a theory on the structure of the Navier Stokes equation singularities is still far from being formulated. Nevertheless, the analysis in terms of multifractals is a first useful tool and allows to give a qualitative description by means of phenomenological models.

We think that one of the most interesting future developments will be given by understanding the link between the renormalization group and generalized exponents. An open problem is the consistence of the usual scaling with a finite number of relevant operators and multifractality. Some authors have shown that the two aspects are compatible at least in some cases [FBT86a, FBT86b]. Moreover there is not yet a clear comprehension of the mechanisms which lead to 'phase transitions' (in the generalized exponents varying the moment order) which have been numerically observed in some dynamical systems and aggregates [BP87b, BGP87, ACL86].

Appendix A. The Kolmogorov–Obukhov lognormal model for intermittency in turbulence

Kolmogorov [K62] and Obukhov [O62] modified the K41 theory assigning a key role to the statistics of the spatial distribution of the energy density dissipation in order to take into account some of the early evidence on small-scale intermittency. Essentially in the new modified theory K62 the first hypothesis of K41 (see section 2.2) is replaced by assuming that in the range $r \ll L$ the n-variate probability distribution of $\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})$ only depends on $\tilde{\varepsilon}_x(r) = (1/r^3) \int_{\Lambda_x(r)} \varepsilon(\boldsymbol{y})\, \mathrm{d}^3\boldsymbol{y}$, r and ν.

The second hypothesis remains the same one as in K41.

Evidently these two hypotheses are quite general and common to all phenomenological cascade models. To obtain explicit results one needs further assumptions on the statistics of $\varepsilon(\boldsymbol{x})$, of course.

Kolmogorov and Obukhov supposed that $\tilde{\varepsilon}_x(r)$ is distributed according to a lognormal in the range $\eta \ll r \ll L$.

By dimensional analysis the first two hypotheses give for structure functions:

$$\langle |\Delta V(r)|^p \rangle \sim r^{p/3} \langle \tilde{\varepsilon}(r)^{p/3} \rangle . \tag{A.1}$$

Note that (A.1) holds also for fractal or multifractal models. According to the third hypothesis on the lognormal distribution, one has [GY67]:

$$\langle |\Delta V(r)|^p \rangle \sim r^{p/3+(1/18)\mu p(3-p)} \tag{A.2}$$

i.e.

$$\zeta_p = p/3 + (1/18)\mu p(3-p) \tag{A.3}$$

where μ is given by the fluctuations of $\tilde{\varepsilon}_x(r)$:

$$\mu = \frac{1}{\ln(L/r)} \langle (\ln \tilde{\varepsilon}(r) - \langle \ln \tilde{\varepsilon}(r) \rangle)^2 \rangle . \tag{A.4}$$

Kolmogorov and Obukhov postulated the lognormal distribution without any justification. It is however not difficult to introduce an argument for it. We briefly repeat the derivation due to Gurvich and Yaglom [GY67]. Let us call ε_n the quantity $\tilde{\varepsilon}_x(r_n)$ with $r_n = 2^{-n}L$, we then obtain ε_n as the product of independent random variables

$$\varepsilon_n = \varepsilon_0 \prod_{i=1}^{n} \varphi_i \,, \qquad \varphi_i = \varepsilon_i/\varepsilon_{i-1} \,. \tag{A.5}$$

Taking the logarithm of (A.5) one has

$$\ln \varepsilon_n = \ln \varepsilon_0 + \sum_{i=1}^{n} \ln \varphi_i \,. \tag{A.6}$$

Assuming that all the variables $\ln \varphi_i$ have finite mean value and finite variance (meaning that the event $\varphi_i = 0$ has probability zero) we can apply for large n ($r \ll L$) the central limit theorem. ε_n is therefore distributed according to a lognormal characterized by two parameters:

$$\frac{1}{\ln(L/r)} \langle \ln \varepsilon_n \rangle \quad \text{and} \quad \frac{1}{\ln(L/r)} \langle (\ln \varepsilon_n - \langle \ln \varepsilon_n \rangle)^2 \rangle \,.$$

Since $\langle \varepsilon_n \rangle$ must depend on n, because of the constancy of the forward energy transfer rate, just a relevant parameter survives: the variance given by (A.4).

We remark that in the Gurvich–Yaglom derivation there is the basic assumption that φ_i vanishes with zero probability. This hypothesis is not made in the fractal (or multifractal) model where there are cubes with $\varepsilon_n = 0$ (non-active fluid) [N69, N70, NS64, M74, FSN78, FP85, BPPV84]. This is the most remarkable difference between the two approaches.

Let us note that, at least formally, the lognormal model is a limiting case of the multifractal one, assuming a parabolic form for $d(h)$ with a maximum at $D_F = 3$ and no restriction on the possible h values. One then obtains $D_F = 3$, $D_I = D_F - \mu/2 = 3 - \mu/2$, $D^* = D_F - \mu = 3 - \mu$. The energy dissipation is concentrated on a three-dimensional support which is therefore not fractal. All the fluid is active but the 'measure' given by $\tilde{\varepsilon}_x(r)$ is fractal since $D_I < 3$.

ζ_p in eq. (A.3) has a parabolic form with a maximum at $p^* = \frac{3}{2}(2 + \mu)/\mu$ and ζ_p becomes negative for large values of p.

This odd feature is due to the fact that the lognormal model gives no restrictions on the possible values of ε_n. This is clearly an unphysical feature of the model. Indeed, we have seen that plausible bounds exist for the values of the singularities h (and therefore for ε_n).

The lognormal distribution, however, is a good approximation for a large class of phenomena but it gives uncorrect values for the moments (see appendix B), since they grow too rapidly with p and thus violate the Carleman's criterion [C22] which has to be satisfied for determining a probability distribution from its moments. This leads to unphysical consequences. If it is not possible to determine the probability distribution by moments of a quantity satisfying some evolution equations, then the initial values of all the moments do not allow to determine uniquely the values of the moments at future times. Orszag [O70b] showed that the non-uniqueness of the moment values at positive times is compatible with the uniqueness of the solution of the evolution equations.

Appendix B. Multiplicative processes and lognormal approximation

Multiplicative processes are the simplest way to get an approximate description of multifractals. Let us recall the random β-model for turbulence (section 2.3) or its equivalent for chaotic attractors [PV84, BPPV84], the response function for temporal intermittency (section 3) or random matrices (section 4). In this appendix we briefly discuss the multiplicative processes and the lognormal approximation in a general context.

Let us consider a variable x_n given by a product of $n \gg 1$ random variables:

$$x_n = \prod_{i=1}^{n} a_i \tag{B.1}$$

where the a_i's are positive independent random variables distributed according to the same distribution of probability laws. Moreover we assume that

$$a_{\min} < a_i < a_{\max} . \tag{B.2}$$

The probability distribution of x_n is close to the lognormal distribution:

$$P(x_n) \simeq P_{\mathrm{LN}}(x_n) = \frac{\exp[-(\ln x_n - \lambda n)^2/(2\mu n)]}{x_n (2\pi\mu n)^{1/2}} \tag{B.3}$$

where λ and μ are given by

$$\lambda = \frac{1}{n}\,\overline{\ln x_n} = \{\ln a\}$$

and

$$\mu = \frac{1}{n}\,\overline{(\ln x_n - \overline{\ln x_n})^2} = \{(\ln a - \{\ln a\})^2\} ,$$

where $\overline{(\cdot)}$ stands for the average on different realizations $[a_1, \ldots a_n]$ and $\{(\cdot)\}$ means average given by the probability distribution of a: $\{(\cdot)\} = \int P(a)\,(\cdot)\,\mathrm{d}a$.

Note that the logarithm of x_n is given by a sum of independent random variables

$$\ln x_n = \sum_{i=1}^{n} \ln a_i . \tag{B.4}$$

Because of the inequalities (B.2) and the independence of the a_i we can apply the central limit theorem. Therefore the probability distribution of the variable $y_n = \ln x_n$ is close to a Gaussian with mean value $n\lambda$ and variance $n\mu$. Equation (B.3) follows from a change of variables.

In spite of the fact that the lognormal distribution is a good approximation, its moments are not close to the right ones.

Using eq. (B.3) one has

$$\overline{x_n^q} = \int_0^\infty x_n^q \, P_{\mathrm{LN}}(x_n) \, \mathrm{d}x_n = \exp\{n \, g_{\mathrm{LN}}(q)\}$$

with

$$g_{\mathrm{LN}}(q) = \lambda q + \tfrac{1}{2} \mu q^2 \, . \tag{B.5}$$

It is trivial to see that (B.5) cannot be valid for large q. Indeed one can compute x_n directly by (B.1):

$$\overline{x_n^q} = \mathrm{e}^{n \, g(q)} \qquad \text{with } g(q) = \ln\{a^q\} \tag{B.6}$$

and because of (B.2) one has:

$$q \ln(a_{\min}) < g(q) < q \ln(a_{\max}) \, . \tag{B.7}$$

Equation (B.5) is not therefore a good approximation for large q. On the contrary, writing $x_n = \exp(q \ln x_n)$ and expanding in series to the second order in q, we see that for small q, $g(q) \simeq g_{\mathrm{LN}}(q)$. This apparent paradox (a probability distribution close to the lognormal whose moments do not follow (B.5)) is related to the fact that the moments of the lognormal distribution grow too fast [C22, O70b, M72].

Without invoking complex mathematical arguments one can convince oneself that the origin of the trouble resides in the tail of $P_{\mathrm{LN}}(x_n)$. Indeed, the true probability distribution $P(x_n)$ is zero for $x_n > (a_{\max})^n$, while for large q the greatest contributions to $\int_0^\infty x_n^q \, P_{\mathrm{LN}}(x_n) \, \mathrm{d}x_n$ come from a region where $P(x_n) = 0$. This becomes evident by computing the value $\tilde{x}^{(q)}$ for which $x_n^q \, P_{\mathrm{LN}}(x_n)$ takes its maximum. One in fact finds for large q

$$\tilde{x}_n^{(q)} \sim \exp(\text{const.} \cdot n \cdot q) \gg (a_{\max})^n \, .$$

We conclude by noting again that a lognormal approximation for moderate values of q (i.e. a parabolic shape of $f(\alpha)$ for α around D_{I}) is rather good, while for large q bounds like (B.2) have to be taken into account.

For example if we repeat the considerations of this appendix for fully developed turbulence, we get an approximate expression for ζ_p, since the random β-model involves a multiplicative process for $\tilde{\varepsilon}(r)$.

Let us remind (eq. 2.3.15) that in the random β-model one has

$$\langle |\Delta V(r_n)|^p \rangle \sim r_n^{p/3} \left\{ \prod_{i=1}^{n} \beta_i^{(1-p/3)} \right\} .$$

One therefore obtains, considering the lognormal approximation for the variable $\Pi_{i=1}^n \beta_i$:

$$\zeta_p \simeq p/3 + D_{\mathrm{I}}(1 - p/3) + (\mu/2)(1 - p/3)^2 \tag{B.8}$$

for $|p - 3|$ not very large.

Note that in (B.8) there are two independent prameters D_{I} and μ while only one is involved in K62.

We have in fact considered the multiplicative process on the fractal and the constraint $\zeta_3 = 1$ is imposed automatically by the model.

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