Dimensional transitions in turbulent flows

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Lecture 1:

2D vs 3D turbulence: Inviscid invariants & turbulent cascades

Thin fluid layers: coexistence of 2D and 3D turbulence

Lecture 2:

Rotation & Stratification effects on thin fluid layers

Turbulent cascade in 3D helical turbulence

Rotating thin fluid layers

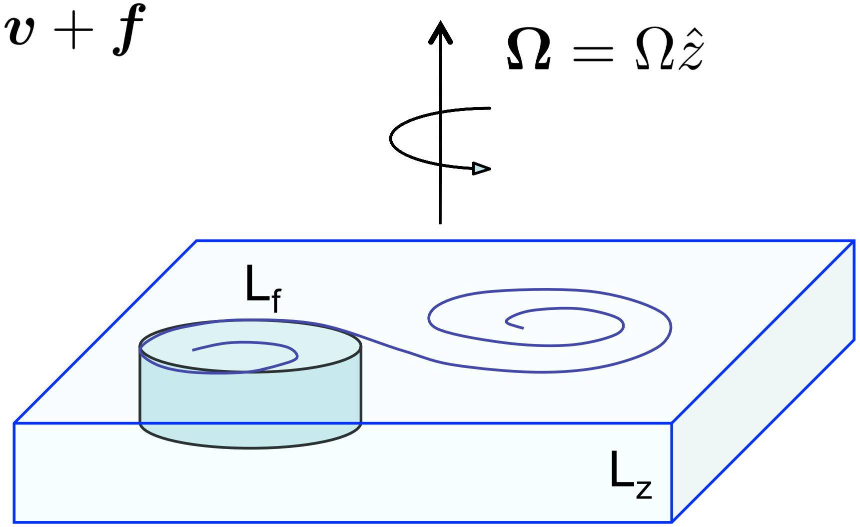
Rotating turbulent layer

Thin fluid layer in a rotating reference frame

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = -\nabla p + \nu \nabla^2 \boldsymbol{v} + \boldsymbol{f}$$

$$\nabla \cdot \boldsymbol{v} = 0$$

$$S = \frac{L_z}{L_f} \qquad R = \Omega \tau_f = Ro^{-1}$$



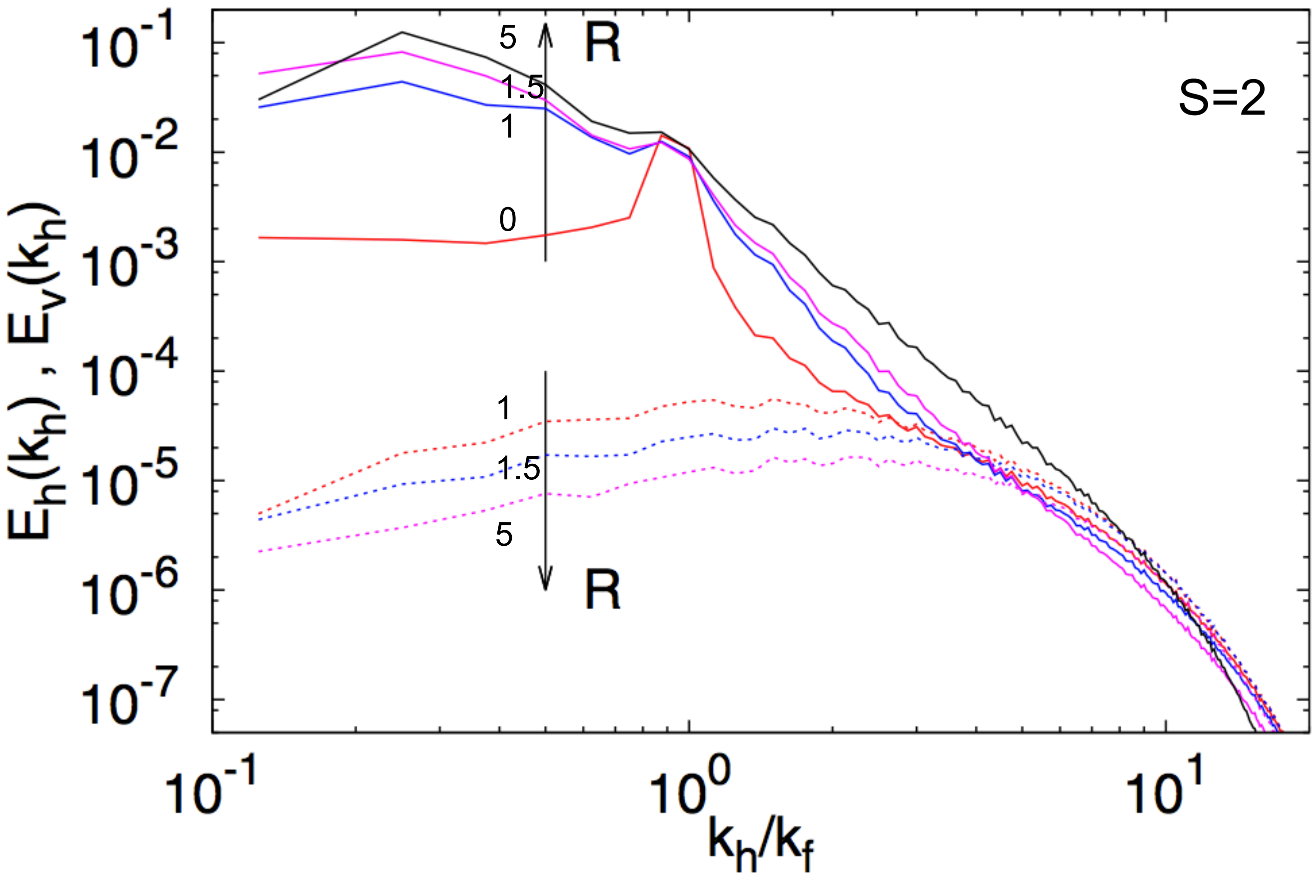
Rotation enhances the inverse energy cascade

L. Smith, J. Chasnov, and F. Waleffe, Phys. Rev. Lett. 77, 2467 (1996)
P. Embid and A. Majda, Geophys. Astrophys. Fluid Dyn. 87, 1 (1998)
Q. Chen, S. Chen, G. Eyink, and D. Holm, J. Fluid Mech. 542, 139 (2005)
L. Bourouiba and P. Bartello, J. Fluid Mech. 587, 139 (2007)
P. Mininni and A. Pouquet, Phys. Rev. E 79, 026304 (2009)
E. Deusebio, G. Boffetta, E. Lindborg, S.M. Phys. Rev. E 90, 023005 (2014)

Energy spectra

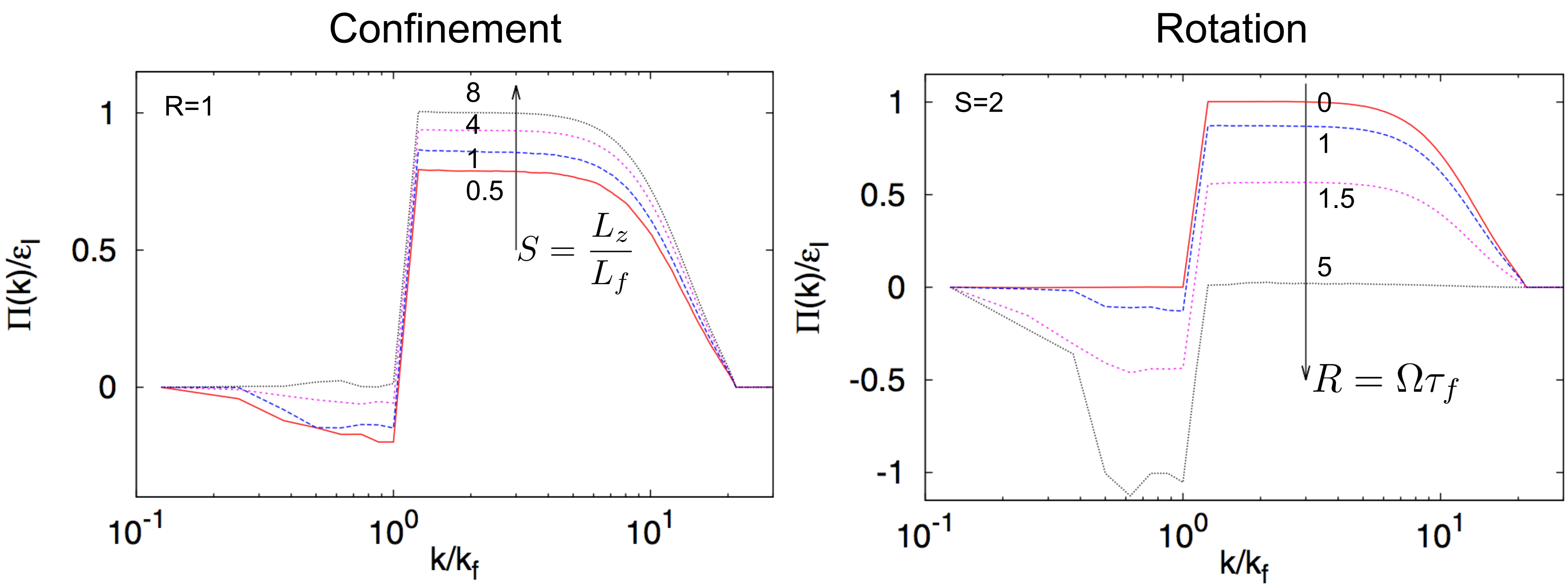
Energy spectra of horizontal (solid lines) and vertical velocities (dotted line)

$k_z = 0 \qquad k_h = (k_x^2 + k_y^2)^{1/2}$



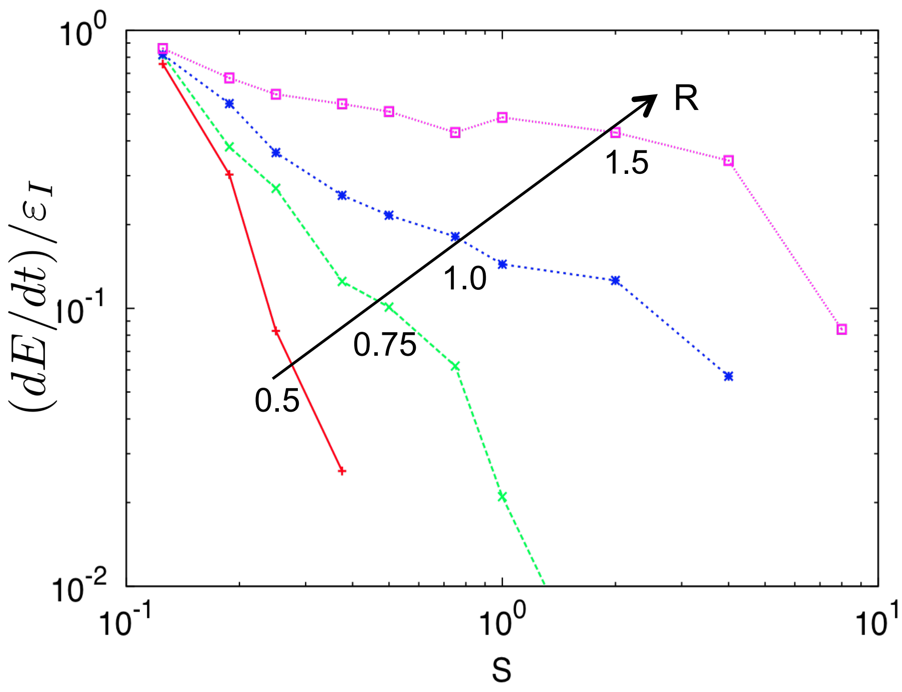
Spectral energy fluxes

$$\Pi_E(k) = \int_k^\infty T(k')dk' \qquad \partial_t E(k) = T(k) + F(k) - \nu k^2 E(k)$$

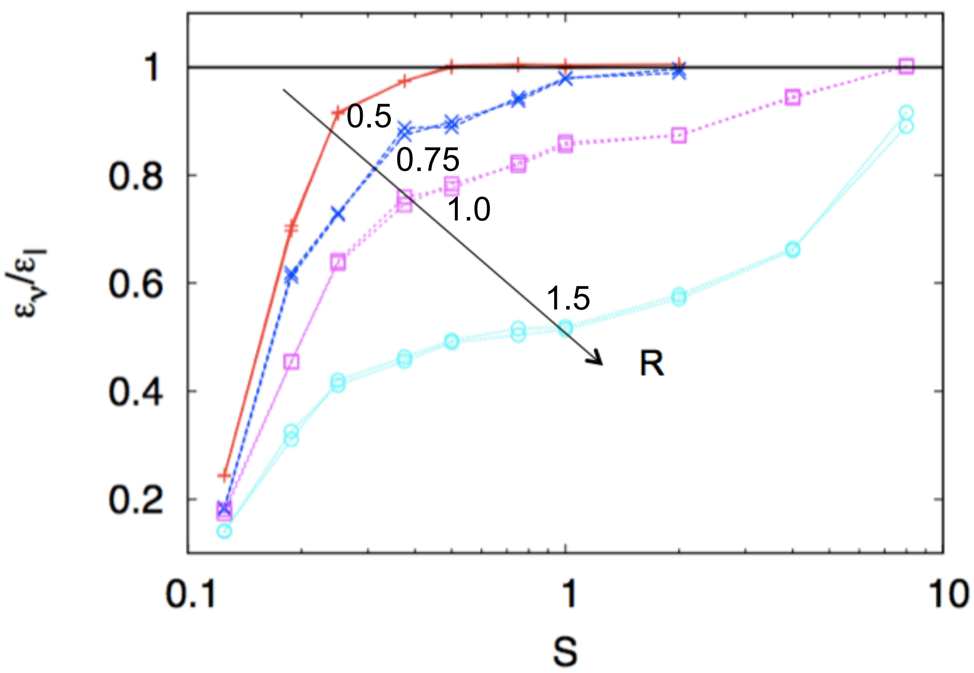


Fluxes of inverse/direct cascade

Energy flux toward large scales



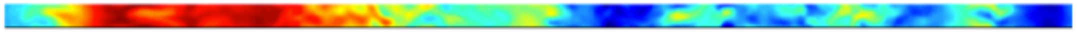
Energy flux toward small scales



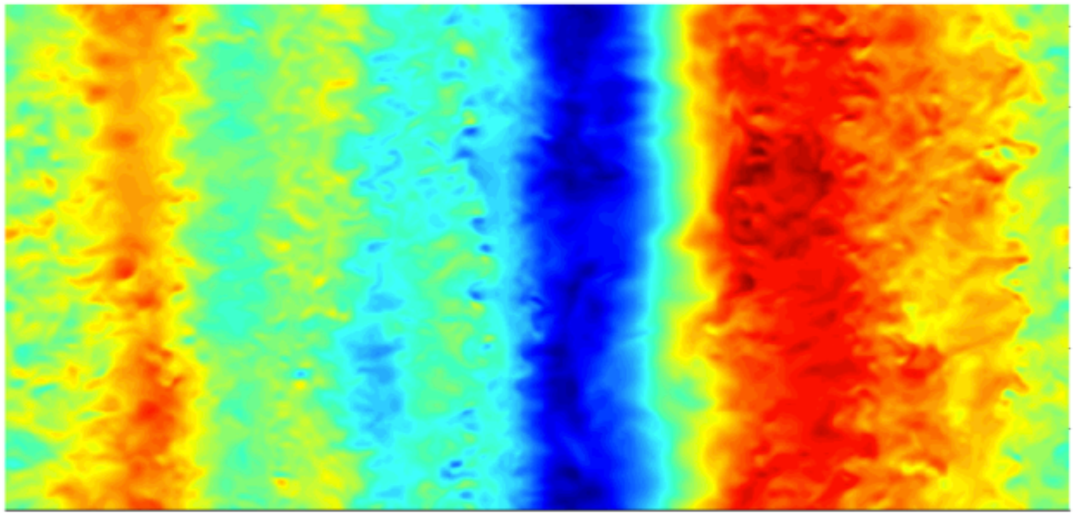
Rotation enhances the inverse cascade, but confinement is necessary.

Confinement vs. Rotation

Non-rotating thin layer (R=0 S=0.1875)

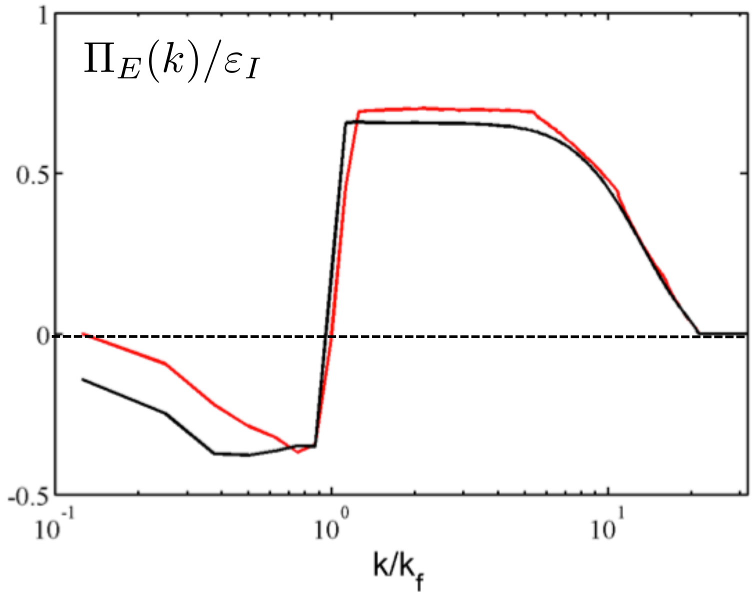


Rotating thick layer (R=1.5 S=4)



Vertical cuts of horizontal velocity

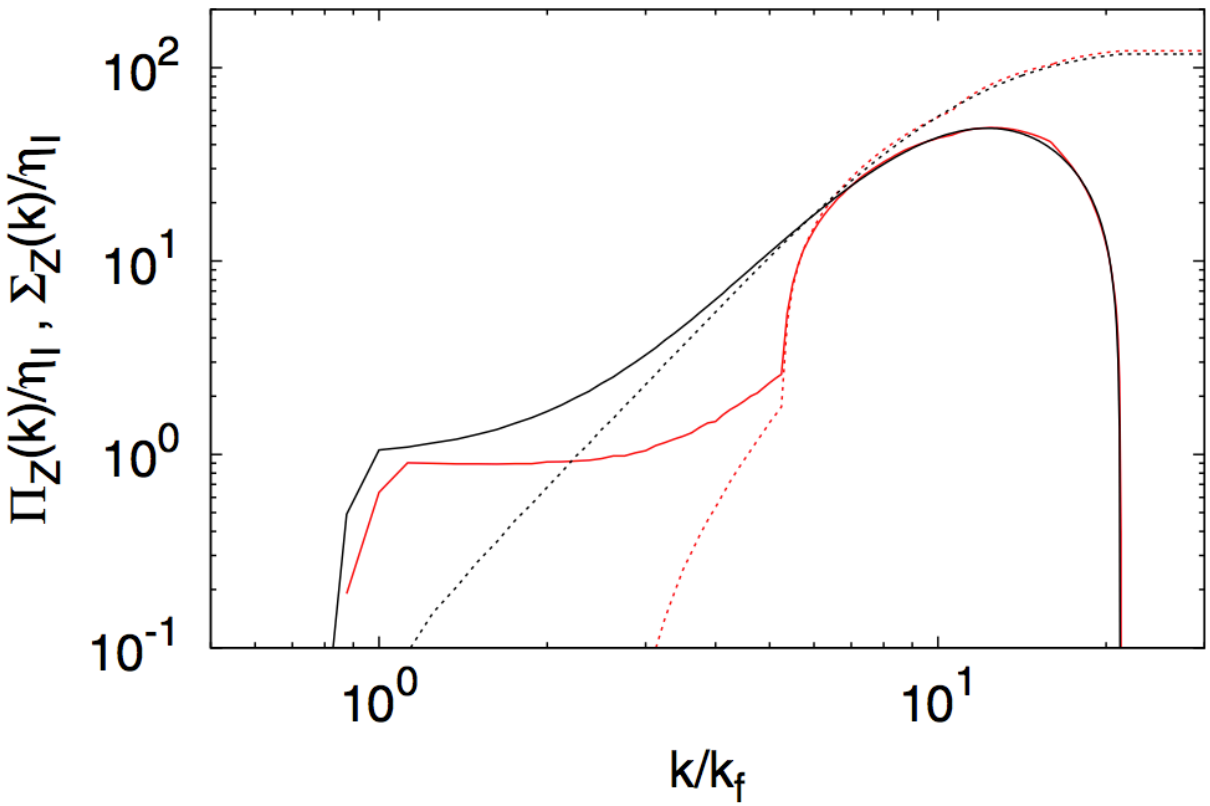
Spectral energy flux



Confinement vs. Rotation

Enstrophy flux $$\Pi(k) = \int_{|q|\le k} d\boldsymbol{q}\ \ (\mathbf{u}\cdot\nabla\boldsymbol{\omega})(q)\boldsymbol{\omega}^*(q)$$

Enstrophy production $$\Sigma(k) = \int_{|q|\le k} d\boldsymbol{q}\ \ (\boldsymbol{\omega}\cdot\nabla\mathbf{u})(q)\boldsymbol{\omega}^*(q)$$



Non-rotating thin layer (R=0 S=0.1875)

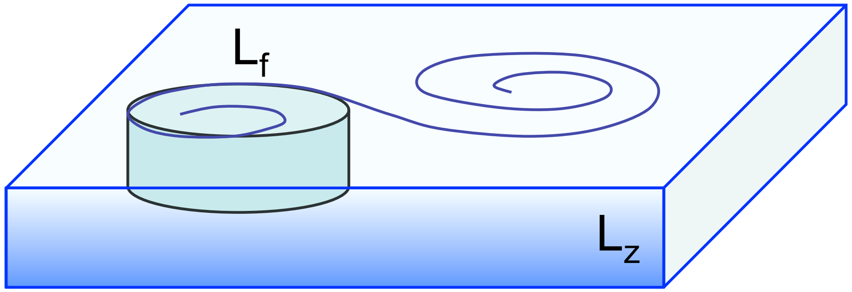
Rotating thick layer (R=1.5 S=4)

Enstrophy production is suppressed by confinement and rotation

Stratified thin fluid layers

Stratified turbulent layer

Thin fluid layer
with a stable mean density gradient



$$\rho(\boldsymbol{x}, t) = \rho_0 - \gamma z + \gamma \phi(\boldsymbol{x}, t)/N$$

Boussinesq eq.

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \nu \nabla^2 \boldsymbol{v} + \boldsymbol{f} - N\phi \hat{e}_z$$

$$\partial_t \phi + \boldsymbol{v} \cdot \nabla \phi = \kappa \nabla^2 \phi + N \boldsymbol{v} \cdot \hat{e}_z$$

$$\nabla \cdot \boldsymbol{v} = 0$$

Brunt-Väisälä frequency $N = \sqrt{g\gamma/\rho_0}$

$$Fr = \frac{\varepsilon^{1/3} k_f^{2/3}}{N} \sim \frac{L_b}{L_f} \qquad S = \frac{L_z}{L_f}$$

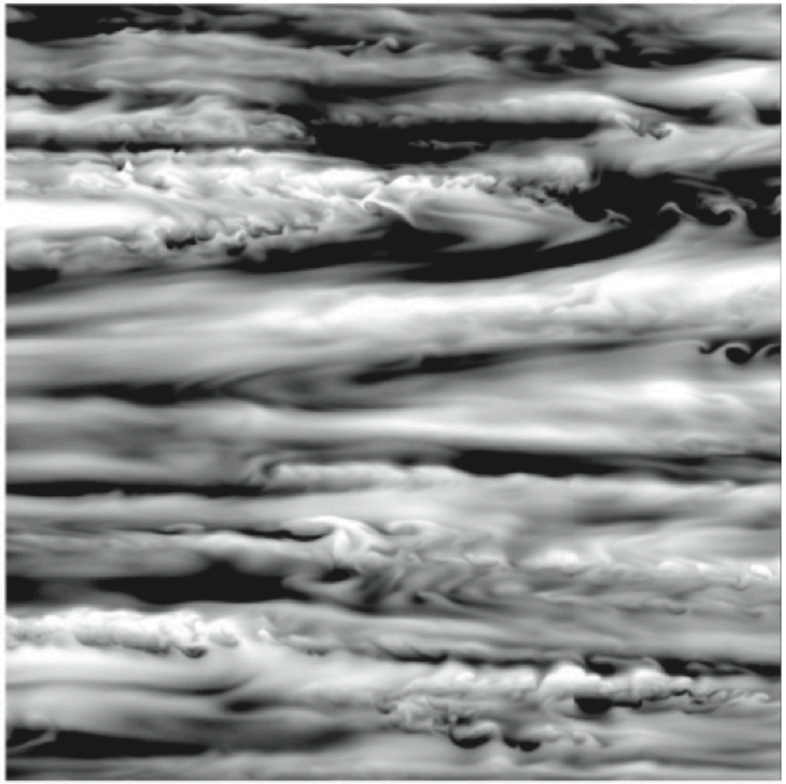
$$Sc = \frac{\nu}{\kappa} \qquad Re = \frac{UL}{\nu}$$

Stably stratified turbulent flows

Layered structures ("pancake vortices")

$$L_b \sim U/N \sim Fr$$

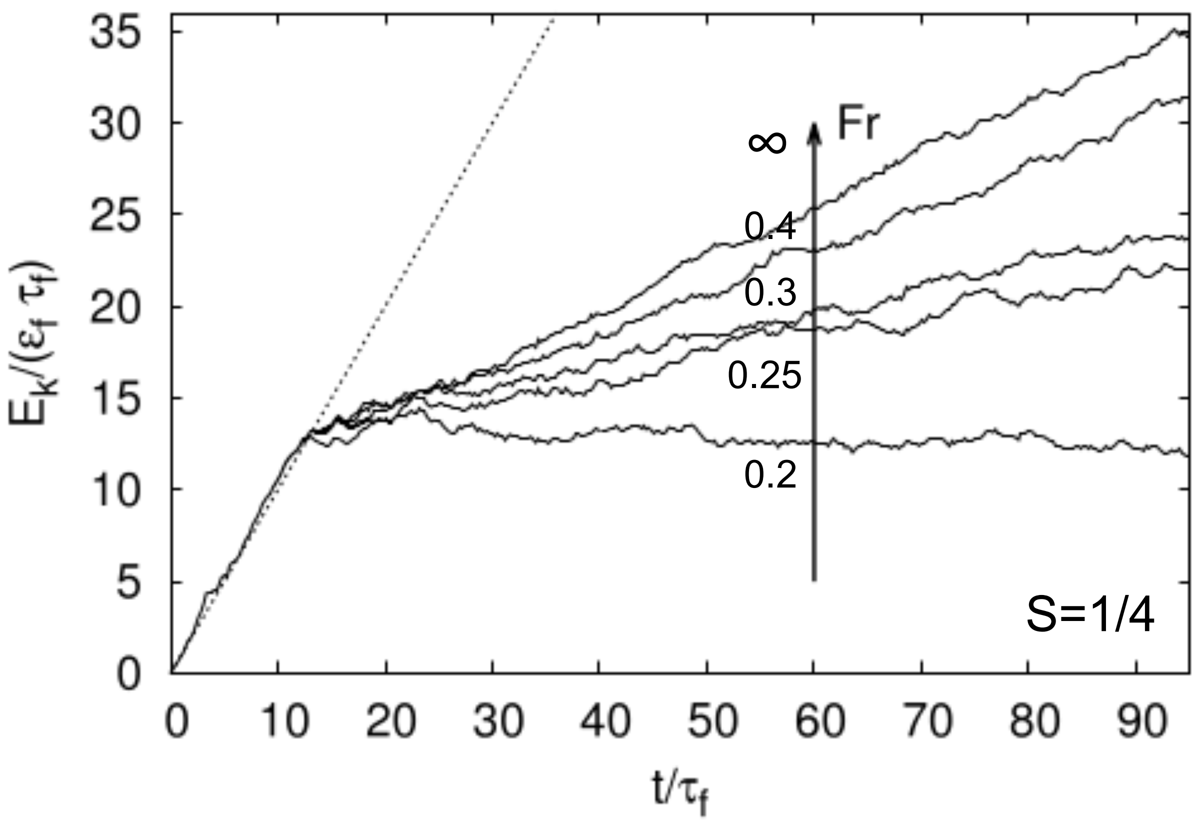
P. Billant, J-M Chomaz, Phys Fluids 13, 1645 (2001)
F. S. Godeferd, C. Staquet, JFM 486, 115 (2004)
R. Godoy-Diana, J-M Chomaz, P. Billant, JFM 504, 229 (2004)
M. L. Waite, P. Bartello, JFM 517, 281 (2004)
O. Praud, A. M Fincham, J. Sommeria, JFM 522, 1 (2005)
G. Brethouwer, P. Billant, E. Lindborg, J.-M. Chomaz, J. Fluid Mech. 585, 343-368 (2007)
C. Rorai, P. D. Mininni, A. Pouquet, PRE **89**, 043002 (2014)



C. Rorai, P. D. Mininni, A. Pouquet, PRE **89**, 043002 (2014)

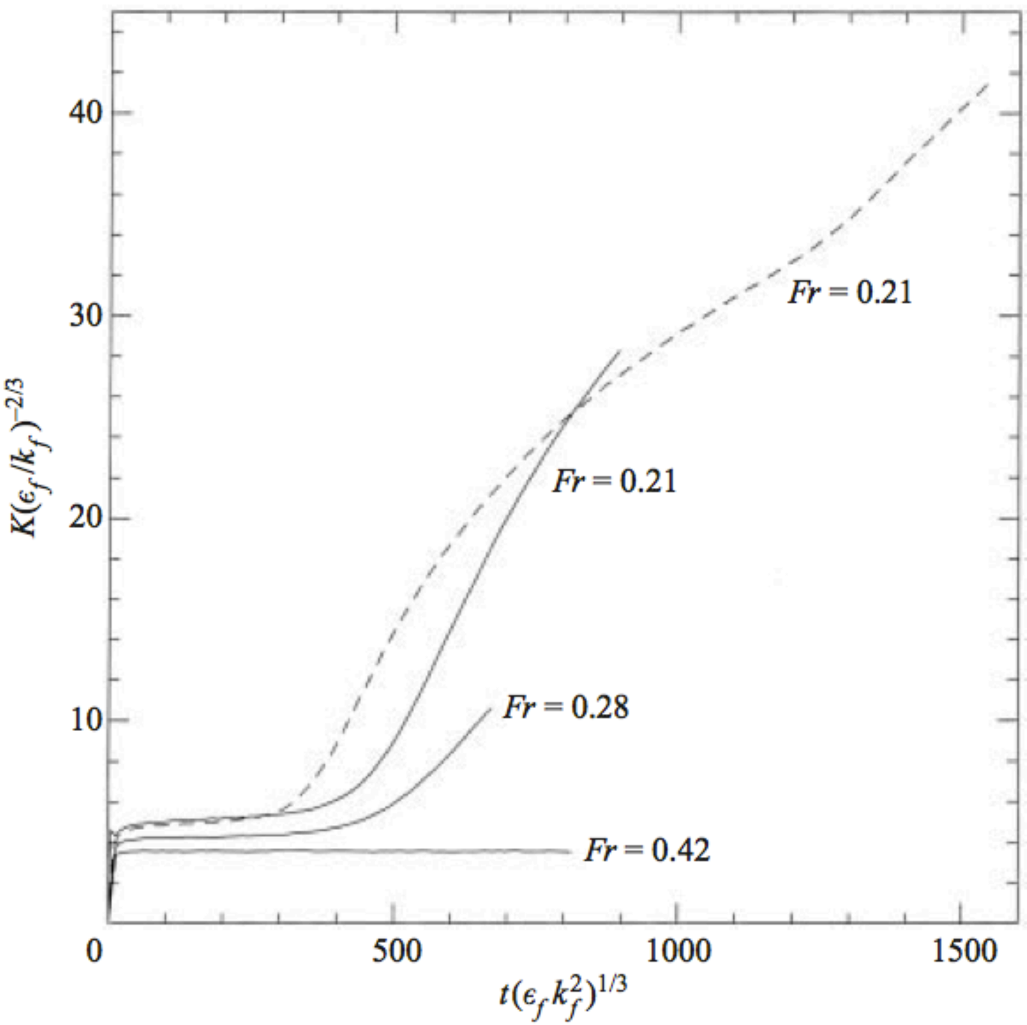
Kinetic energy growth rates

At fixed aspect ratio S the energy growth rate reduces as the stratification increases $Fr \to 0$



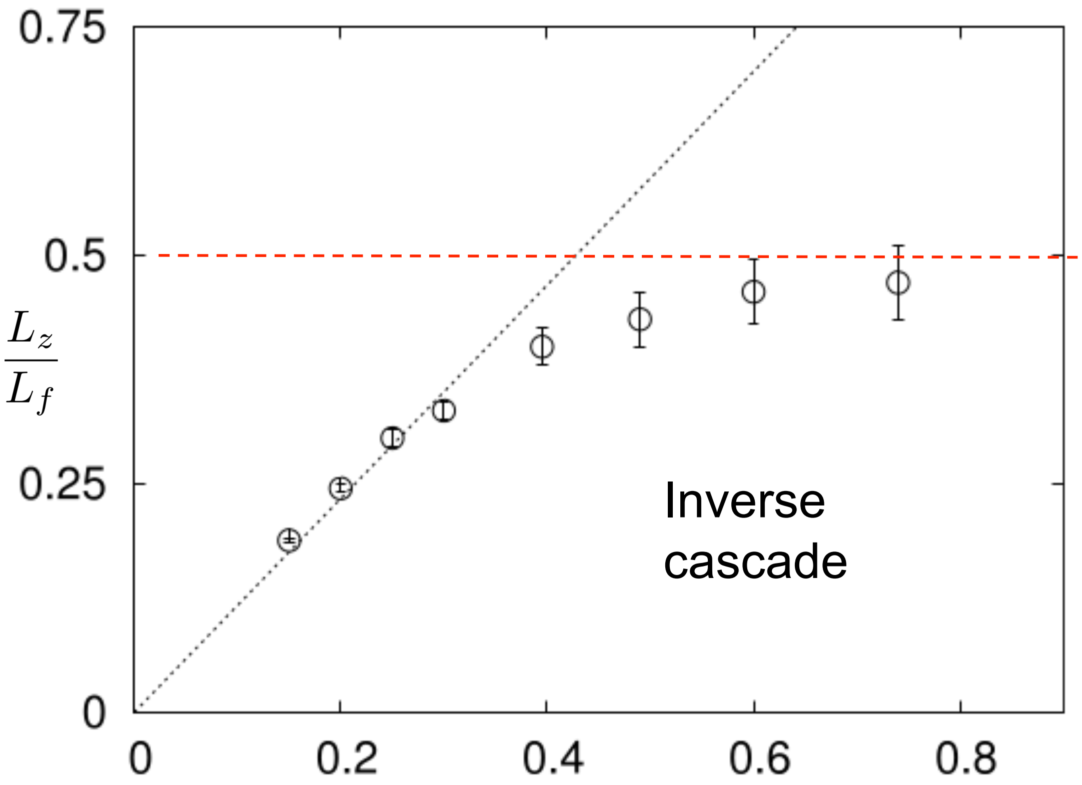
A. Sozza, G. Boffetta, P. Muratore-Ginanneschi, SM, Phys. Fluids 27, 035112 (2015)

Vertically sheared horizontal flows (VSHF)



L. M. Smith, F. Waleffe, Fluid Mech. (2002), vol. 451, pp. 145–168

Parameter space (S, Fr)



$$S = \frac{L_z}{L_f}$$

$$Fr = \frac{\varepsilon^{1/3} k_f^{2/3}}{N} \sim \frac{L_b}{L_f}$$

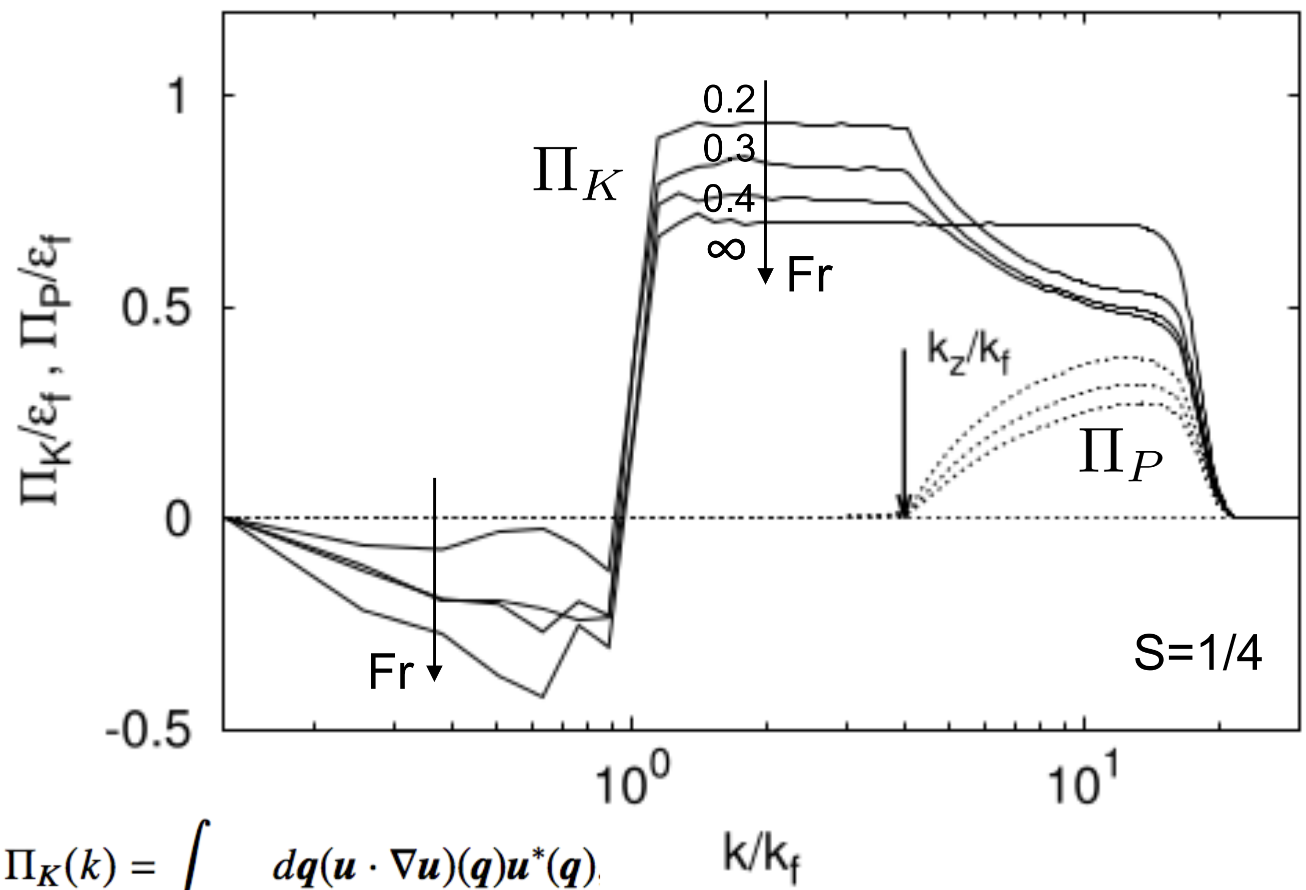
The complete suppression of the inverse cascade occurs when

$$L_z \simeq L_b \Rightarrow S_c \simeq Fr$$

$$S_c \simeq 1.17 Fr$$

Spectral fluxes of kinetic and potential energy

$$E = K + P = \frac{1}{2}\langle|\boldsymbol{u}|^2\rangle + \frac{1}{2}\langle\phi^2\rangle$$

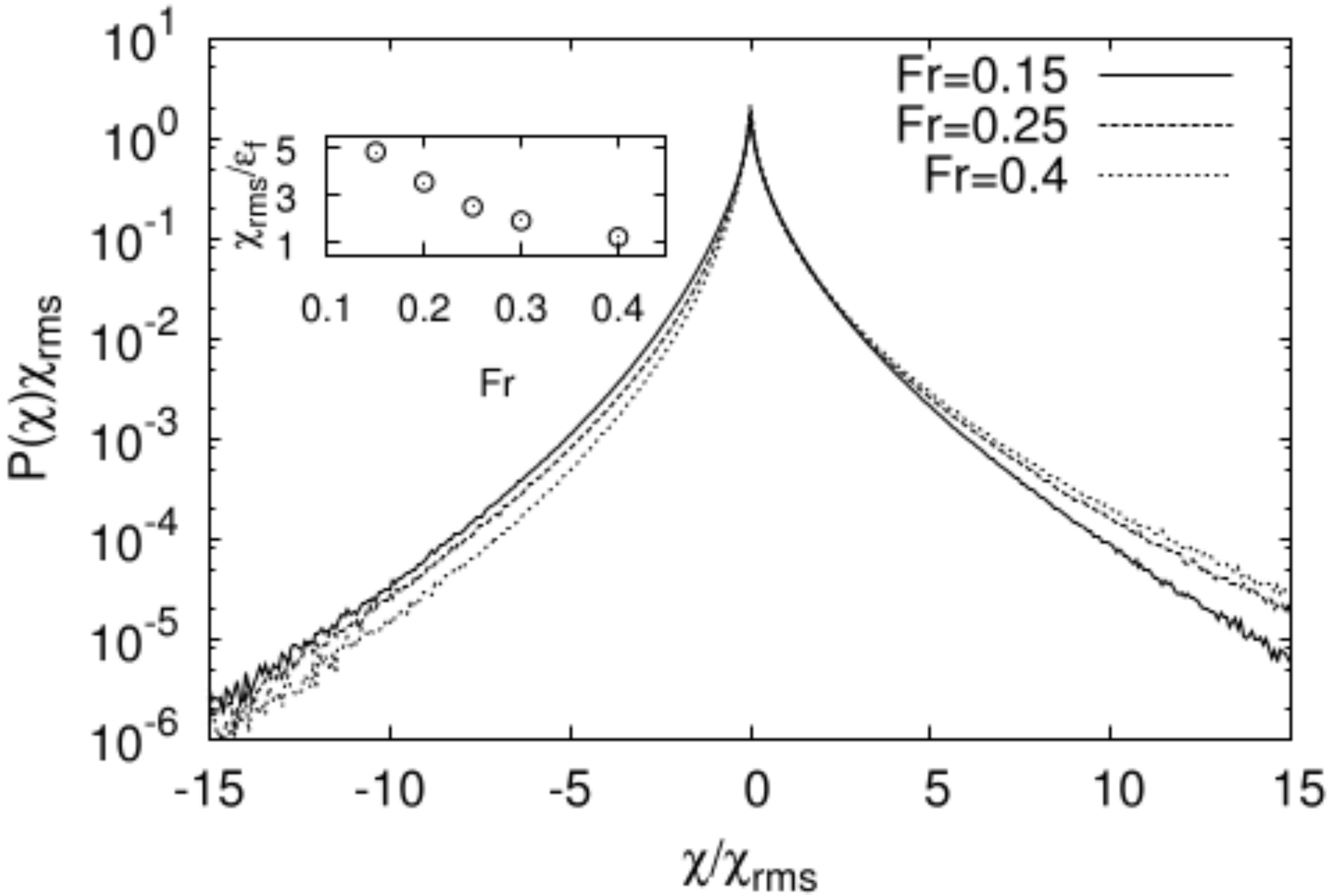


$$\Pi_K(k) = \int_{|\boldsymbol{q}|\le k} d\boldsymbol{q}(\boldsymbol{u}\cdot\nabla\boldsymbol{u})(\boldsymbol{q})\boldsymbol{u}^*(\boldsymbol{q}).$$

$$\Pi_P(k) = \int_{|\boldsymbol{q}|\le k} d\boldsymbol{q}(\boldsymbol{u}\cdot\nabla\boldsymbol{\phi})(\boldsymbol{q})\boldsymbol{\phi}^*(\boldsymbol{q}).$$

PDF of exchange rates

exchange rate $\chi = N u_3 \phi$ $\langle \chi \rangle > 0$

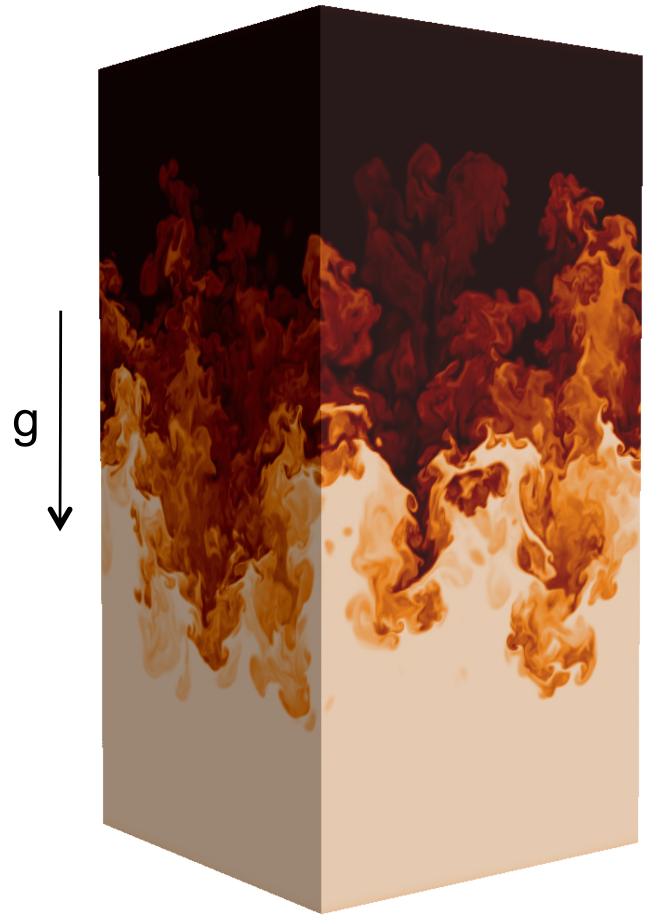


Dimensional transitions in confined Rayleigh-Taylor Turbulence

Rayleigh-Taylor Turbulence

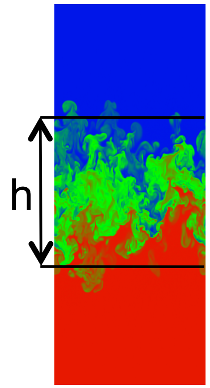
Turbulence in the mixing layer
between two fluids with different densities $\rho_1 > \rho_2$

Rayleigh (1883): unstable stratification in gravitational field
Taylor (1950): generalization to all acceleration mechanisms



Quadratic growth of the width of the mixing layer

$$h(t) = \alpha A g t^2$$

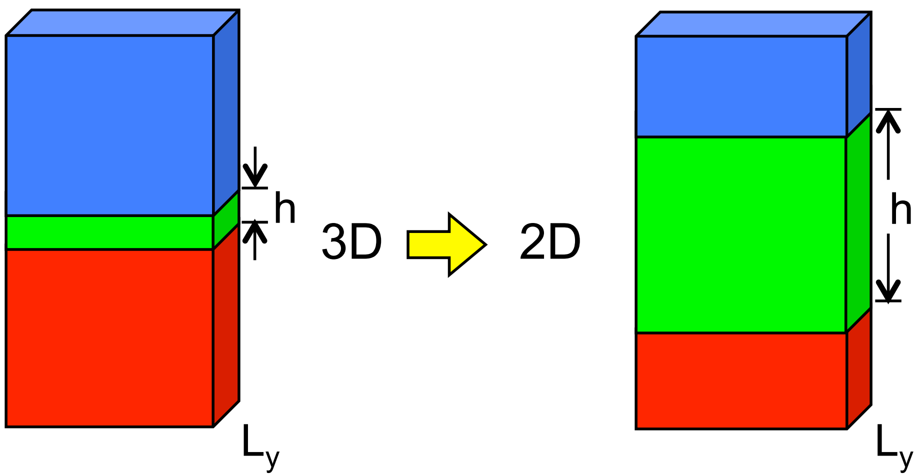


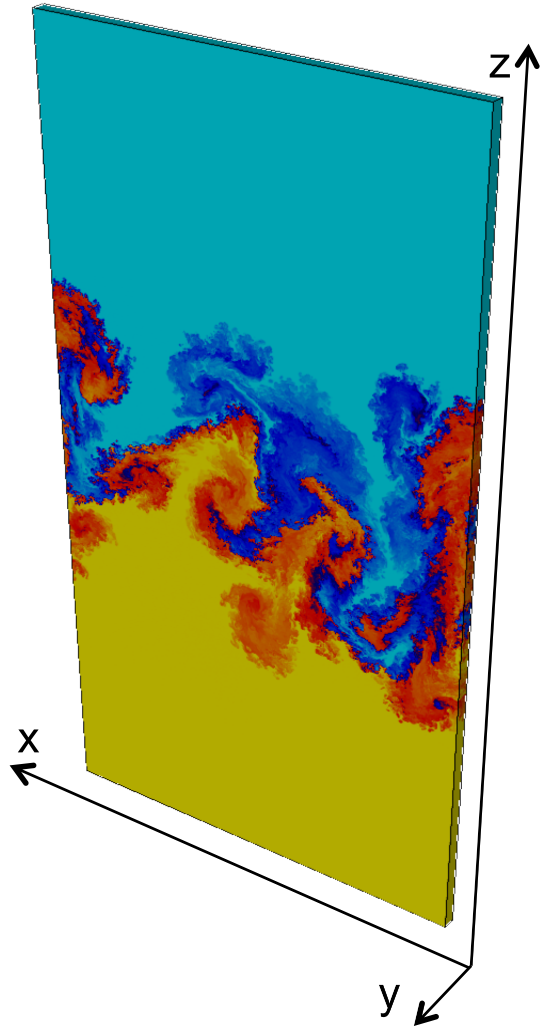
E. Fermi & J. von Neumann (1955)
US Atomic Energy Commission Report AECU-2979

Confined Rayleigh-Taylor Turbulence

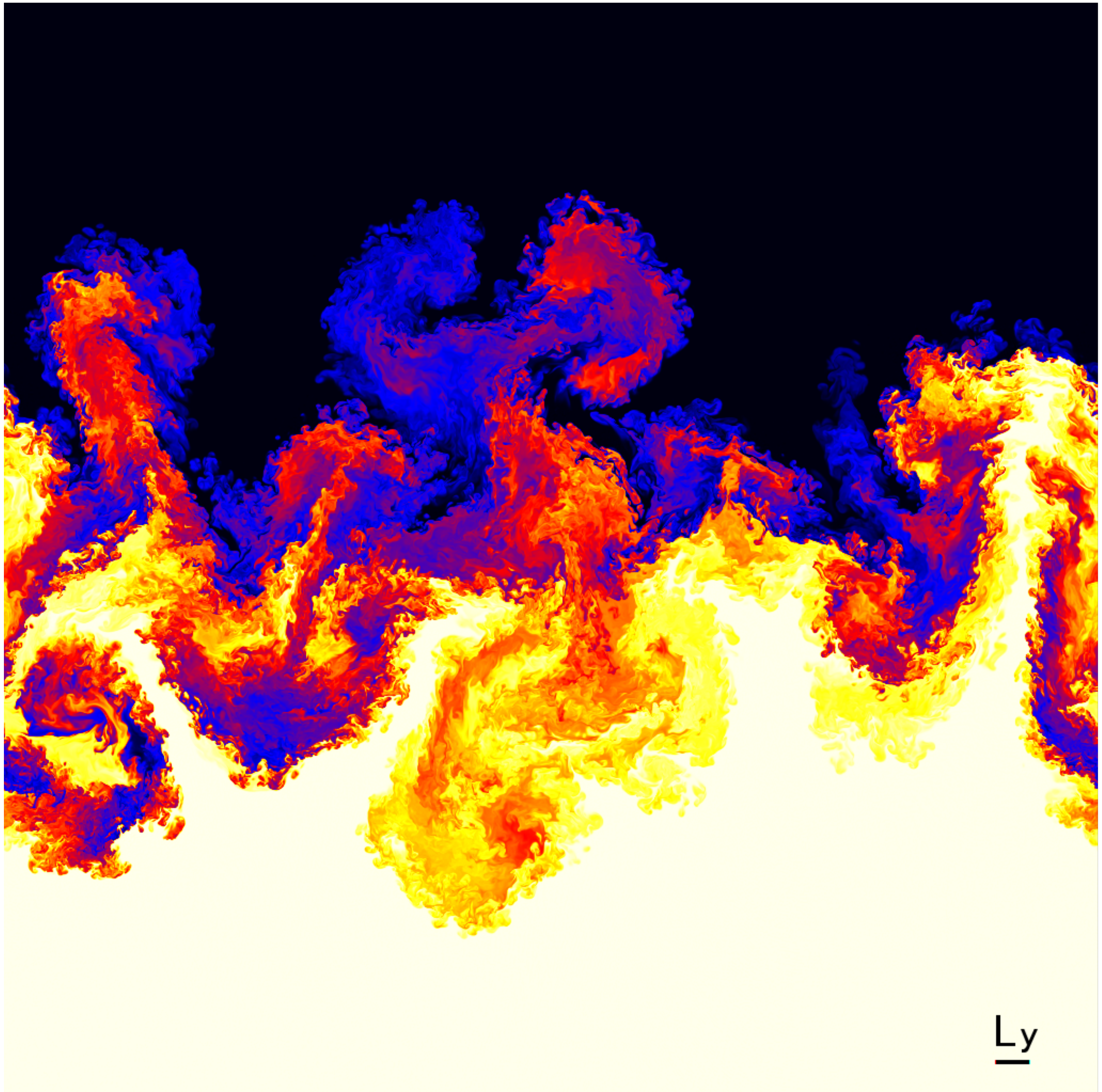
Confinement in thin vertical layer

Dimensional transition when the width of the mixing layer h(t) becomes larger than the confining scale L_y



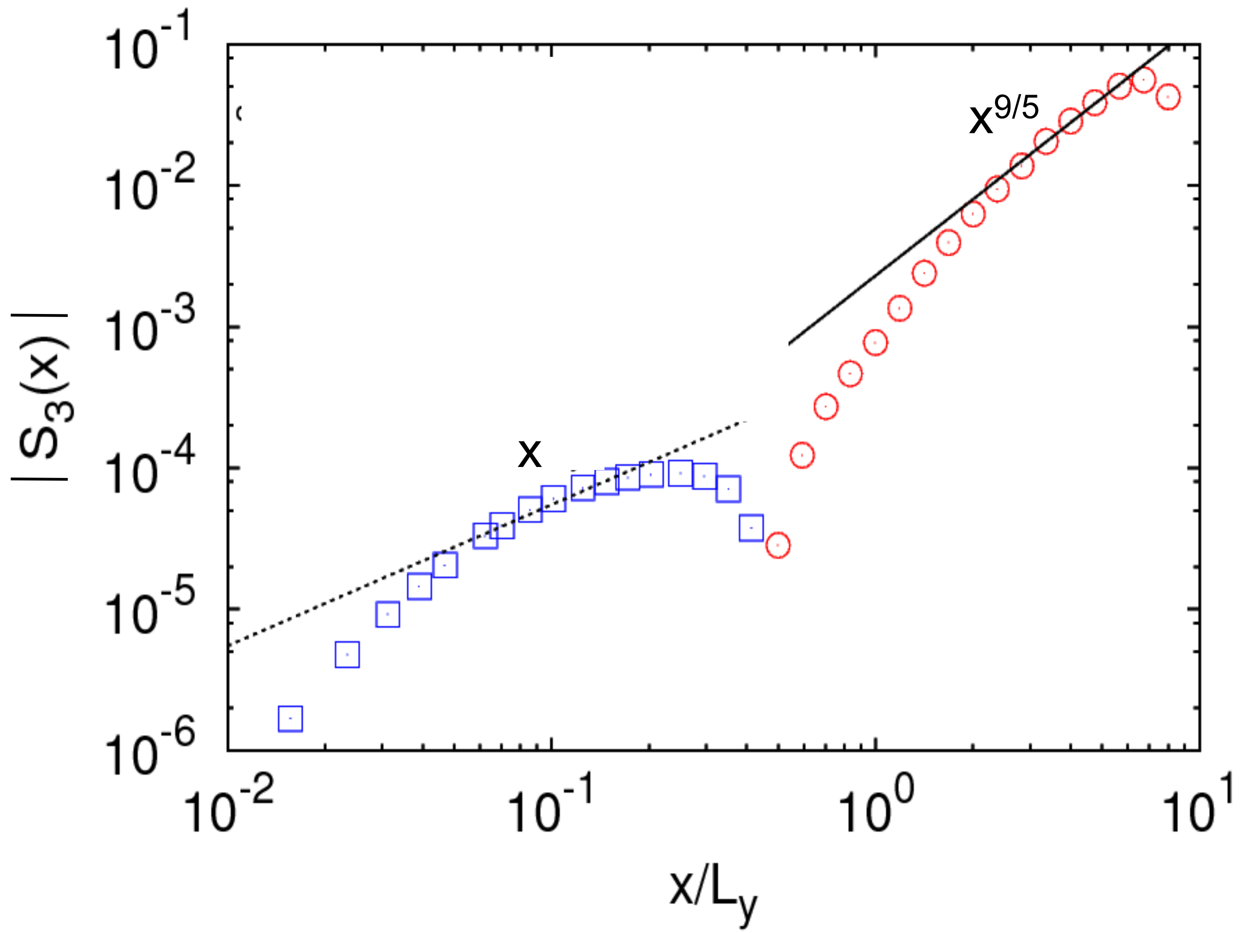


Confined Rayleigh-Taylor Turbulence



Confined Rayleigh-Taylor Turbulence

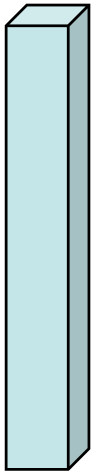
G. Boffetta, F. De Lillo, A. Mazzino, SM, J. Fluid Mech. 690 426-440 (2012)



3D turbulence at small scales
Kolmogorov-Obukhov

2D turbulence at large scales
Bolgiano-Obukhov

Confined Rayleigh-Taylor Turbulence

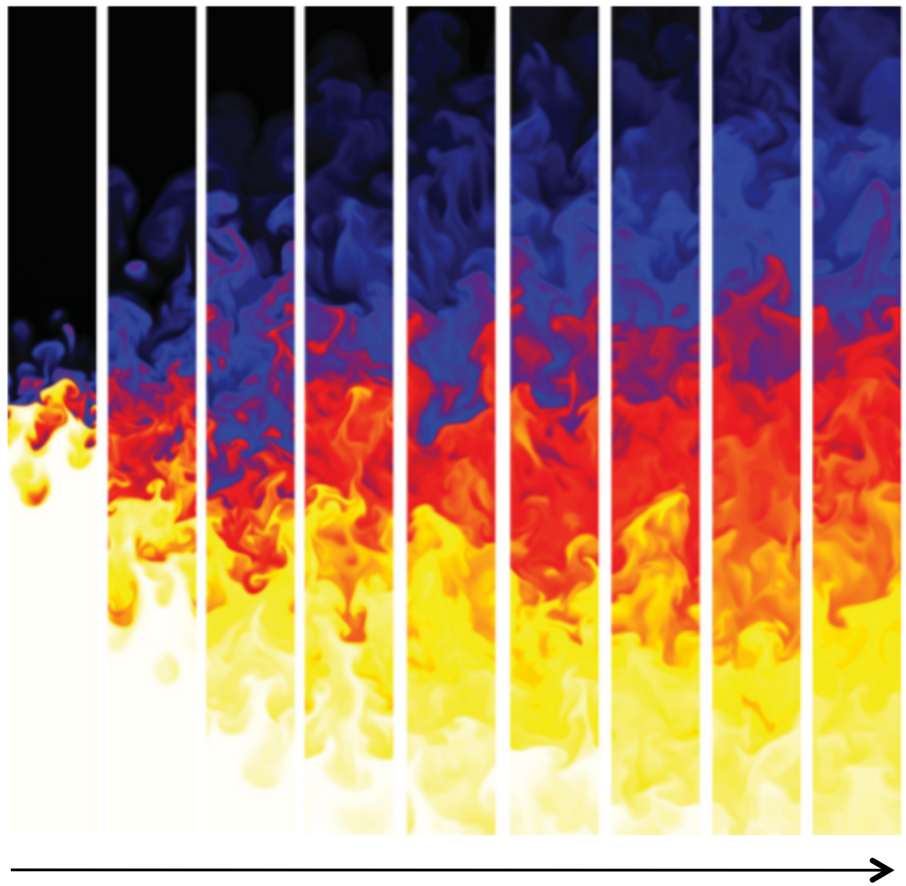


Quasi 1D geometry
$L_x = L_y << L_z$

Transition from super-diffusive to sub-diffusive growth of the mixing layer

$$h(t) \sim g t^2$$

$$h(t) \sim g^{1/5} L^{4/5} t^{2/5}$$



N. Inogamov, et al., J. Exp. Theor. Phys. 92, 715 (2001)
S. Dalziel, M. Patterson, C. Caulfield, and I. Coomaraswamy, Phys. Fluids 20, 065106 (2008)
G. Boffetta, F. De Lillo, SM, Phys. Rev. E 85 066322 (2012)

Inverse energy cascade
in
3D isotropic helical turbulence

Conservation of Helicity

H. K. Moffat, J. Fluid Mech. 35, 117 (1969)

$$H = \frac{1}{V}\int_V dV\, \boldsymbol{u}\cdot\boldsymbol{\omega} = \langle \boldsymbol{u}\cdot\boldsymbol{\omega} \rangle$$

Helicity is an inviscid invariant of 3D Navier-Stokes equation

J. Fluid Mech. (1969), *vol.* 35, *part* 1, *pp.* 117–129 117
Printed in Great Britain

The degree of knottedness of tangled vortex lines

By H. K. MOFFATT
Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge

(Received 17 May 1968)

Let $\mathbf{u}(\mathbf{x})$ be the velocity field in a fluid of infinite extent due to a vorticity distribution $\boldsymbol{\omega}(\mathbf{x})$ which is zero except in two closed vortex filaments of strengths κ_1, κ_2. It is first shown that the integral

$$I = \int \mathbf{u}.\boldsymbol{\omega}\, dV$$

is equal to $\alpha\kappa_1\kappa_2$ where α is an integer representing the degree of linkage of the two filaments; $\alpha = 0$ if they are unlinked, ± 1 if they are singly linked. The invariance of I for a continuous localized vorticity distribution is then established for barotropic inviscid flow under conservative body forces. The result is interpreted in terms of the conservation of linkages of vortex lines which move with the fluid.

Some examples of steady flows for which $I \neq 0$ are briefly described; in particular, attention is drawn to a family of spherical vortices with swirl (which is closely analogous to a known family of solutions of the equations of magnetostatics); the vortex lines of these flows are both knotted and linked.

Two related magnetohydrodynamic invariants discovered by Woltjer (1958*a*, *b*) are discussed in §5.

Helicity cascade

A. Brissaud, U. Frisch, J. Leorat, M. Lesieur, and M. Mazure,
Phys. Fluids 16, 1366 (1973)

1) Joint direct cascade of E and H

$$E(k) \sim \varepsilon^{2/3} k^{-5/3}$$

$$H(k) \sim h \varepsilon^{-1/3} k^{-5/3}$$

2) Direct cascade of H
Inverse cascade of E

$$E(k) \sim h^{2/3} k^{-7/3}$$

$$H(k) \sim h^{2/3} k^{-4/3}$$

THE PHYSICS OF FLUIDS VOLUME 16, NUMBER 8 AUGUST 1973

Research Notes

Research Notes published in this Section include important research results of a preliminary nature which are of special interest to the physics of fluids and new research contributions modifying results already published in the scientific literature. Research Notes cannot exceed five printed columns in length including space allowed for title, abstract, figures, tables, and references. The abstract should have three printed lines. Authors must shorten galley proofs of Research Notes longer than five printed columns before publication.

Helicity cascades in fully developed isotropic turbulence

A. Brissaud
Ecole Nationale Superieure de l'Aeronautique et de l'Espace, Toulouse, France

U. Frisch
Centre National de la Recherche Scientifique, Observatoire de Nice, 06300 Nice, France

J. Leorat
Observatoire de Meudon, 92190 Meudon, France

M. Lesieur
Centre National de la Recherche Scientifique, Observatoire de Nice, 06300 Nice, France

A. Mazure
Observatoire de Meudon, 92190 Meudon, France
(Received 12 June 1972; final manuscript 7 March 1973)

Based on total helicity conservation in inviscid incompressible flows, the existence of simultaneous energy and helicity cascades is envisaged.

Helicity cascade: EDQN Closures

J. C. Andre and M. Lesieur, J. Fluid Mech. 81, 187 (1977).
F. Waleffe, Phys. Fluids A 4, 350 (1992).

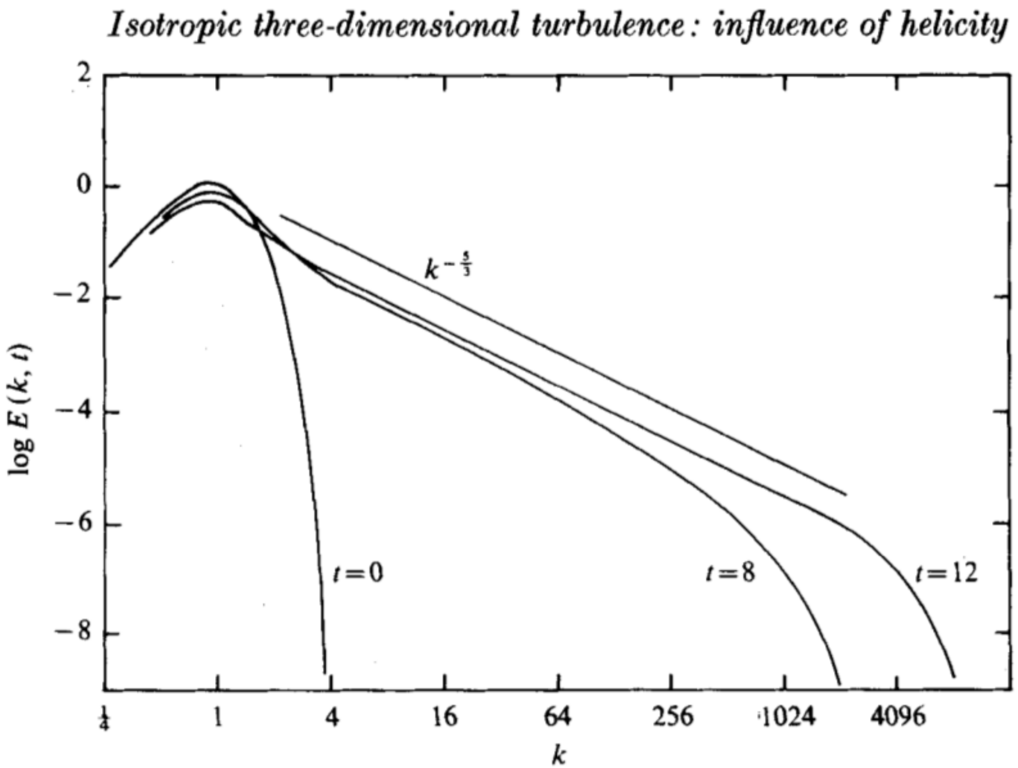


FIGURE 6. Temporal evolution of the energy spectrum $E(k,t)$; initial energy spectrum $E(k,0) \sim k^4 \exp(-2k^2)$, Reynolds number $R = 524000$, maximal initial helicity.

Joint direct cascade of E and H with Kolmogorow spectrum

$$E(k) \sim \varepsilon^{2/3} k^{-5/3}$$

Helicity cascade: numerical simulations

Q. Chen, S. Chen, and G. L. Eyink, Phys. Fluids 15, 361(2003)
Q. Chen, S. Chen, G. L. Eyink, and D. D. Holm ,Phys. Rev. Lett. 90, 214503 (2003)

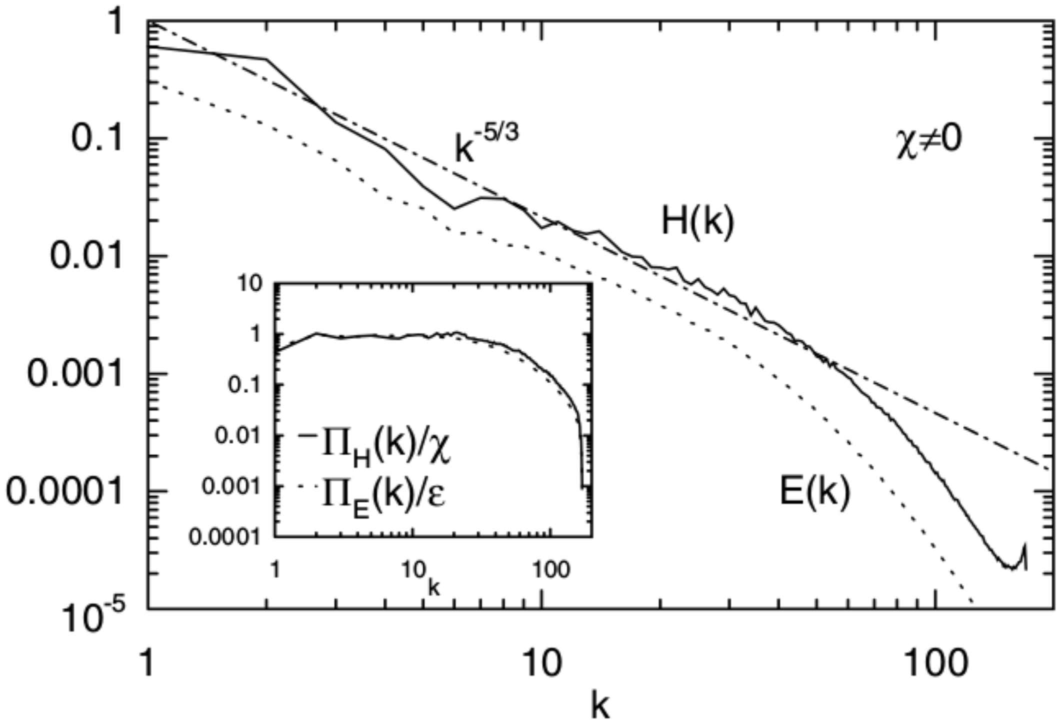


FIG. 1. Energy and helicity spectra. In the inset is shown normalized energy and helicity fluxes.

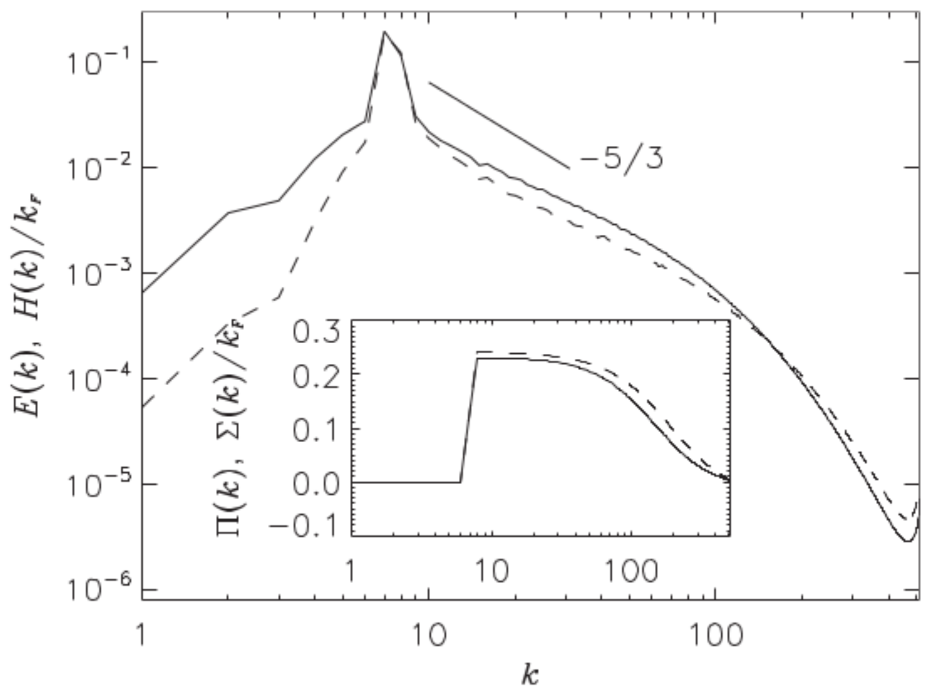
Joint direct cascade of E and H with Kolmogorov spectrum

$$E(k) \sim \varepsilon^{2/3} k^{-5/3}$$

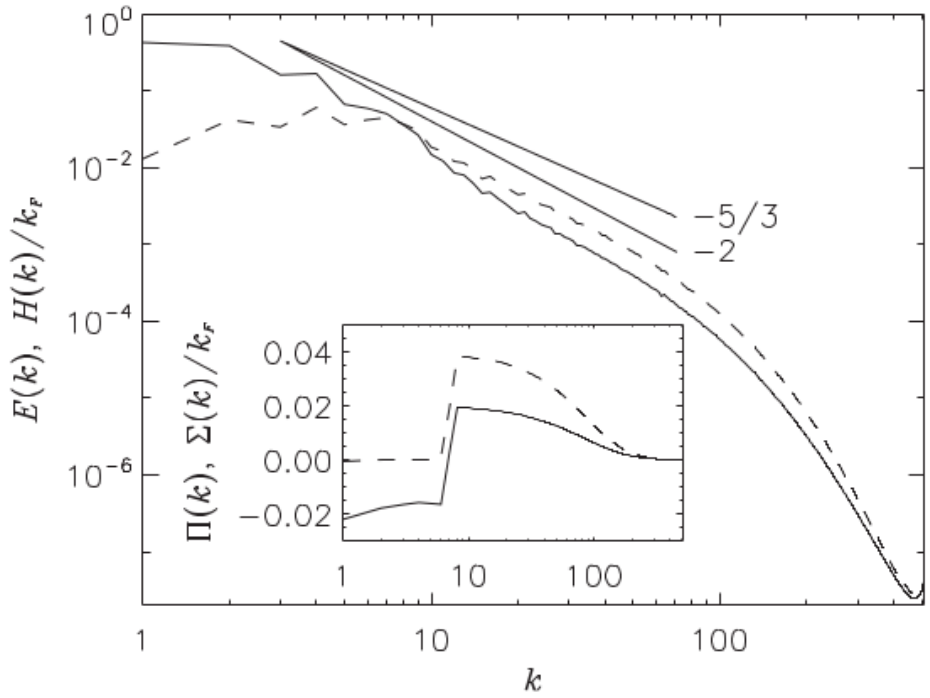
Helicity cascade: numerical simulations

P.D. Mininni and A. Pouquet, Phys. Fluids 22, 035105 (2010)

No rotation



Rotation



Rotation can induce an inverse energy cascade + direct helicity cascade

Helical-Fourier decomposition of Navier-Stokes eq.

P. Constantin & A. Majda 1988 Commun. Math. Phys. 115, 435-456
F. Waleffe 1992 Phys. Fluids A 4, 350–363

$$\boldsymbol{u}(\boldsymbol{k}) = u^{+}(\boldsymbol{k})\boldsymbol{h}^{+}(\boldsymbol{k}) + u^{-}(\boldsymbol{k})\boldsymbol{h}^{-}(\boldsymbol{k})$$

Eigenvectors of curl operator $i\boldsymbol{k} \times \boldsymbol{h}^{\pm} = \pm k \boldsymbol{h}^{\pm}$

Energy $E = \int d^3\boldsymbol{x} |\boldsymbol{u}(\boldsymbol{x})|^2 = \sum_{\boldsymbol{k}} |\boldsymbol{u}^{+}(\boldsymbol{k})|^2 + |\boldsymbol{u}^{-}(\boldsymbol{k})|^2$

Helicity $H = \int d^3\boldsymbol{x} \boldsymbol{u} \cdot \boldsymbol{\omega} = \sum_{\boldsymbol{k}} k(|\boldsymbol{u}^{+}(\boldsymbol{k})|^2 - |\boldsymbol{u}^{-}(\boldsymbol{k})|^2)$

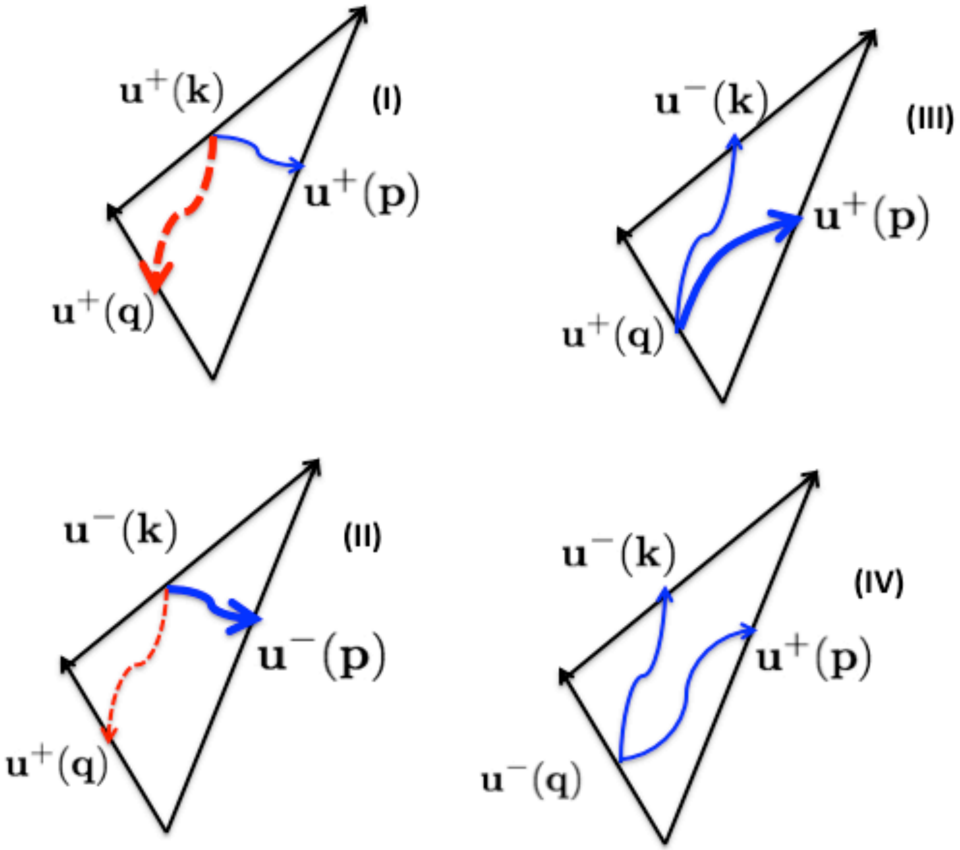
Navier-Stokes equations

$$(\partial_t + \nu k^2)\overline{u}^{s_k}(\boldsymbol{k}) = -\frac{1}{4} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0} \sum_{s_p, s_q} (s_p p - s_q q)[\boldsymbol{h}^{s_p} \times \boldsymbol{h}^{s_q} \cdot \boldsymbol{h}^{s_k}] u^{s_p}(\boldsymbol{p}) u^{s_q}(\boldsymbol{q})$$

$$(s_k = \pm, s_p = \pm, s_q = \pm)$$

Classes of triadic interactions in Navier-Stokes eq.

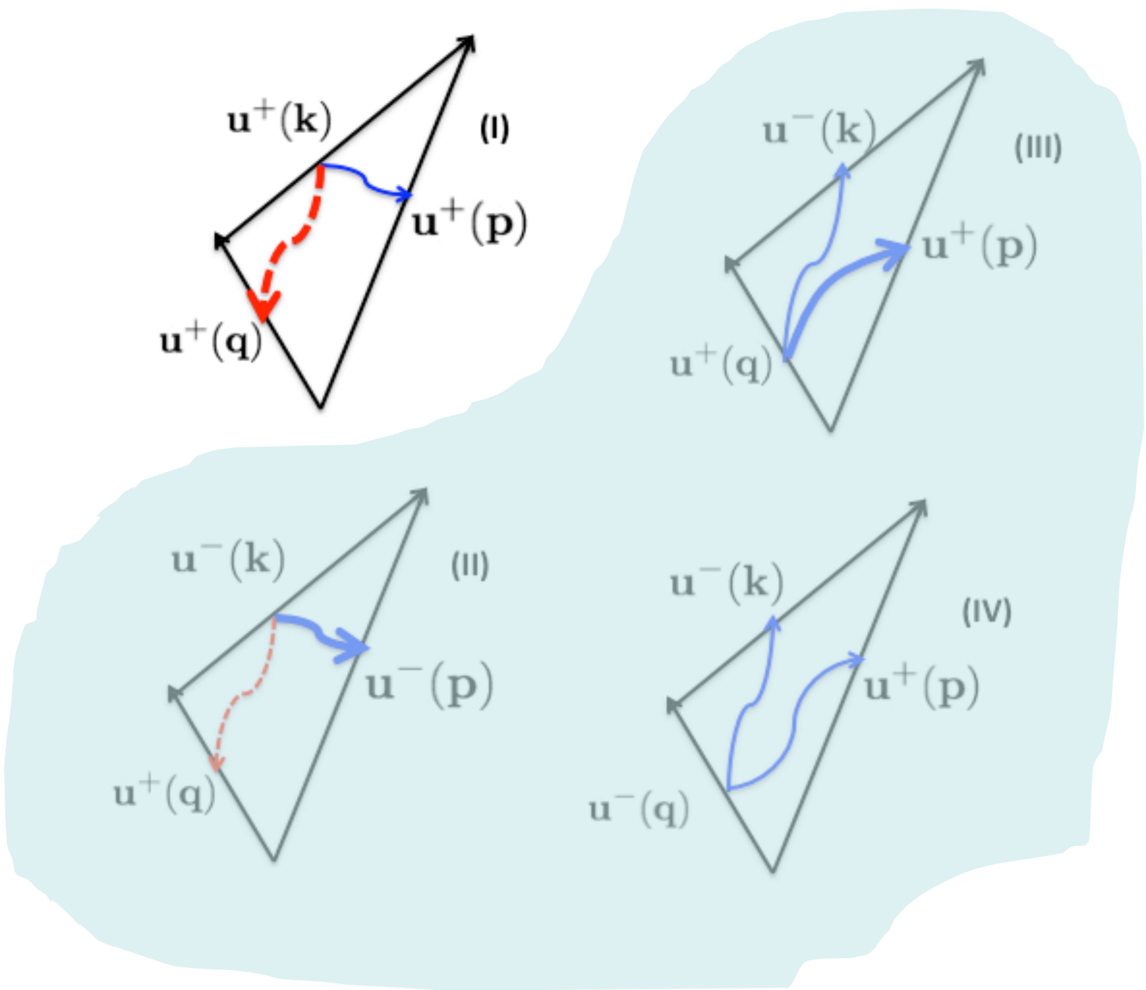
F. Waleffe 1992 Phys. Fluids A 4, 350–363



Class I:
Large-scale
energy transfer

Class II, III, IV:
Small-scale
energy transfer

Decimated triadic interactions



Class I:
Large-scale
energy transfer

Decimate Navies-Stokes eq. keeping only the interactions between modes with positive helicity

Decimated Navier-Stokes equation

Projector on positive/negative helicity states

$$\mathcal{P}^{\pm}_{ij}(\boldsymbol{k}) \equiv \frac{h^{\pm}_i(\boldsymbol{k})\overline{h^{\pm}_j}(\boldsymbol{k})}{\overline{\boldsymbol{h}^{\pm}}(\boldsymbol{k}) \cdot \boldsymbol{h}^{\pm}(\boldsymbol{k})}$$

$$u^{+}_j(\boldsymbol{x}) \equiv P^{+}_{jm} u_m(\boldsymbol{x}) \equiv \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \mathcal{P}^{+}_{jm}(\boldsymbol{k}) u_m(\boldsymbol{k})$$

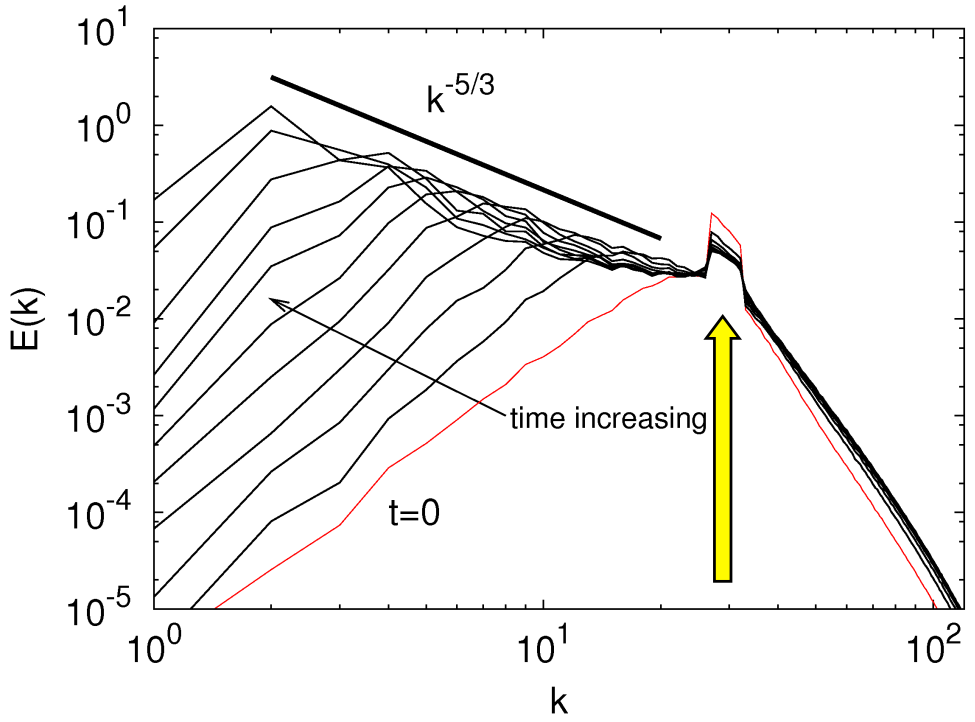
Decimated Navier-Stokes equation (only triadic interactions with positive helicity)

$$\partial_t v^{+}_i = -P^{+}_{im} \partial_j (v^{+}_j v^{+}_m) + \nu \Delta v^{+}_i + f^{+}_i$$

Two positive-defined inviscid invariants

$$E = \sum_{\boldsymbol{k}} |\boldsymbol{u}^{+}(\boldsymbol{k})|^2 \qquad H = \sum_{\boldsymbol{k}} k |\boldsymbol{u}^{+}(\boldsymbol{k})|^2$$

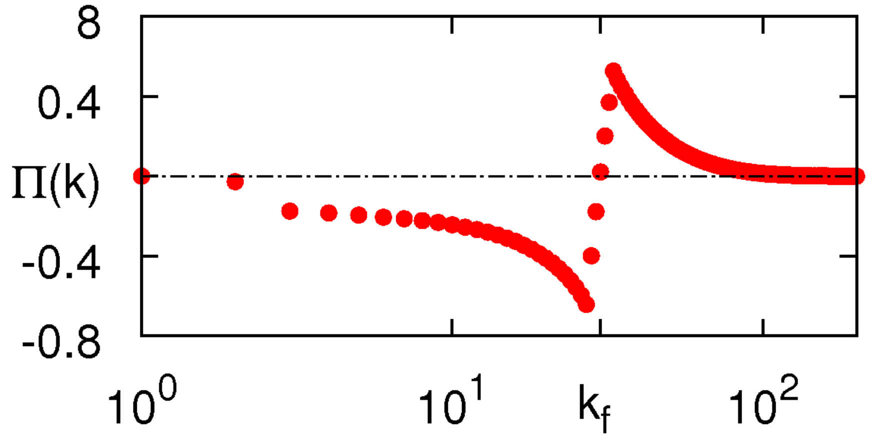
Small-scale forcing: Inverse energy cascade



Power-law energy spectrum

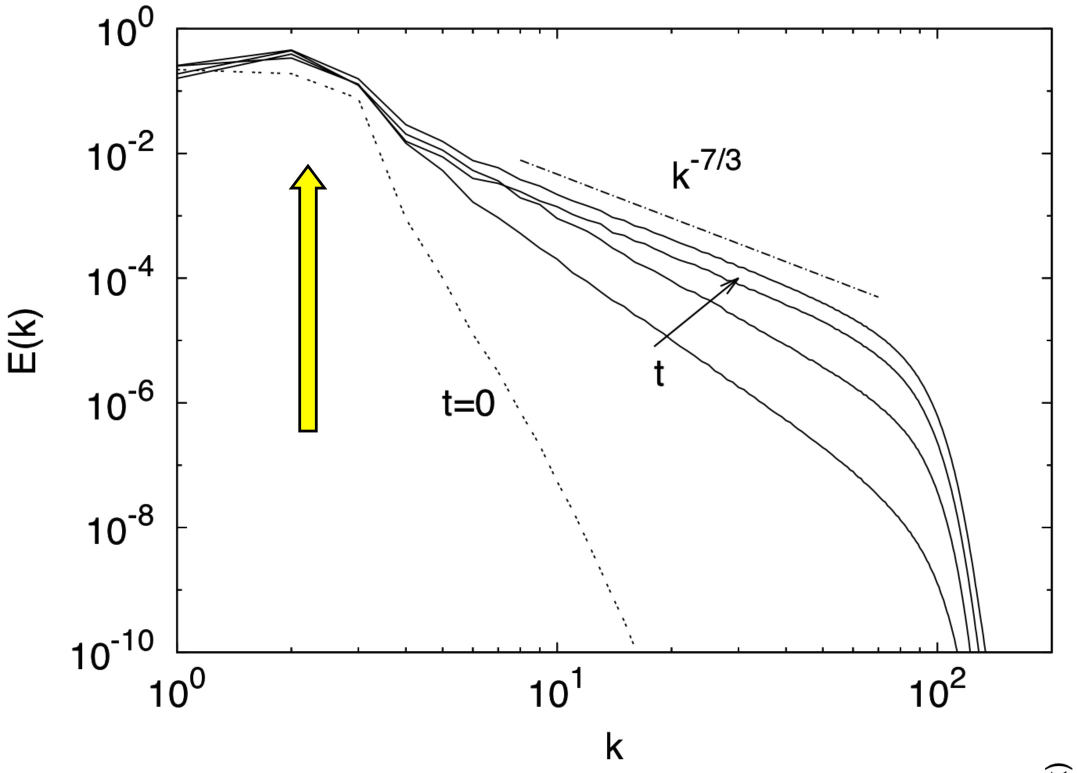
$$E(k) \sim k^{-5/3}$$

Negative spectral flux of energy



L. Biferale, SM, F. Toschi, Phys. Rev. Lett. 108 164501 (2012)

Large-scale forcing: Direct helicity cascade

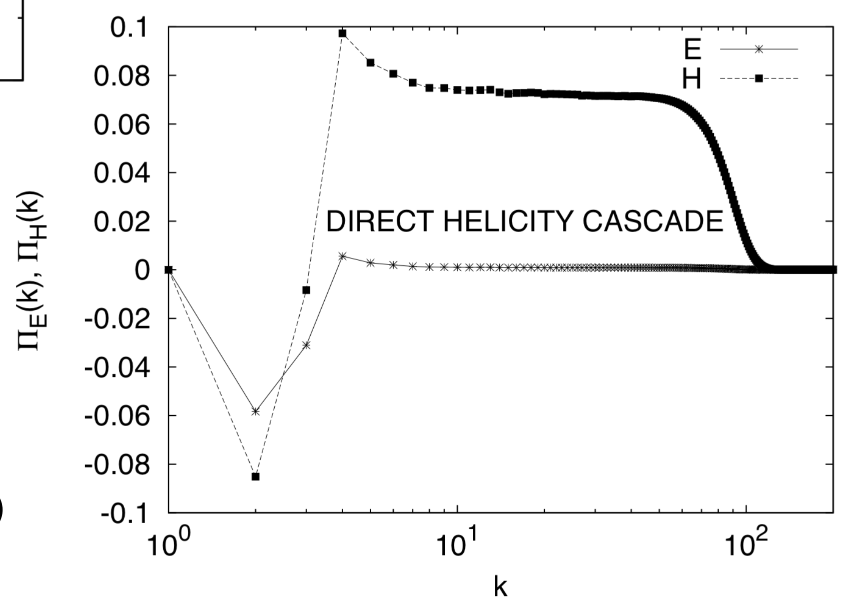


Power-law energy spectrum

$$E(k) \sim k^{-7/3}$$

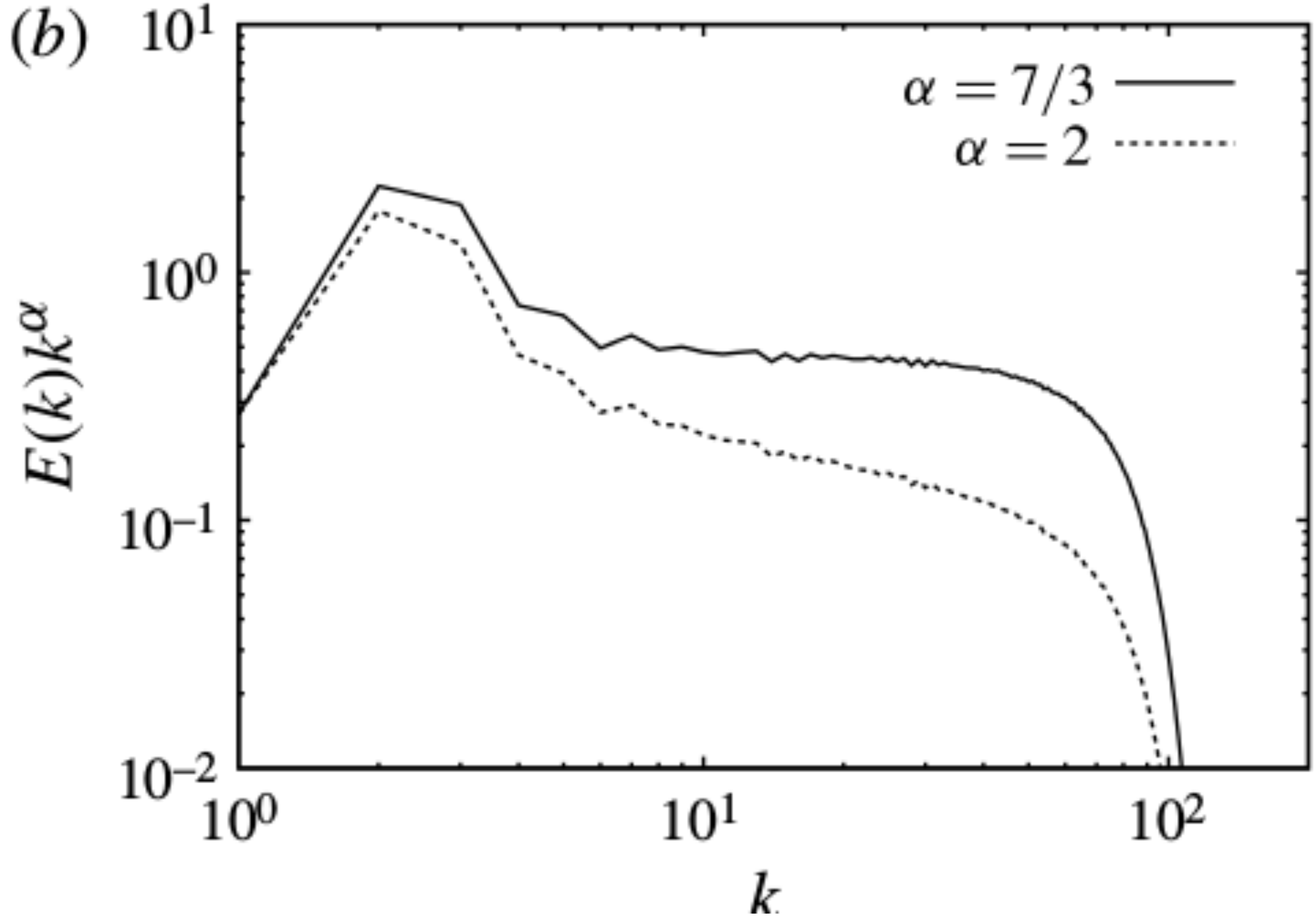
No dissipative anomaly for kinetic energy

Positive spectral flux of helicity



L. Biferale, SM, F. Toschi, J. Fluid Mech. 730, 309 (2013)

Large-scale forcing: Direct helicity cascade



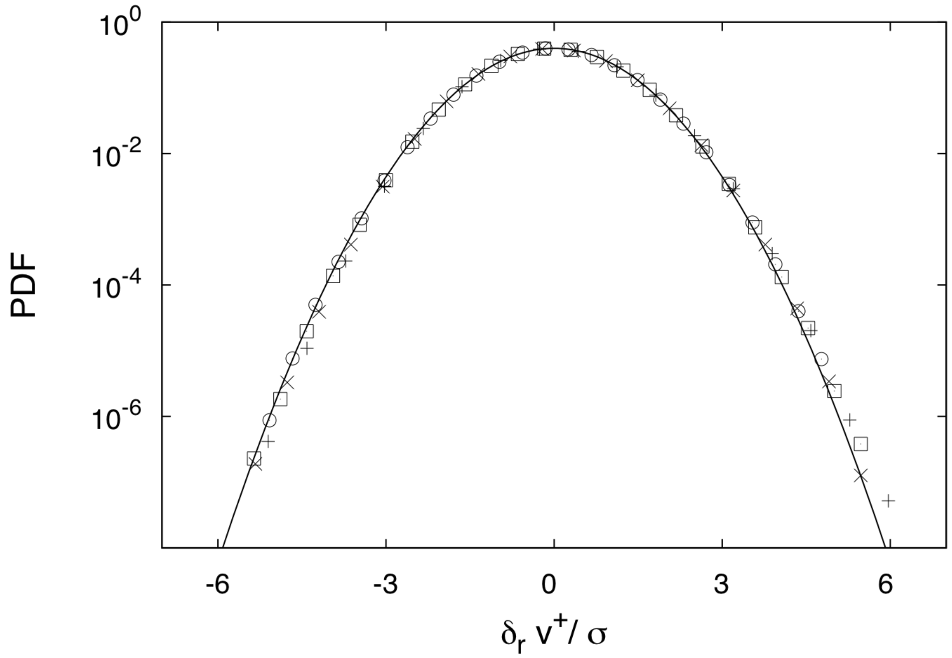
L. Biferale, SM, F. Toschi, J. Fluid Mech. 730, 309 (2013)

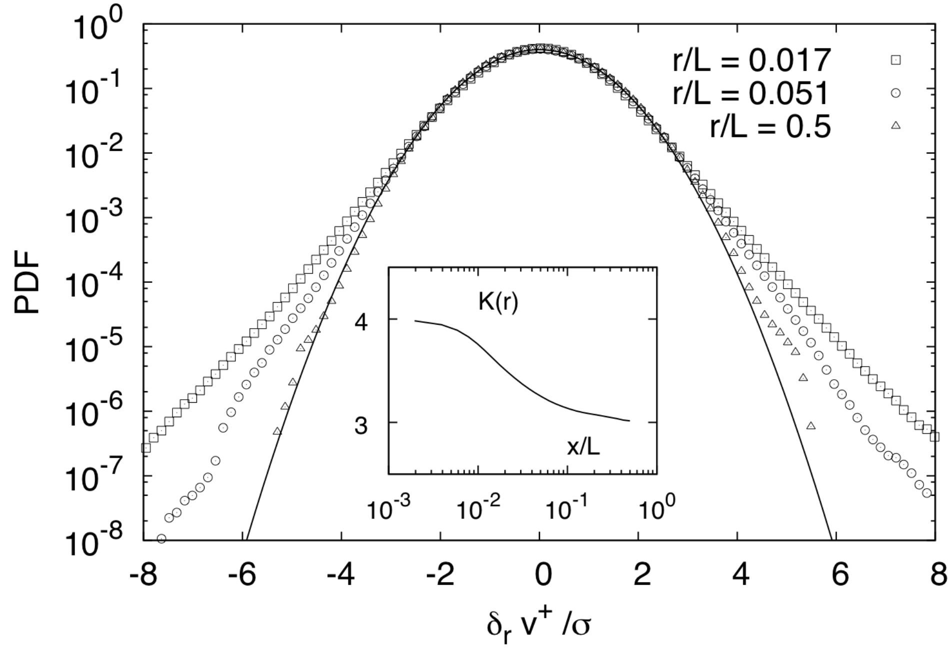
PDF of longitudinal velocity increments

$$\delta u_{\parallel}(r) = [\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})] \cdot \frac{\boldsymbol{r}}{|\boldsymbol{r}|}$$

Gaussian PDF for velocity
in the inverse energy cascade

Small-scale intermittency
in the direct helicity cascade





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Exact 3rd order relation: inverse energy cascade

Exact relation for 3rd order correlation function

$$C_j(\boldsymbol{r}) = -[\langle v_i^+(0)F_{ij}(\boldsymbol{r})\rangle - \langle v_i^+(\boldsymbol{r})F_{ij}(0)\rangle]$$

$$F_{ij} = P_{im}^+(v_j^+ v_m^+)$$

$$C_j(\boldsymbol{r}) = \frac{1}{3}\epsilon r_j$$

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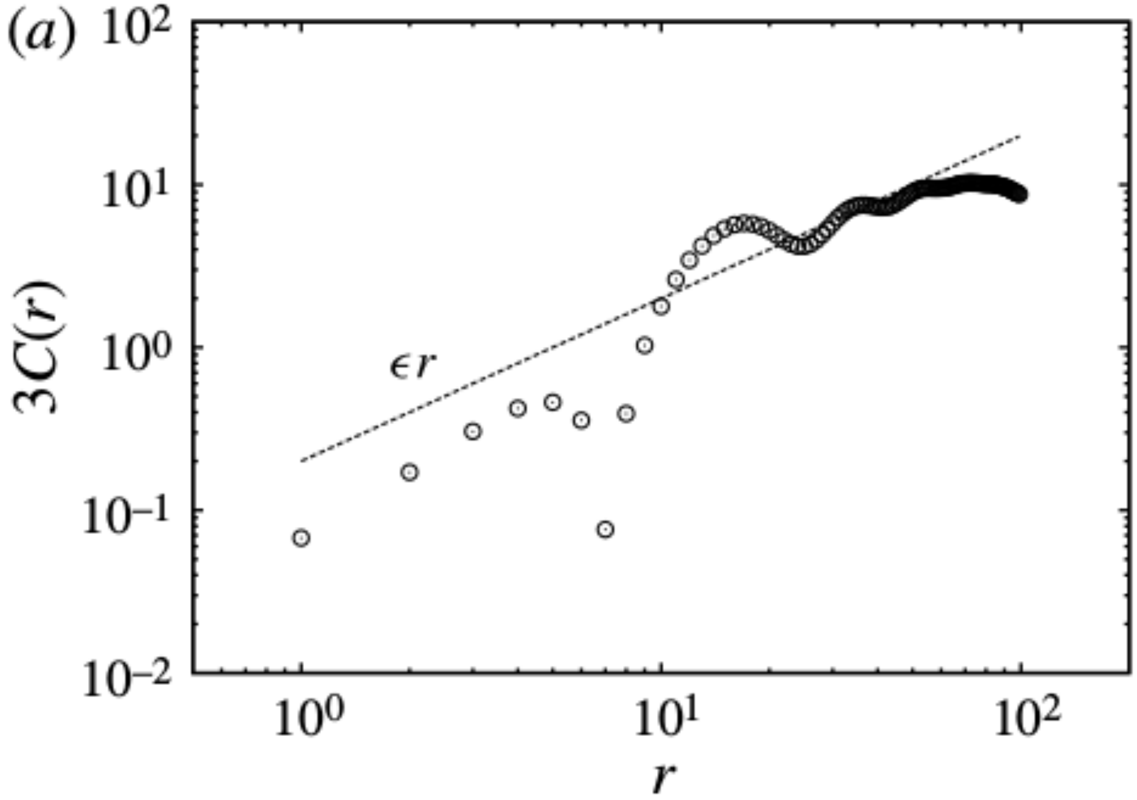

Exact 3rd order relation: direct helicity cascade

Exact relation for 3rd order correlation function

$$H_{jl}(\mathbf{r}) = \epsilon_{ijk}[\langle F'_{kl} v_i \rangle - \langle v'_k F_{il} \rangle]$$

$$F_{ij} = P^+_{im}(v^+_j v^+_m)$$

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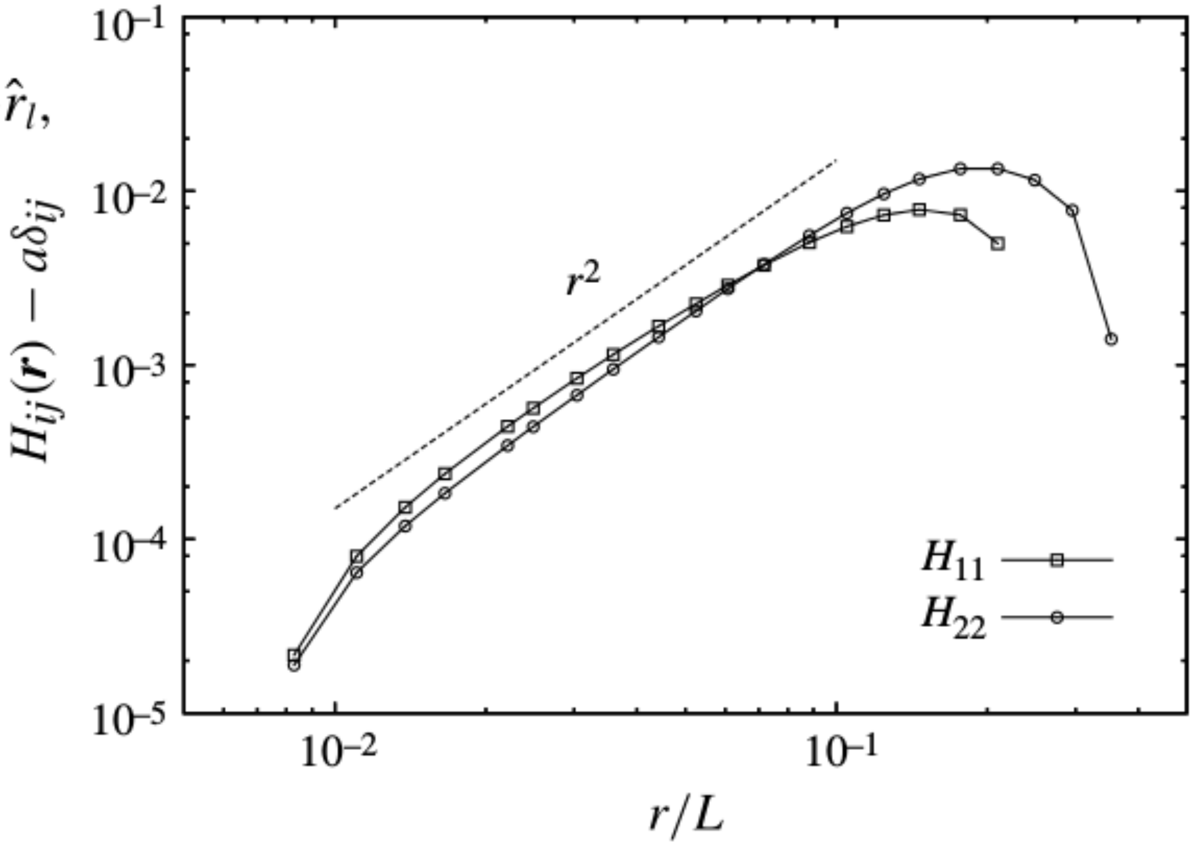
$$H_{jl}(\mathbf{r}) = (a + br^2)\delta_{jl} + cr^2 \hat{r}_j \hat{r}_l,$$

$$6b + 12c = h$$



Conclusion

Helical decimated Navier-Stokes

Inverse energy cascade in 3D isotropic turbulence

Helicity has a definite sign

Partial decimation? $P_{ij}^{+} + \alpha P_{ij}^{-}$ (see Sahoo's talk)

Helical forcing alone is not enough

Rotation (two-dimensionalization)
P.D. Mininni and A. Pouquet,
Phys. Fluids 22, 035105 (2010)

Conclusions

Two cases of 3D turbulent flows in which a positive inviscid invariant causes a reversal of the energy cascade.

Thin fluid layers:

Enstrophy is a quasi-invariant (conserved by large-scale dynamics)
Split energy cascade (inverse + direct cascade)
Direct enstrophy cascade at intermediate scales

Rotation favours the inverse cascade
Stratification suppresses the inverse cascade

Helical-decimated Navier-Stokes:

Helicity is positive defined
Inverse energy cascade in 3D isotropic turbulence

Turbulence in Fractal Dimensions

Turbulence in fractal dimensions

J-D. Fournier and U. Frisch, Phys. Rev. A **17**, 747 (1978).

EDQNM Closure: Energy-inertial solutions with spectrum 5/3
exist for arbitrary dimension

The direction of the cascade reverses at d = 2.05

V. L'vov, A. Pomyalov, and I. Procaccia, Phys. Rev. Lett.89, 064501 (2002).

Turbulence in d = 4/3 Equilibrium state with equipartition of enstrophy
Gaussian statstics

U. Frisch, A. Pomyalov, I. Procaccia, S.S. Ray, Phys. Rev. Lett 108, 074501 (2012)

Numerical simulations of Turbuence in non-integer dimensions

Fractal Fourier decimation

Turbulence in fractal dimensions

U. Frisch, A. Pomyalov, I. Procaccia, S.S. Ray, Phys. Rev. Lett 108, 074501 (2012)

Fractal Fourier decimation

If $\boldsymbol{u} = \sum_{\boldsymbol{k} \in Z^2} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{\boldsymbol{u}}_{\boldsymbol{k}}$, then $P_D \boldsymbol{u} = \sum_{\boldsymbol{k} \in Z^2} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \theta_{\boldsymbol{k}} \hat{\boldsymbol{u}}_{\boldsymbol{k}}$.

Here, $\theta_{\boldsymbol{k}}$ are random numbers such that

$$\theta_k = \begin{cases} 1 & \text{with probability } h_k \\ 0 & \text{with probability } 1 - h_k, \end{cases} \qquad k \equiv |\boldsymbol{k}|.$$

To obtain D-dimensional dynamics we choose

$$h_k = C(k/k_0)^{D-2}, \qquad 0 < D \leq 2, \qquad 0 < C \leq 1,$$

Equilibrium Gibbs state with Kolmogorov spectrum at D=4/3

Inverse cascade persists below D=2
Kolmogorov constant diverge as $(D - 4/3)^{-2/3}$

THE END