# Dimensional transitions in turbulent flows

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Lecture 1:

2D vs 3D turbulence: Inviscid invariants & turbulent cascades

Thin fluid layers: coexistence of 2D and 3D turbulence

Lecture 2:

Rotation & Stratification effects on thin fluid layers

Turbulent cascade in 3D helical turbulence

# Turbulence & Dimensions

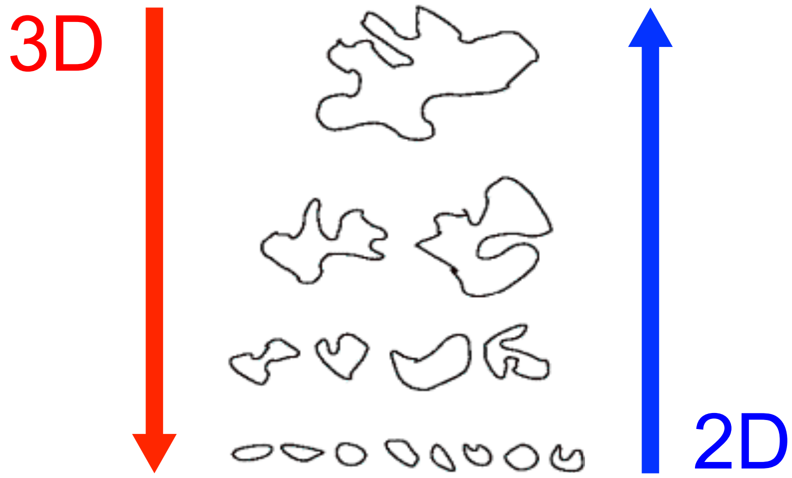
Many physical phenomena can change of the dimensionality of a turbulent flow:

Confinement in thin fluid layers

Rotation

Stable stratification

Helical flows

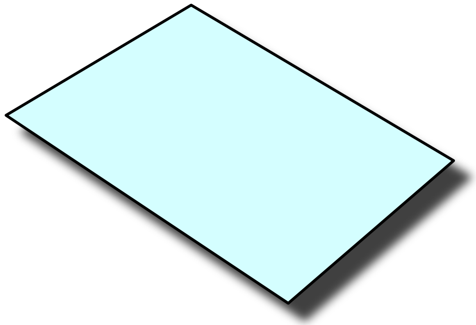


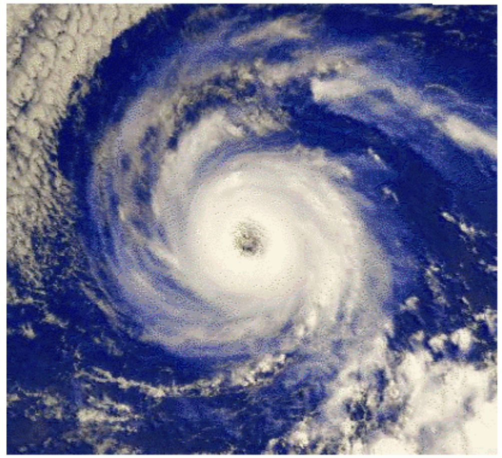

3D: Kinetic energy is transferred from large to small eddies

2D: Kinetic energy is transferred from small to large eddies

# Geophysical flows

Many geophysical flows (e.g. oceans, atmosphere) have quasi-2d aspect ratios



A4 paper ($80 gr/m^2$)
$L_1$ = 210 mm
$L_2$ = 297 mm
h = 0.1 mm

Pacific Ocean
N-S = 15000 km
E-W = 19800 km
average depth = 4.28 km

Complex systems:

Turbulence
Waves
Stratification
Convection
Rotation (Coriolis)
Boundaries
Cloud physics

# 3D TURBULENCE

# Turbulence

The turbulent flow of a viscous incompressible fluid is described by Navier-Stokes equation (1823)

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{\nabla P}{\rho} + \nu \Delta \boldsymbol{u} + \boldsymbol{f}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$\nu = \mu / \rho$ kinematic viscosity

$$\nu_{water} = 10^{-6} m^2/s$$

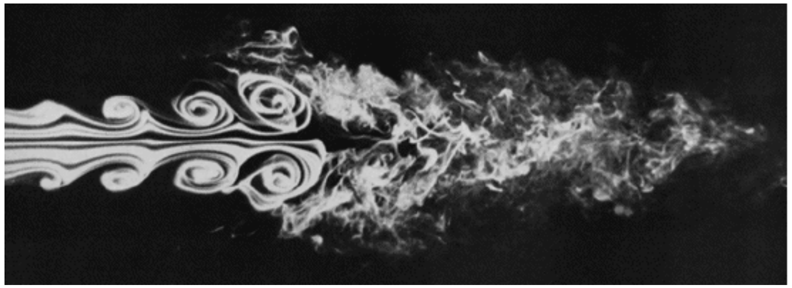
$$\nu_{air} = 1.5 \cdot 10^{-5} m^2/s$$

Reynolds number

$$\boxed{Re = \frac{UL}{\nu}}$$

Low Re: laminar flow

High Re: turbulent flow



# Energy balance

$\boldsymbol{f} = 0 \; ; \; \nu = 0$

Inviscid invariant

Kinetic energy (per unit mass of fluid)

$$E = \frac{1}{2V} \int_V dV \; |\boldsymbol{u}|^2 = \frac{1}{2} \langle \boldsymbol{u}^2 \rangle$$

$\boldsymbol{f} = 0 \; ; \; \nu \neq 0$

Dissipative anomaly

$$\frac{dE}{dt} = -2\nu Z = -\varepsilon_\nu$$

Enstrophy $Z = \frac{1}{2} \langle \boldsymbol{\omega}^2 \rangle$

Vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$

$\boldsymbol{f} \neq 0 \; ; \; \nu \neq 0$

Steady state

$$\frac{dE}{dt} = \varepsilon_f - \varepsilon_\nu$$

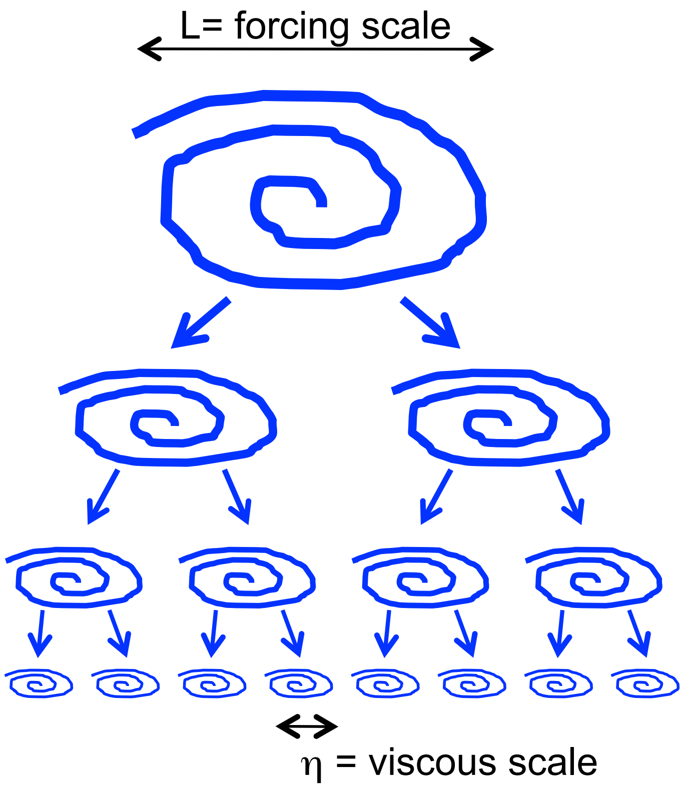
energy input rate $\varepsilon_f = \langle \boldsymbol{f} \cdot \boldsymbol{u} \rangle$

dissipation rate $\varepsilon_\nu = 2\nu Z$

# Turbulent cascade of kinetic energy

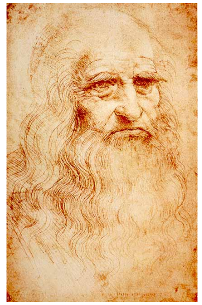
*"Big whorls have little whorls*
*That feed on their velocity,*
*And little whorls have lesser whorls*
*And so on to viscosity."*

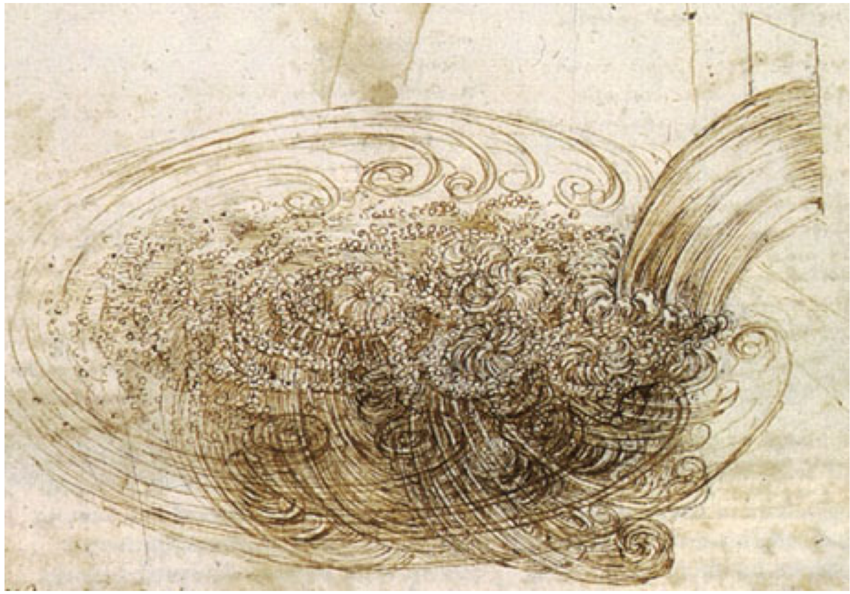
L.F. Richardson 1922




*"Doue la turbolenza dellacqua rigenera,*
*doue la turbolenza dellacqua*
*simantiene plugho,*
*doue la turbolenza dellacqua siposa"*

Leonardo da Vinci 1507





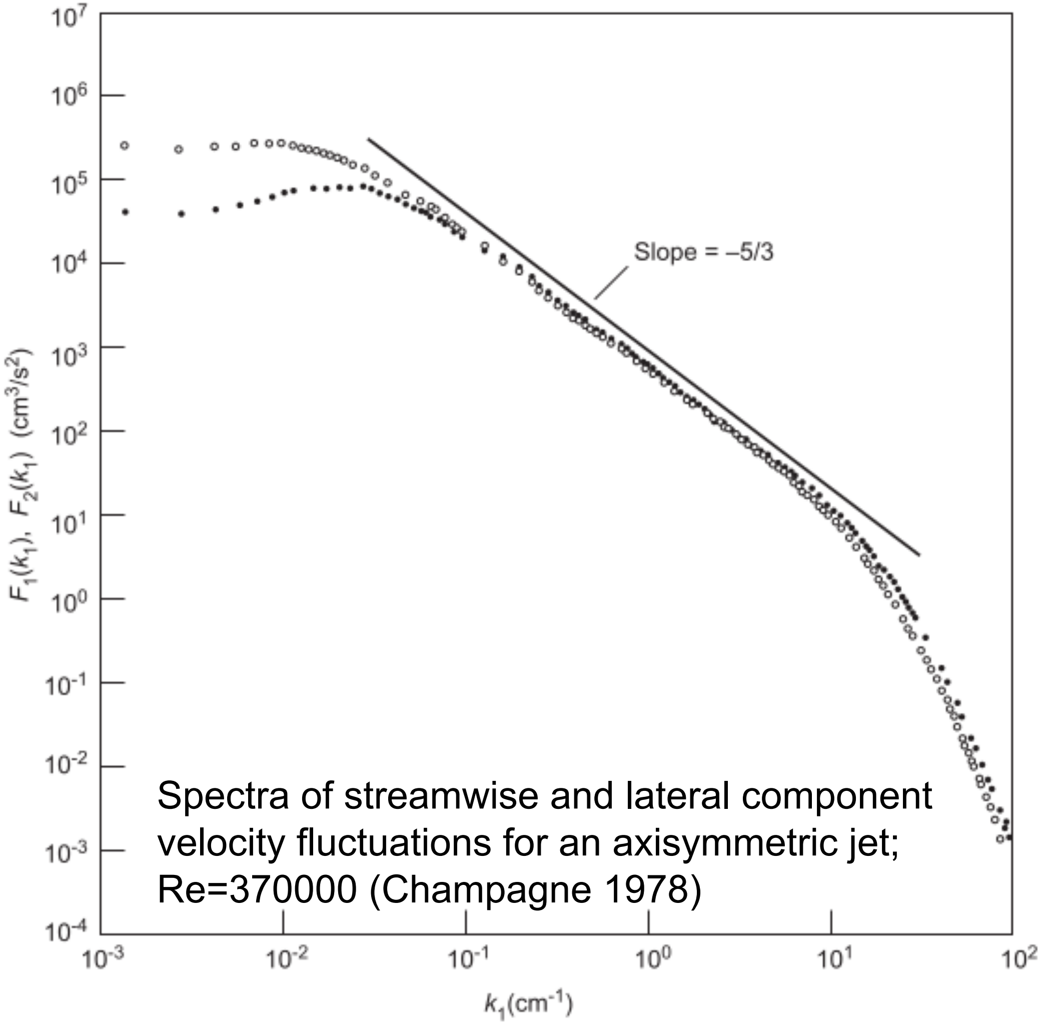
# Energy spectrum: Kolmogorov 41

A. N. Kolmogorov, 1941

Constant energy flux
Scale invariance
of the velocity field
in the inertial range

Kolmogorov spectrum

$$E(k) = C\varepsilon^{2/3}k^{-5/3}$$



# Kolmogorov 1941

Longitudinal velocity increments $\delta u_{\parallel}(r) = [\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})] \cdot \frac{\boldsymbol{r}}{|\boldsymbol{r}|}$

Structure functions $S_n(r) = \langle \delta u_{\parallel}(r)^n \rangle$

Hp: isotropy, homogeneity, stationarity

$\ell_\nu \ll r \ll \ell_f$

$$S_3(r) = \langle \delta u_{\parallel}(r)^3 \rangle = -\frac{4}{5}\varepsilon r$$

Hp: scale invariance, self similarity

$$\delta_r u \sim (\varepsilon r)^{1/3}$$

$$S_n(r) = \langle \delta u_{\parallel}(r)^n \rangle \sim r^{\zeta_n} \qquad \zeta_n = n/3$$

Kolmogorov scale $\eta = (\nu^3/\varepsilon)^{1/4}$

$$Re_\eta = \frac{\eta \delta_\eta u}{\nu} = 1$$

$$Re \sim (L/\eta)^{4/3}$$

# 2D TURBULENCE

# Two-dimensional turbulence

Navier-Stokes eq. for velocity field in 2D $\boldsymbol{u}(\boldsymbol{x},t) = (u_x(x,y,t), u_y(x,y,t))$

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\frac{\nabla P}{\rho} + \nu\Delta\boldsymbol{u} + \boldsymbol{f}$$

$$\nabla\cdot\boldsymbol{u} = 0$$

Stream function $\psi(\boldsymbol{x},t)$ $\boldsymbol{u} = (\partial_y\psi, -\partial_x\psi)$

Vorticity $\omega = [\nabla\times\boldsymbol{u}]_z = \partial_x u_y - \partial_y u_x = -\Delta\psi$

2D $\partial_t\omega + \boldsymbol{u}\cdot\nabla\omega = \nu\Delta\omega + f_\omega$

3D $\partial_t\boldsymbol{\omega} + \boldsymbol{u}\cdot\nabla\boldsymbol{\omega} = \underline{\boldsymbol{\omega}\cdot\nabla\boldsymbol{u}} + \nu\Delta\boldsymbol{\omega} + \boldsymbol{f}_\omega$

In 2D enstrophy $Z = \frac{1}{2}\langle\omega^2\rangle$ is an inviscid invariant

# Energy & Enstrophy balance

$\boldsymbol{f} = 0 \; ; \; \nu = 0$

2 inviscid invariants

Energy $E = \frac{1}{2}\langle|\boldsymbol{u}|^2\rangle$

Enstrophy $Z = \frac{1}{2}\langle\omega^2\rangle$

$\boldsymbol{f} = 0 \; ; \; \nu \neq 0$

No dissipative anomaly for kinetic energy

$$\frac{dE}{dt} = -2\nu Z = -\varepsilon_\nu$$

$$\frac{dZ}{dt} = -2\nu P = -\eta_\nu$$

Palinstrophy

$P = \frac{1}{2}\langle|\nabla\omega|^2\rangle$

$\boldsymbol{f} \neq 0 \; ; \; \nu \neq 0$

Energy grows for $t < L^2/\nu$

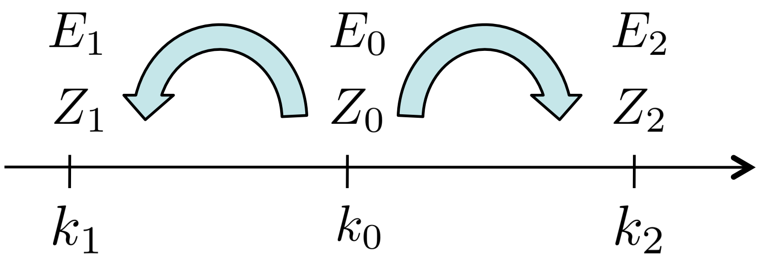
$$\frac{dE}{dt} = \varepsilon_f - \varepsilon_\nu$$

Energy input $\varepsilon_f = \langle \boldsymbol{f} \cdot \boldsymbol{u}\rangle$

$$\frac{dZ}{dt} = \eta_f - \eta_\nu$$

Enstrophy input $\eta_f = \langle \omega f_\omega \rangle$

# Fjørtoft 1953

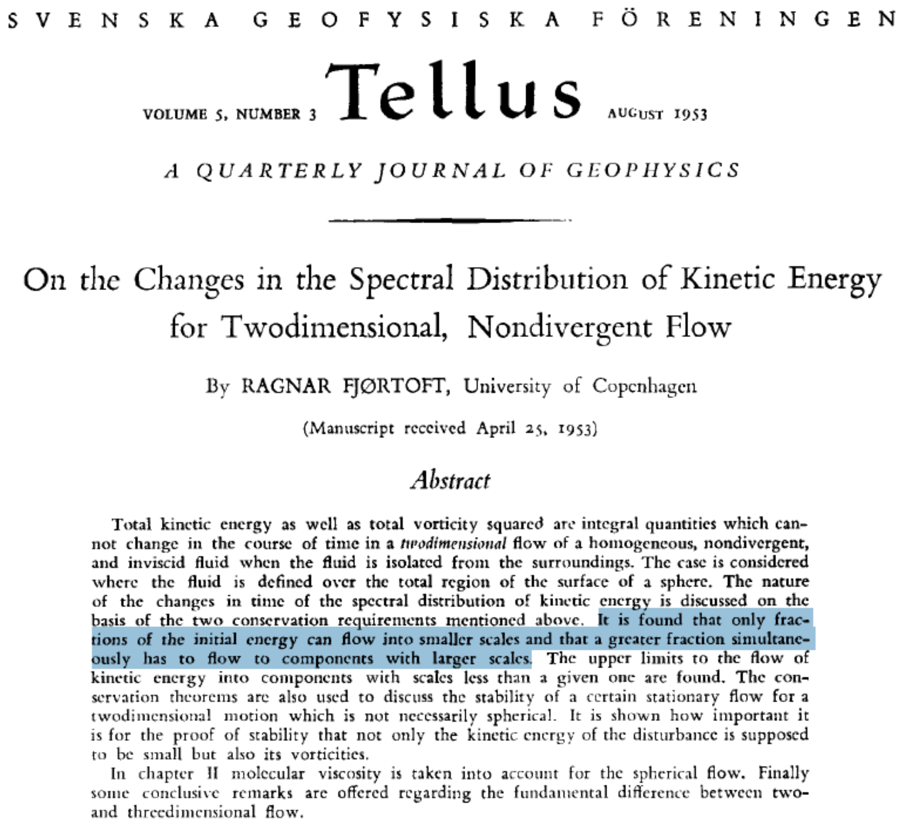



$k_1 = 1/2\ k_0$ Large scale

$k_2 = 2\ k_0$ Small scale

$$E_0 = E_1 + E_2$$

$$Z_0 = Z_1 + Z_2 \Rightarrow k_0^2 E_0 = k_1^2 E_1 + k_2^2 E_2$$

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On the Changes in the Spectral Distribution of Kinetic Energy for Twodimensional, Nondivergent Flow


By RAGNAR FJØRTOFT, University of Copenhagen



(Manuscript received April 25, 1953)



*Abstract*

Total kinetic energy as well as total vorticity squared are integral quantities which cannot change in the course of time in a *twodimensional* flow of a homogeneous, nondivergent, and inviscid fluid when the fluid is isolated from the surroundings. The case is considered where the fluid is defined over the total region of the surface of a sphere. The nature of the changes in time of the spectral distribution of kinetic energy is discussed on the basis of the two conservation requirements mentioned above. It is found that only fractions of the initial energy can flow into smaller scales and that a greater fraction simultaneously has to flow to components with larger scales. The upper limits to the flow of kinetic energy into components with scales less than a given one are found. The conservation theorems are also used to discuss the stability of a certain stationary flow for a twodimensional motion which is not necessarily spherical. It is shown how important it is for the proof of stability that not only the kinetic energy of the disturbance is supposed to be small but also its vorticities.

In chapter II molecular viscosity is taken into account for the spherical flow. Finally some conclusive remarks are offered regarding the fundamental difference between two- and threedimensional flow.


$E_1/E_2 = 4$ Energy is transferred toward large scales

$Z_1/Z_2 = 1/4$ Enstrophy is transferred toward small scales

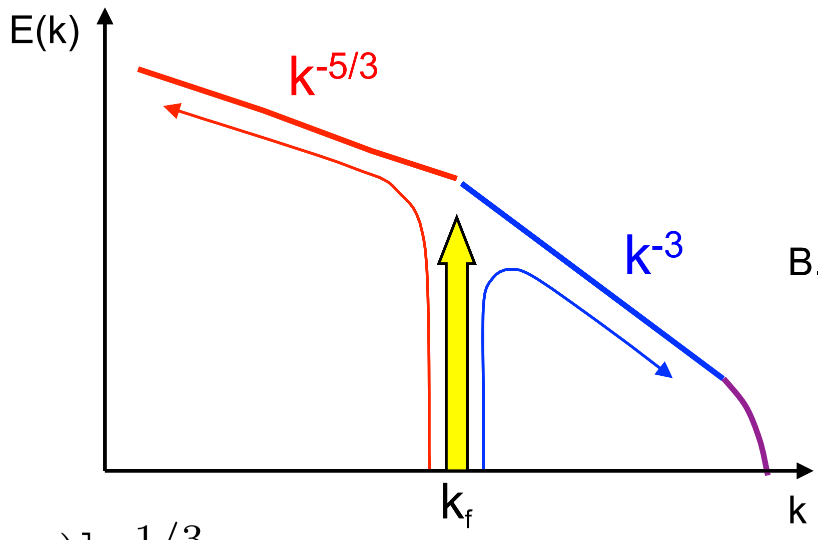
# Double energy - enstrophy cascade

Inverse cascade:
constant flux of energy

$$E(k) = C\varepsilon^{2/3}k^{-5/3}$$

Direct cascade:
constant flux of enstrophy

$$E(k) = C'\eta^{2/3}k^{-3}[\ln(k/k_{min})]^{-1/3}$$




B. Kraichnan, 1967

Exact relations for third order structure functions
D. Bernard, Phys. Rev. E 60, 6184 (1999)
V. Yakhot Phys. Rev. E 60, 5544 (1999)

$$S_3(r) = \frac{3}{2}\varepsilon r \qquad r \gg \ell_f$$

$$S_3(r) = \frac{1}{8}\eta r^3 \qquad r \ll \ell_f$$

G. Boffetta, R.E. Ecke (2012)
Two-dimensional turbulence.
Annu. Rev. Fluid Mech. 44, 427-451

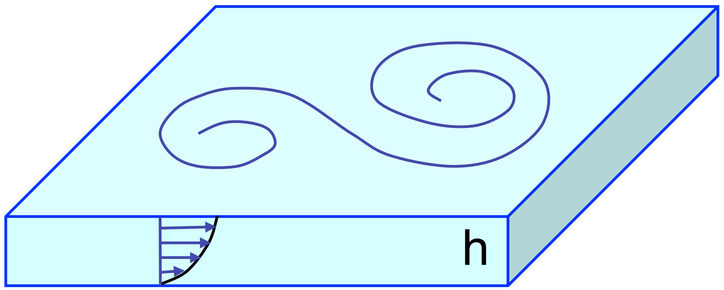
# 2D Navier-Stokes + Friction

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{\nabla P}{\rho} + \boldsymbol{f} + \nu \Delta \boldsymbol{u} - \underline{\alpha \boldsymbol{u}}$$

Ekman friction (rotating flow)
Rayleigh friction (stratified flow)
Hartmann friction (MHD)
air friction (soap film)

Thin fluid layer
No slip b.c. at bottom
Viscous velocity profile

$$\alpha \sim \nu / h^2$$

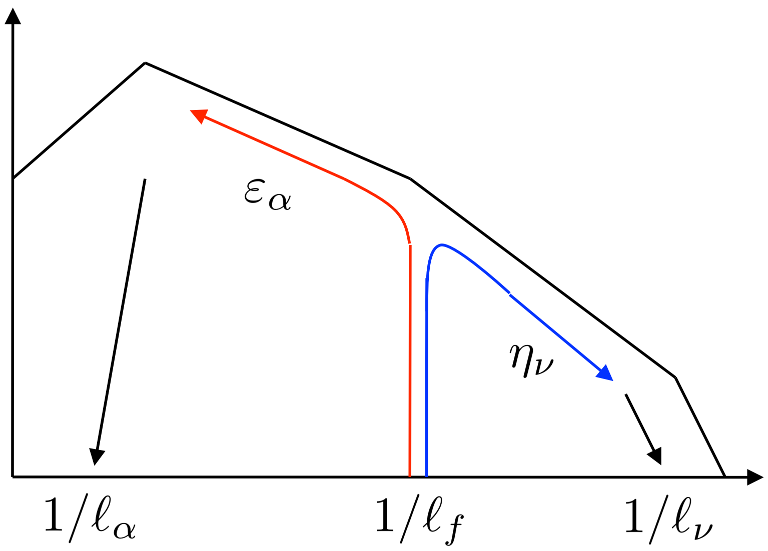



Friction dissipates energy at large scale
and stops the inverse energy cascade

$$\ell_\alpha \simeq \varepsilon_\alpha^{1/2} \alpha^{-3/2} \sim h^3$$

# Heuristic argument for fluxes ratios

R.H. Kraichnan, Phys. Fluids 10 (1967) 1417
G. L. Eyink, Physica D 91, 97-142 (1996)




forcing scale $\ell_f^2 = \varepsilon_f/\eta_f$

friction scale $\ell_\alpha^2 = \varepsilon_\alpha/\eta_\alpha$

viscous scale $\ell_\nu^2 = \varepsilon_\nu/\eta_\nu$

Energy balance: $\varepsilon_f = \varepsilon_\nu + \varepsilon_\alpha$

Enstrophy balance: $\eta_f = \eta_\nu + \eta_\alpha \Rightarrow \varepsilon_f/\ell_f^2 = \varepsilon_\nu/\ell_\nu^2 + \varepsilon_\alpha/\ell_\alpha^2$

$$\frac{\varepsilon_\nu}{\varepsilon_\alpha} = \left(\frac{\ell_\nu}{\ell_f}\right)^2 \left(\frac{\ell_f}{\ell_\alpha}\right)^2 \frac{(\ell_\alpha/\ell_f)^2 - 1}{1 - (\ell_\nu/\ell_f)^2}$$

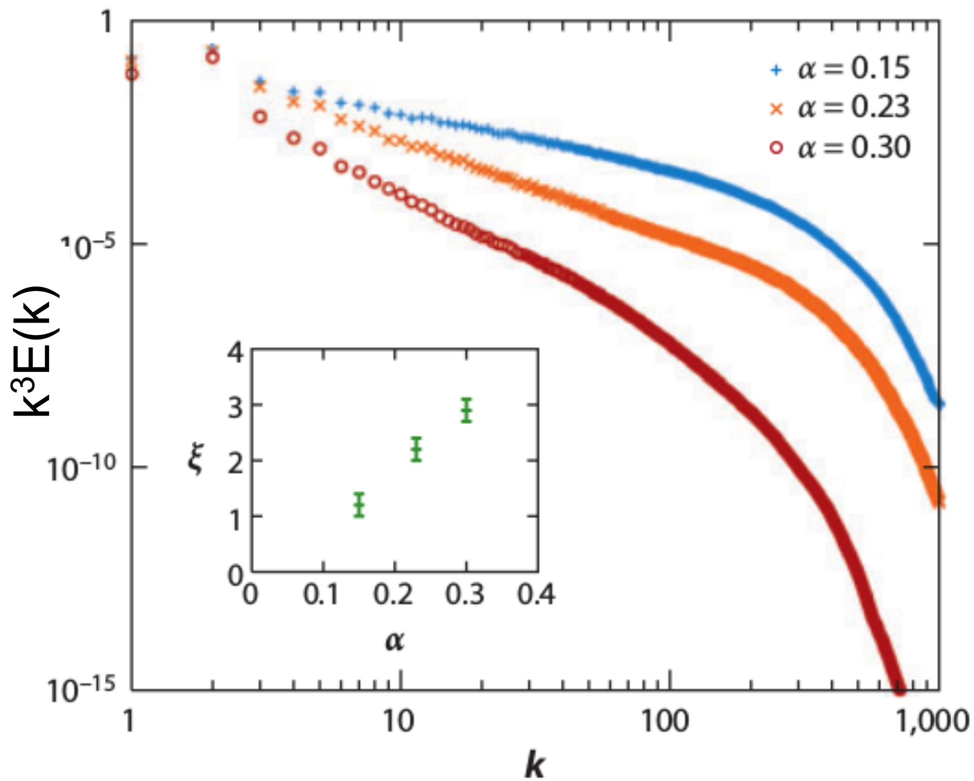
$$\frac{\eta_\nu}{\eta_\alpha} = \frac{(\ell_\alpha/\ell_f)^2 - 1}{1 - (\ell_\nu/\ell_f)^2}$$

# Effects of friction on the direct enstrophy cascade

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{\nabla P}{\rho} + \boldsymbol{f} + \nu \Delta \boldsymbol{u} - \underline{\alpha \boldsymbol{u}}$$

Friction causes a steepening of the spectrum of the enstrophy cascade

$$E(k) \sim \eta^{2/3} k^{-(3+\xi)}$$




Bernard D. Europhys. Lett. 50, 333 (2000)
Nam K, Ott E, Antonsen TM, Guzdar PN. Phys. Rev. Lett. 84:5134 (2000)
Boffetta G, Celani A, SM, Vergassola M.. Phys. Rev. E 66, 026304 (2002)

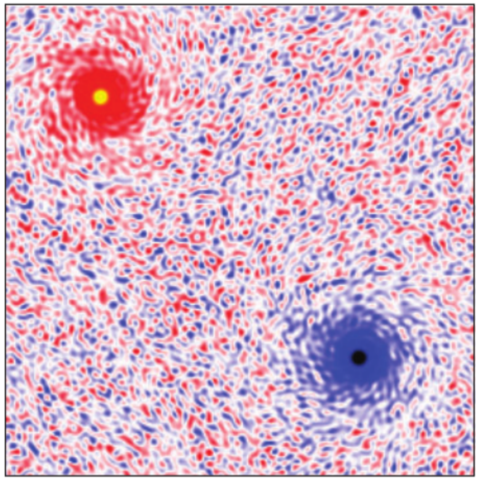
# Condensate

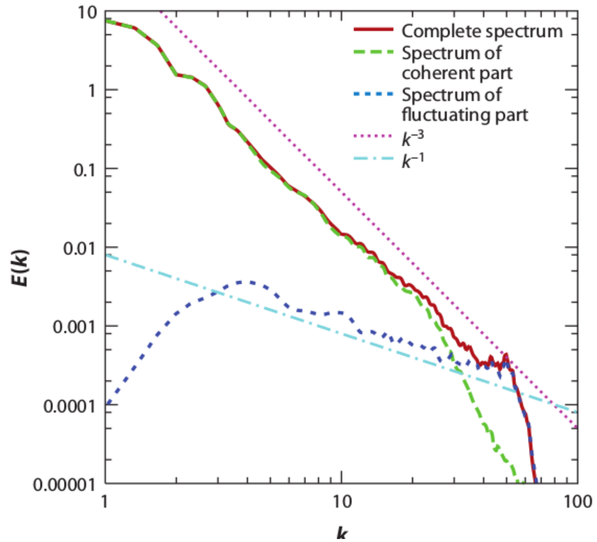
## Numerics

L.M. Smith and V. Yakhot,
PRL 71, 352 (1993), Fluid Mech. 214,115-138 (1994)

M. Chertkov, C. Connaughton, I. Kolokolov, and V. Lebedev, PRL 99, 084501 (2007)

J. Laurie, G. Boffetta, G. Falkovich, I. Kolokolov, and V. Lebedev, PRL 113, 254503 (2014)



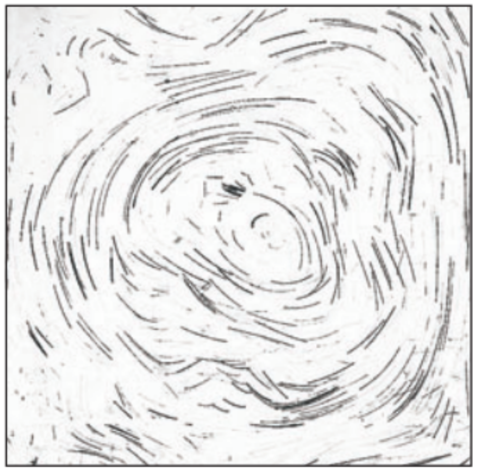



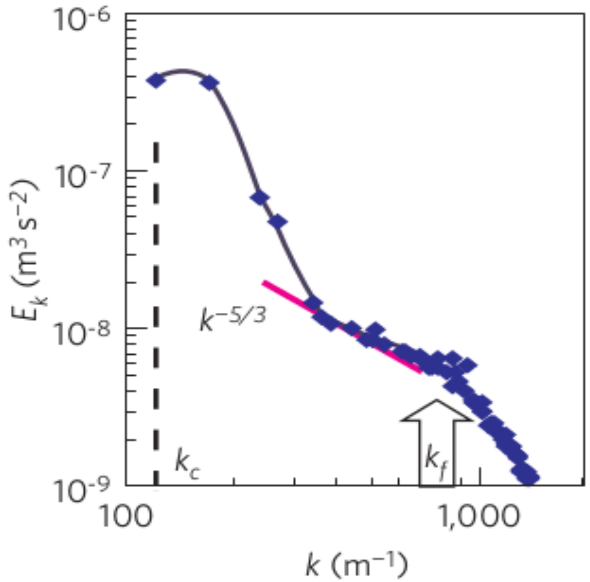

## Experiments

H. Xia, H. Punzmann, G. Falkovich, and M. G. Shats PRL 101, 194504 (2008)

H. Xia, M. Shats, and G. Falkovich, Phys. Flyids 21, 125101 (2009)

H. Xia, D. Byrne, G. Falkovich, and M. Shats, Nat. Phys. 7, 321 (2011)





# Reviews of turbulence in Flatland

R.H. Kraichnan, D. Montgomery (1980)
Two-dimensional turbulence.
Rep. Prog. Phys. 43, 547-619

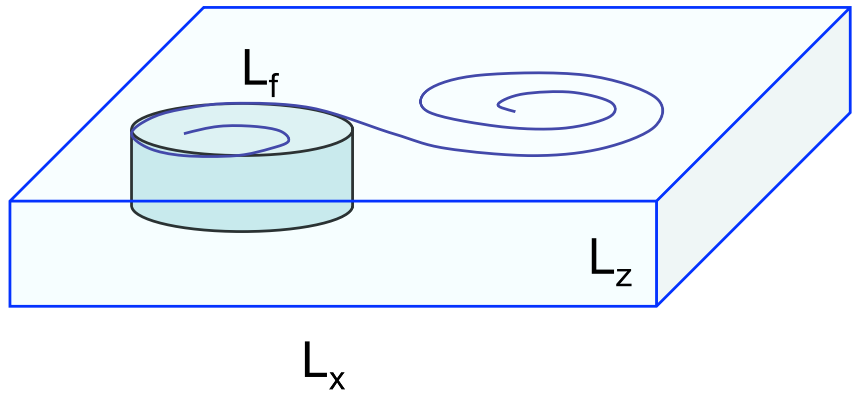
H. Kellay, W. I. Goldburg (2002)
Two-dimensional turbulence:
a review of some recent experiments.
Rep. Prog. Phys. 65, 845–894

G. Boffetta, R.E. Ecke (2012)
Two-dimensional turbulence.
Annu. Rev. Fluid Mech. 44, 427-451

# 2D – 3D TURBULENCE

# Dimensional transition in thin fluid layers

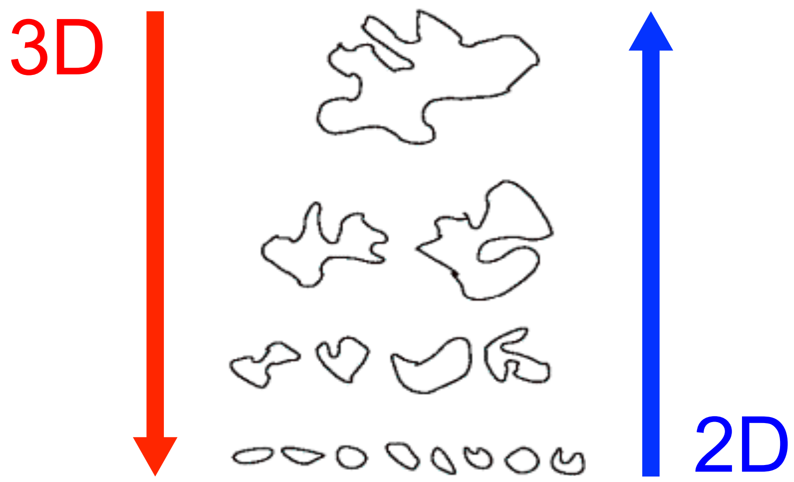
Transition from 2D to 3D turbulence as the thickness of the layer increases




$L_z = 0$ 2D turbulence
inverse energy cascade

$L_z$ small: 2D turbulence ( + friction)

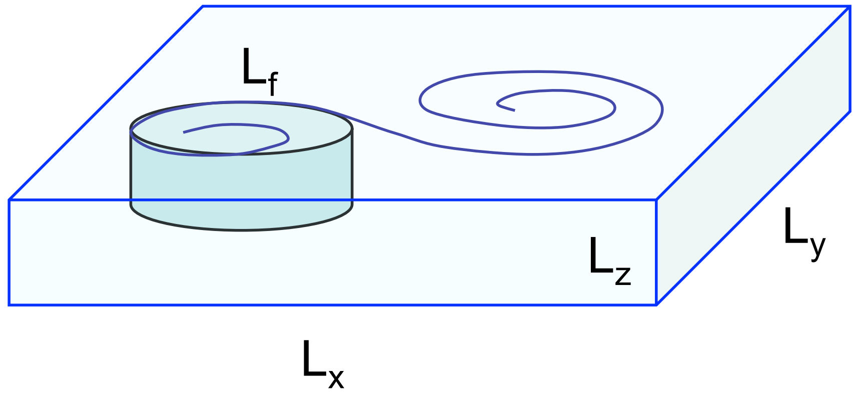
$L_z$ large: 3D turbulence
direct energy cascade



# Numerical simulations of thin fluid layers

L. M. Smith, J. R. Chasnov, and F. Waleffe, Phys. Rev. Lett. **77**, 2467 (1996).
L. M. Smith and F. Waleffe, Phys. Fluids **11**, 1608 (1999)
A. Celani, SM, D. Vincenzi, Phys.Rev. Lett. 104, 184506 (2010)

3D Navier-Stokes equation for a thin layer of incompressible fluid.




Periodic b.c: no wall turbulence
no friction

Forcing: random in time (constant energy input)
two dimensional force (2D2C)

$$\boldsymbol{f}(\boldsymbol{x}) = (f_x(x,y), f_y(x,y), 0)$$

Aspect ratio $S = \dfrac{L_z}{L_f}$

Hyperviscosity $\nu \Delta \boldsymbol{u} \to (-1)^{p+1} \nu_p \Delta^p \boldsymbol{u}$

# Energy balance

L. M. Smith, J. R. Chasnov, and F. Waleffe, Phys. Rev. Lett. **77**, 2467 (1996)

$$\frac{dE}{dt} = \varepsilon_f - \varepsilon_\nu$$

energy input rate $\varepsilon_f = \langle \boldsymbol{f} \cdot \boldsymbol{u} \rangle$

dissipation rate $\varepsilon_\nu = 2\nu Z$

3D: dE/dt = 0

2D: dE/dt ≠ 0

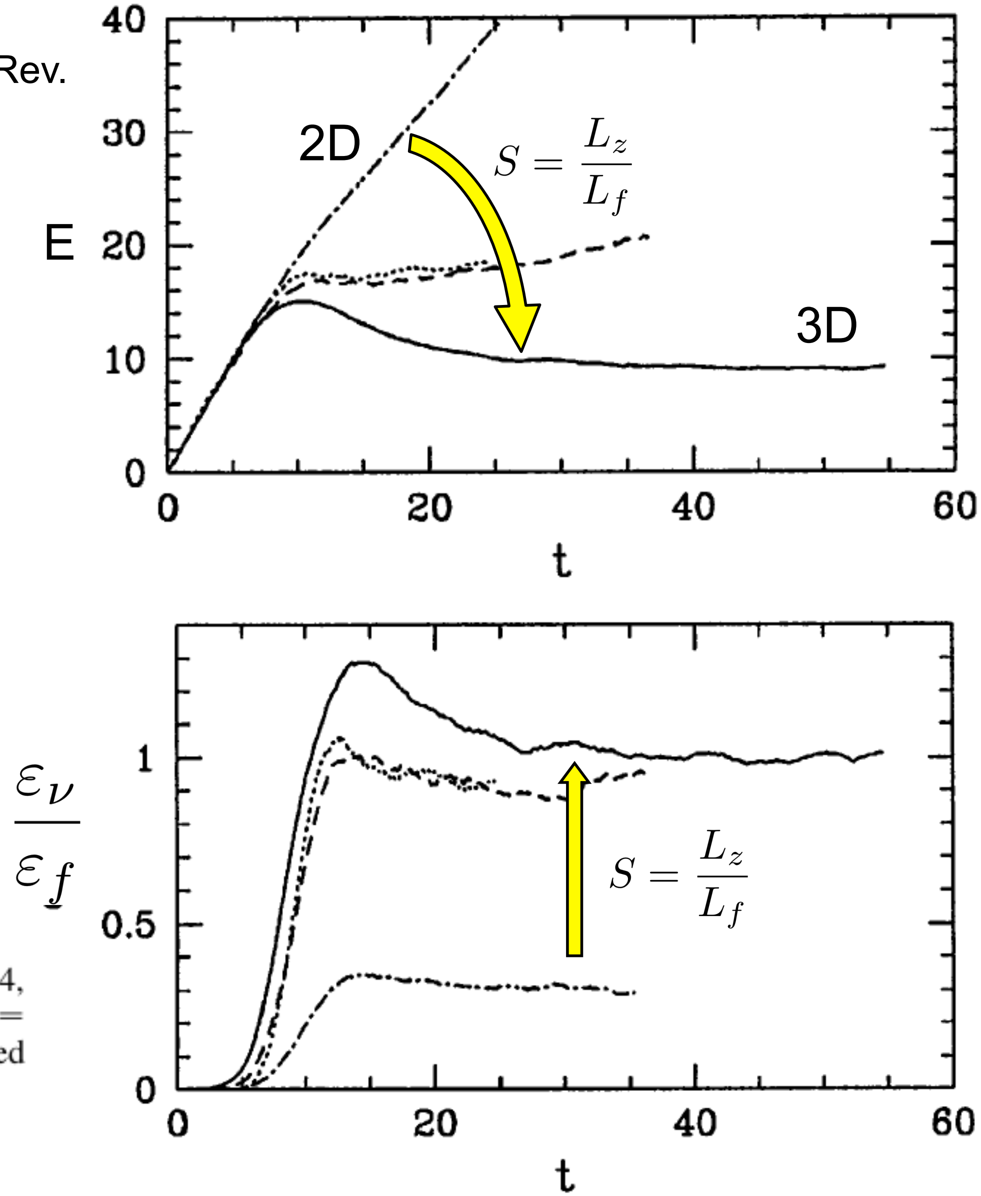
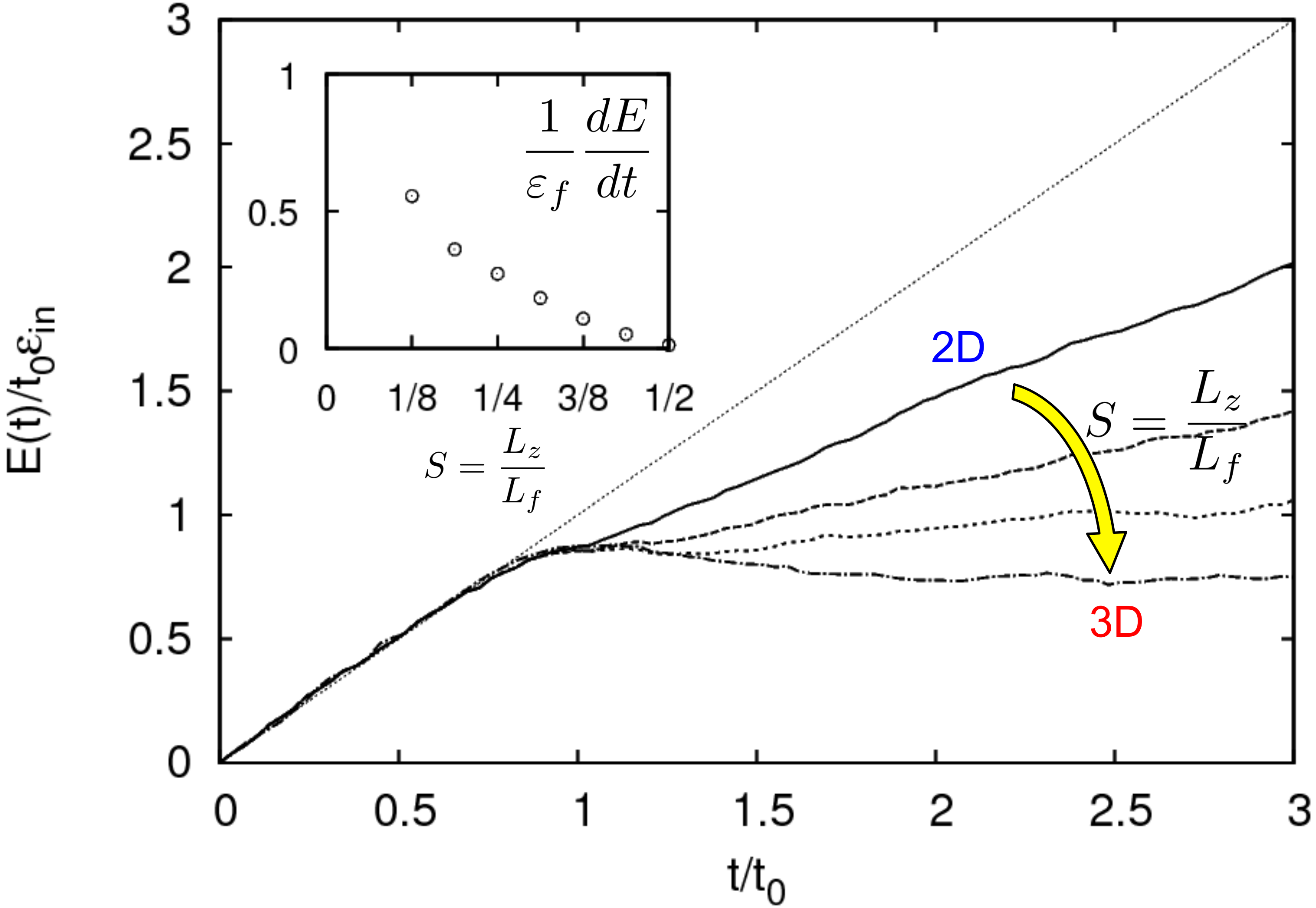



FIG. 1. $A = 1/64$, Ro $= \infty$, $S = 0.75$ (solid line); $A = 1/64$, Ro $= \infty$, $S = 0.375$, eddy viscosity model (dashed line); $A = 1/64$, Ro $= \infty$, $S = 0.375$, hyperviscosity operator $\nabla^4$ (dotted line); $A = 1/64$, Ro $= 0.5$, $S = 0.75$ (dot-dashed line).

# Energy growth rates

A. Celani, SM, D. Vincenzi, PRL 104, 184506 (2010)



# Energy spectra

L. M. Smith, J. R. Chasnov, and F. Waleffe, Phys. Rev. Lett. **77**, 2467 (1996)

Thick layer (a) S = 0.75
No inverse cascade

Thin layer (b) S = 0.375
Inverse cascade

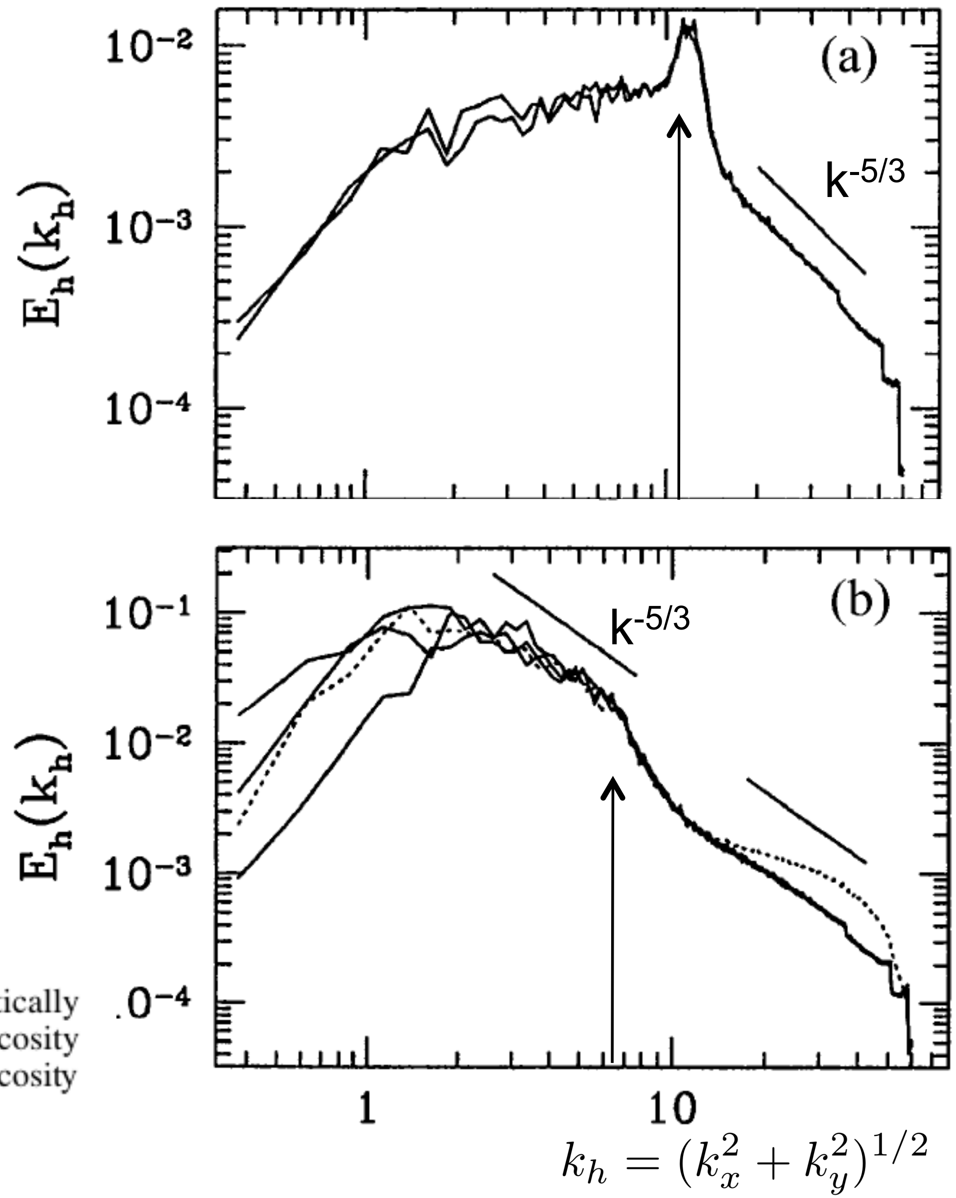
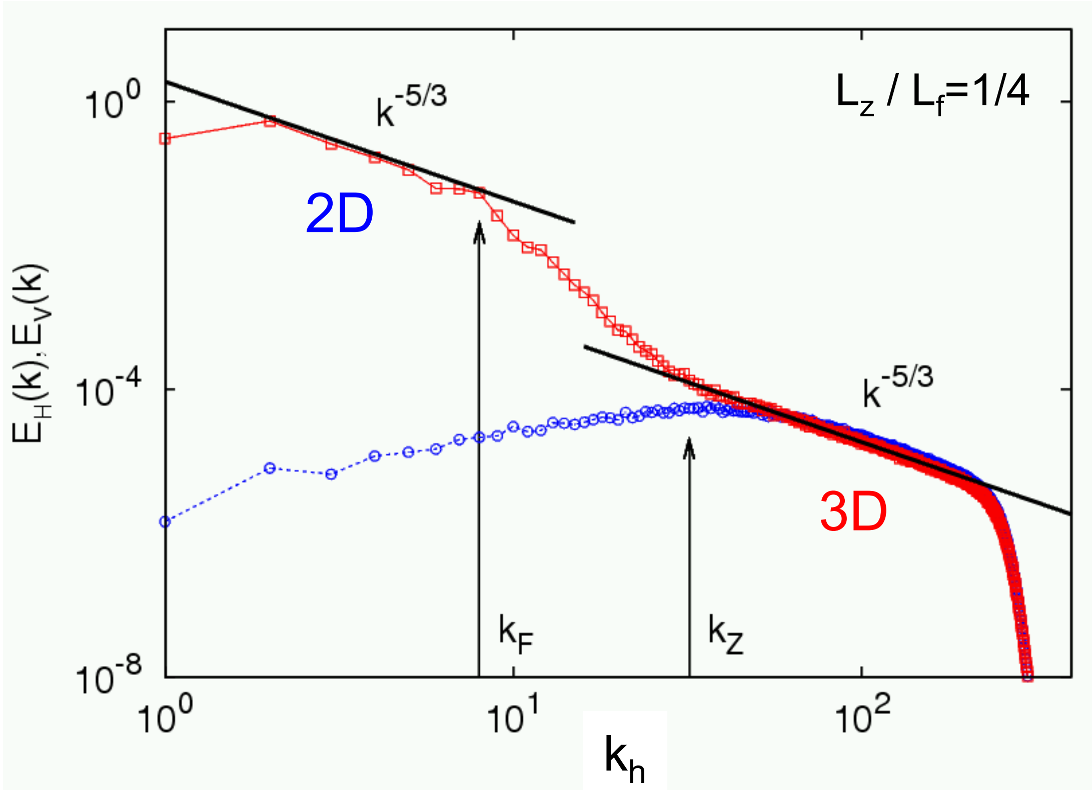



FIG. 2. (upper) $A = 1/64$, Ro $= \infty$, $S = 0.75$ (statistically steady); (lower) $A = 1/64$, Ro $= \infty$, $S = 0.375$: eddy viscosity (solid line) with time increasing upwards; hyperviscosity (dotted line). The lines are $E_h \propto k_h^{-5/3}$.

# Energy spectra

A. Celani, SM, D. Vincenzi,
PRL 104, 184506 (2010)




2D spectra of horizontal and vertical velocities averaged in the vertical direction

$$E_{h,v}(k_h) = \int_{|\boldsymbol{q}_h|=k_h} d\boldsymbol{q}_h^2 \, |\boldsymbol{u}_{h,v}(\boldsymbol{q}_h, q_z = 0)|^2$$

as a function of horizontal wavenumber $k_h = (k_x^2 + k_y^2)^{1/2}$

# Spectral energy fluxes

Spectral energy balance

$$\partial_t E(k) = T(k) + F(k) - \nu k^2 E(k)$$

Spectral energy flux

$$\Pi_E(k) = \int_k^\infty T(k')dk'$$

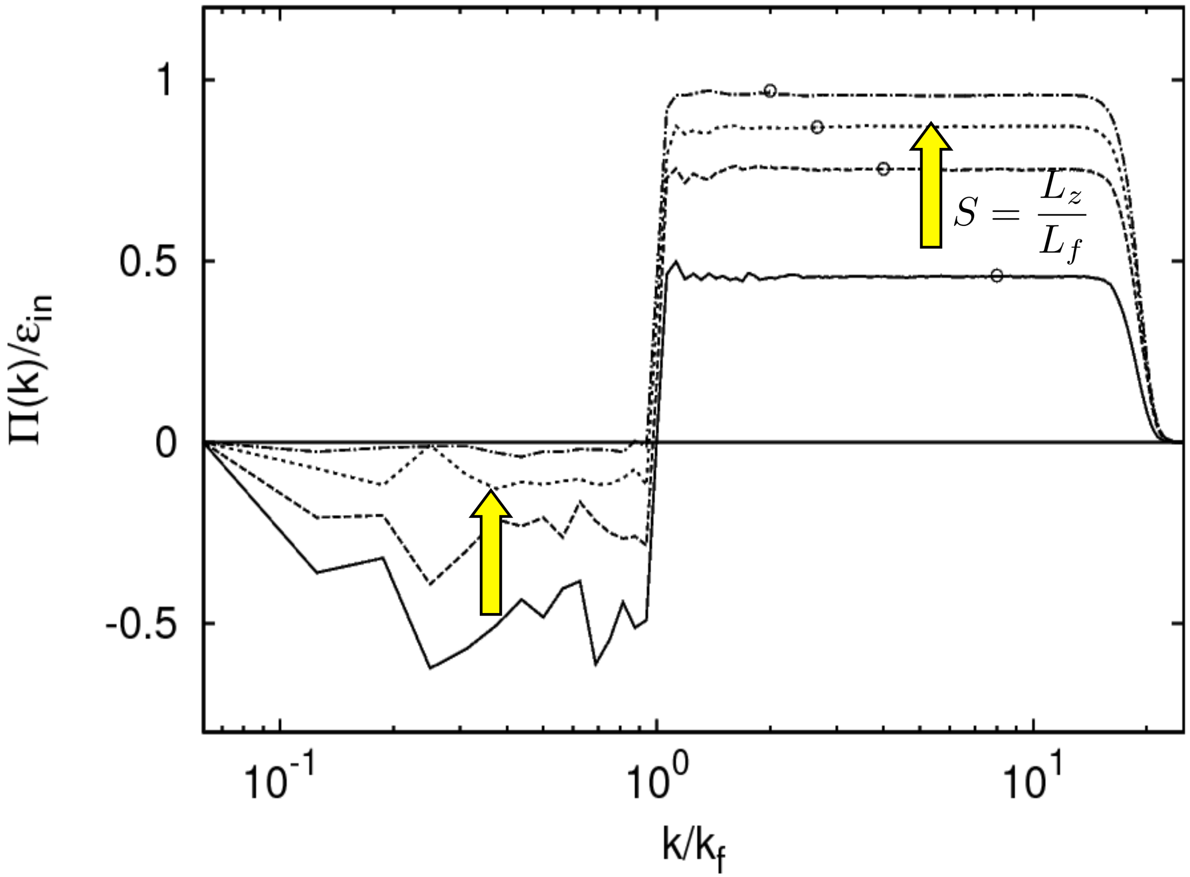
$$\Pi_E(k) > 0$$

Direct cascade

$$\Pi_E(k) < 0$$

Inverse cascade

A. Celani, SM, D. Vincenzi, PRL 104, 184506 (2010)

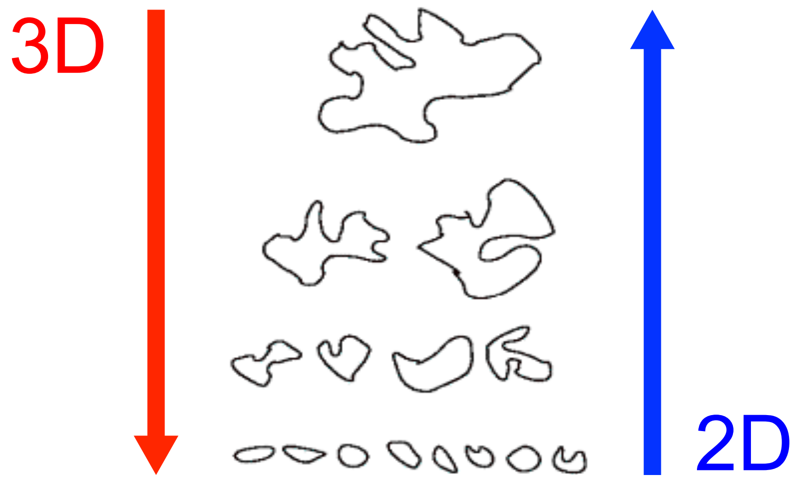


# Turbulent cascades & invariants

The direction of the cascade is determined by positive-defined inviscid invariants.

3D: Energy

2D: Energy & Enstrophy




Navier-Stokes equation for vorticity $\omega = \nabla \times \mathbf{u}$

2D $\partial_t \omega + \boldsymbol{v} \cdot \nabla \omega = \nu \nabla^2 \omega + f_\omega$

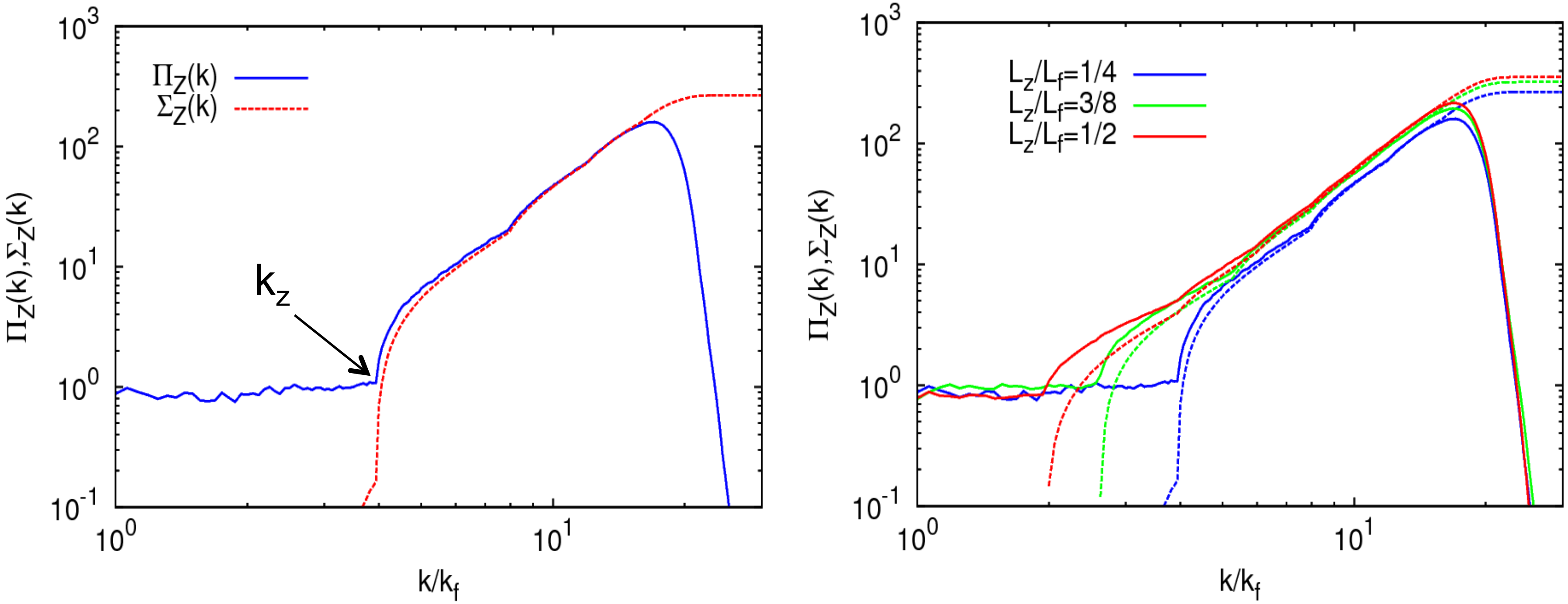
3D $\partial_t \boldsymbol{\omega} + \boldsymbol{v} \cdot \nabla \boldsymbol{\omega} = \underline{\boldsymbol{\omega} \cdot \nabla \boldsymbol{v}} + \nu \nabla^2 \boldsymbol{\omega} + \boldsymbol{f}_\omega$

Is there a suppression of enstrophy production in thin fluid layers?

# Enstrophy flux & vortex stretching

Enstrophy flux
$$\Pi(k) = \int_{|q|\leq k} d\boldsymbol{q} \; (\mathbf{u}\cdot\nabla\boldsymbol{\omega})(q)\boldsymbol{\omega}^*(q)$$

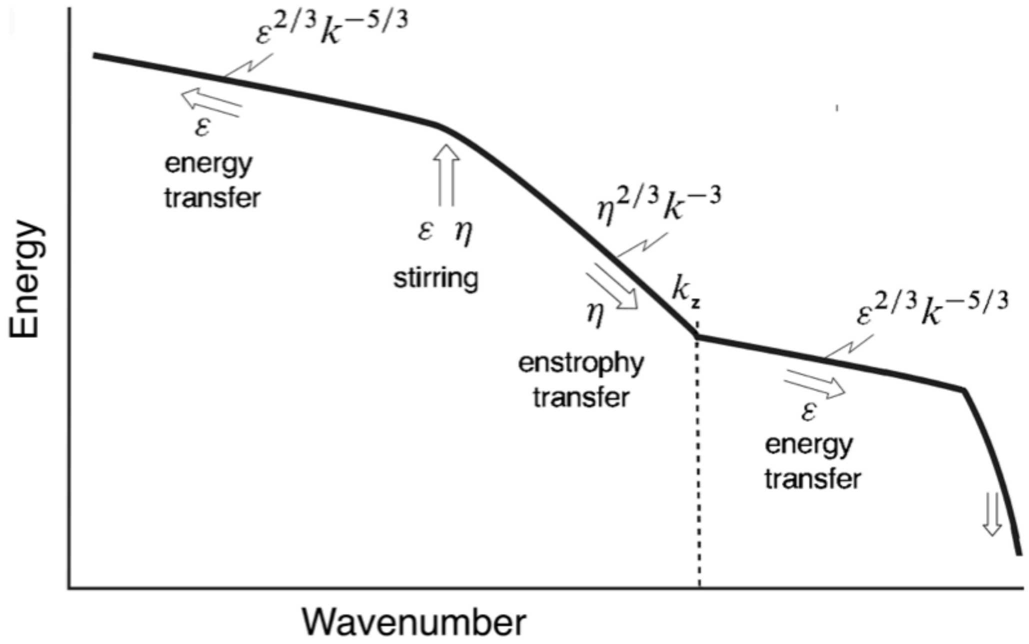
Enstrophy production
$$\Sigma(k) = \int_{|q|\leq k} d\boldsymbol{q} \; (\boldsymbol{\omega}\cdot\nabla\mathbf{u})(q)\boldsymbol{\omega}^*(q)$$




Constant enstrophy flux for $k_f < k < k_z$

Enstrophy production only for $k < k_z$

# Energy-Enstrophy cascades in thin fluid layers




2D inverse energy cascade at large scales $L > L_f$

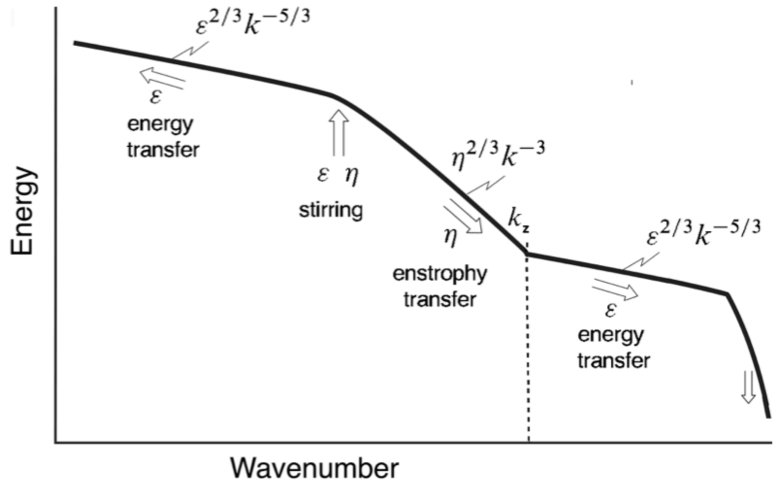
2D direct enstrophy cascade at intermediate scales $L_z < L < L_f$

3D direct energy cascade at small scales $L < L_z$

# Prediction for the energy fluxes

Energy balance $\varepsilon_f = \varepsilon_\alpha + \varepsilon_\nu$

Enstrophy balance $\eta_f = \eta_\alpha + \eta_z$
(for $\ell > \ell_z$ )




Forcing scale $\ell_f^2 = \varepsilon_f / \eta_f$

Friction scale $\ell_\alpha^2 = \varepsilon_\alpha / \eta_\alpha$

Flux of the direct energy cascade =
Residual energy flux carried by the
enstrophy cascade at the scale $\ell_z$

$$\varepsilon_\nu = \eta_z \ell_z^2$$

$$\ell_\alpha / \ell_f \to \infty$$

Direct energy cascade flux $\frac{\epsilon_\nu}{\epsilon_f} = \left(\frac{\ell_z}{\ell_f}\right)^2 = S^2$

Inverse energy cascade flux $\frac{\epsilon_\alpha}{\epsilon_f} = 1 - S^2$

# Shell model for thin fluid layers

Models which mimic Navier-Stokes dynamics

L. Biferale, Annu. Rev. Fluid Mech. 35, 441 (2003)

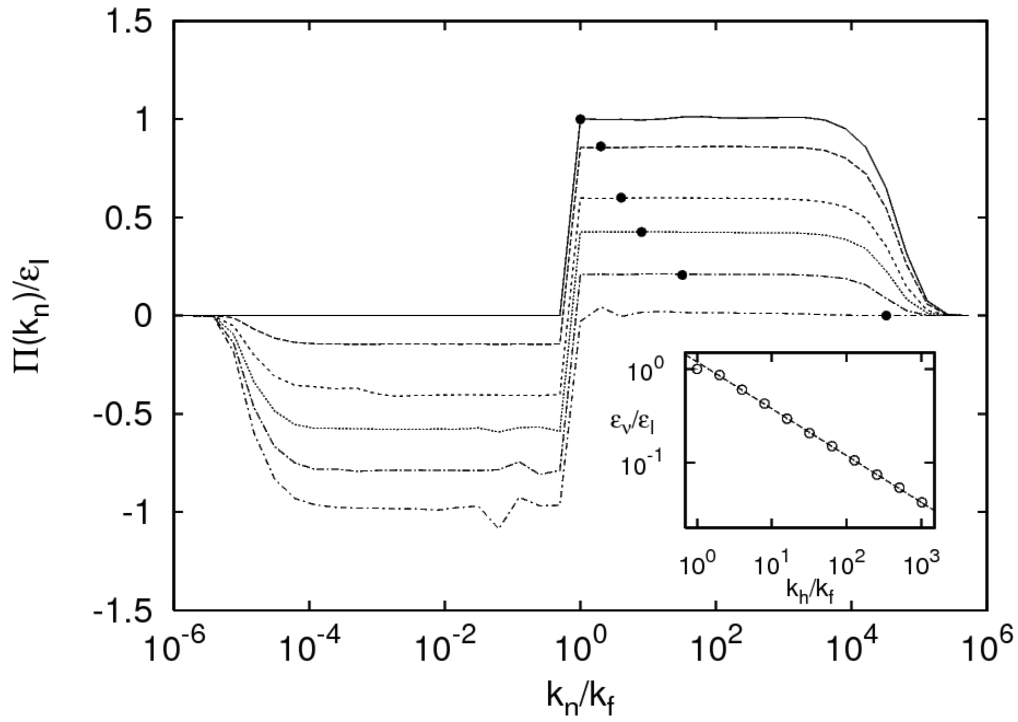
Global conservation of energy

$$E = \frac{1}{2} \sum_{n=1}^{N} |u_n|^2$$

Conservation of

$$H = \frac{1}{2} \sum_{n=1}^{N} k_n^{\beta} |u_n|^2$$

for $1 \leq n \leq n_h$

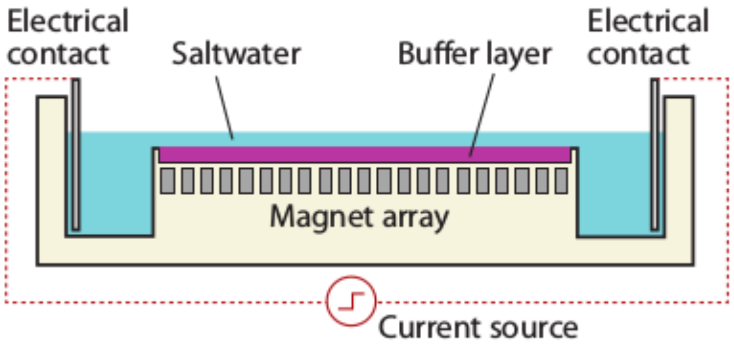



G. Boffetta, F. De Lillo, SM, Phys. Rev. E 83, 066302 (2011)

# 2D Turbulence: Experiments

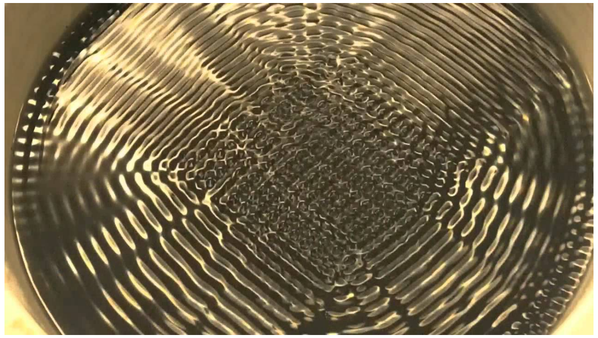
## Electromagetically driven layers

J. Sommeria
P. Tabeling
J. Gollub
A. Cenedese
R. Ecke
M. Shats
H. Xia




## Faraday-waves driven layers

A. von Kameke
M. Shats
H. Xia



## Soap films

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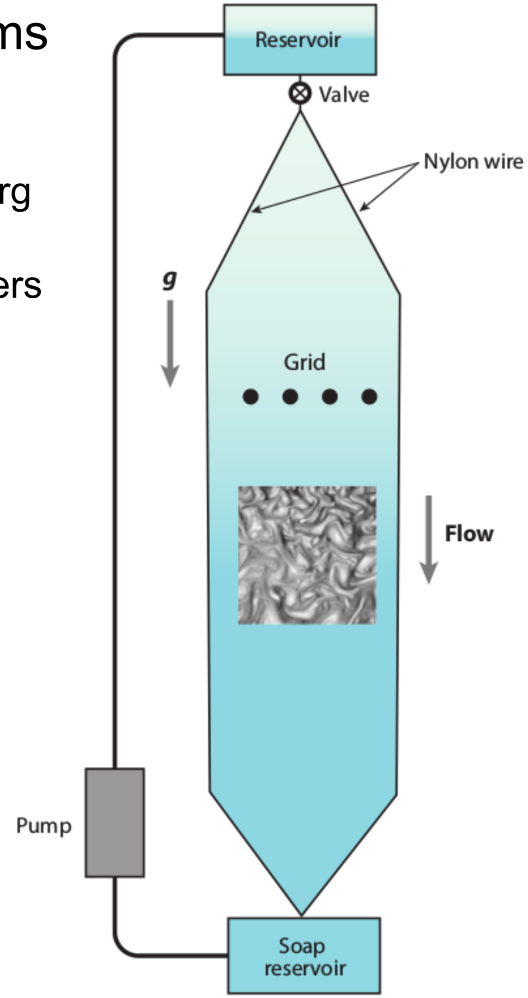
W. Goldburg
H. Kellay
M.A. Rutgers
M. Rivera
R. Ecke

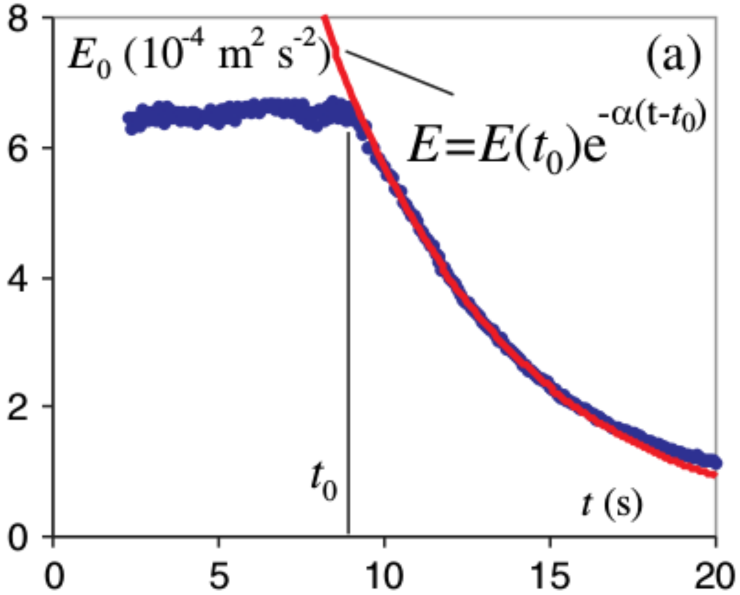


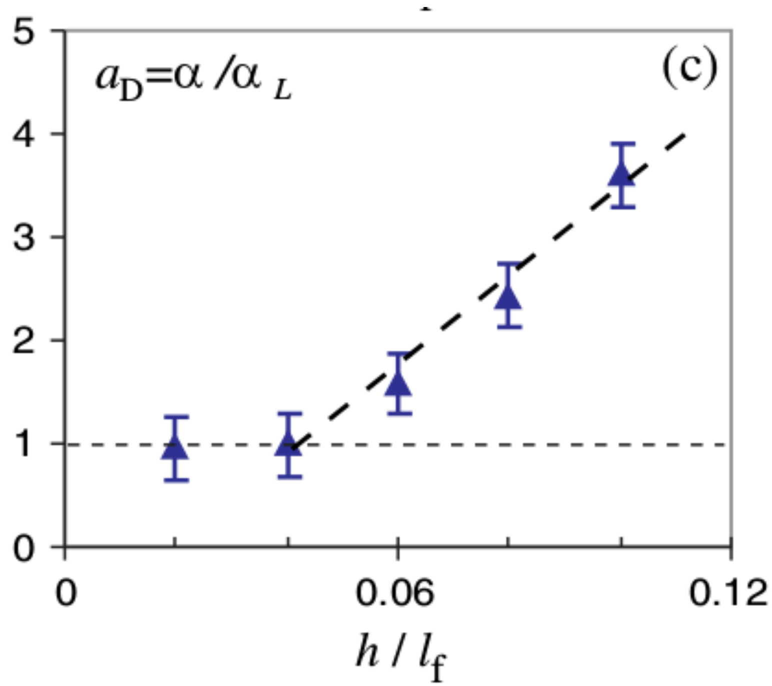
# Dimensional transition: Experiments

M. Shats, D. Byrne, and H. Xia Phys. Rev. Lett. 105, 264501 (2010)

2D = Viscous friction $\alpha_L = \nu\pi^2/2h^2$

3D = Viscous friction + Turbulent eddy viscosity $\alpha_t = (\nu + K)\pi^2/2h^2$.



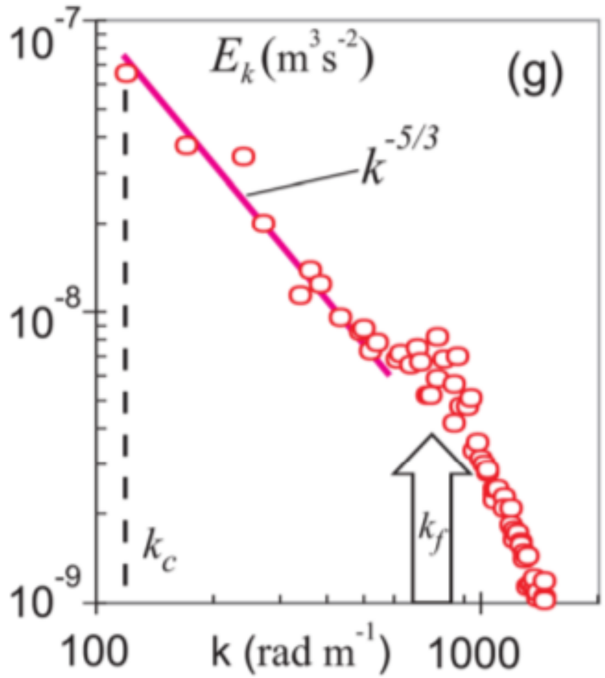



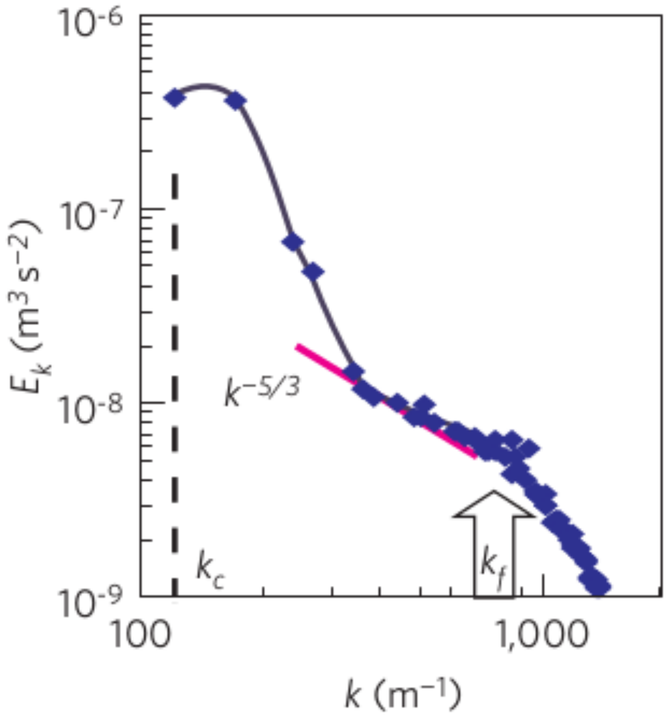
# Thick fluid layers: Experiments

nature physics
LETTERS
PUBLISHED ONLINE: 6 FEBRUARY 2011 | DOI: 10.1038/NPHYS1910

**Upscale energy transfer in thick turbulent fluid layers**

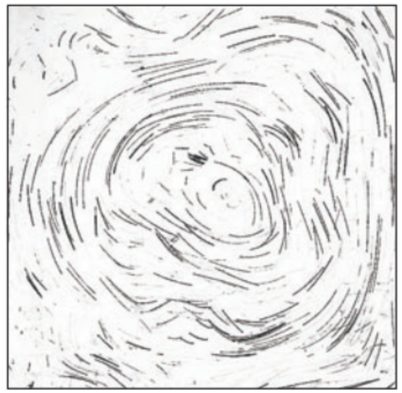
H. Xia[1], D. Byrne[1], G. Falkovich[2] and M. Shats[1]*




Inverse cascade

Condensate



H. Xia, D. Byrne, G. Falkovich, and M. Shats, Nat. Phys. 7, 321 (2011)
D. Byrne, H. Xia and M. Shats Phys. Fluids, 23, 095109 (2011)

# Conclusions (Part 1)

Dimensional transition in turbulent fluid layers from 2D inverse energy cascade to 3D direct energy cascade as the thickness of the layer $L_z$ increases.

Splitting of the energy cascade: coexistence of inverse & direct cascade.
Ratio of fluxes depend on the aspect ratio $L_z / L_f$

Enstrophy is a quasi-invariant (conserved by large-scale dynamics)
Direct enstrophy cascade at intermediate scales.

The development of the inverse cascade is due to the presence of a second positive-defined (quasi) invariant.

Is it possible to observe an inverse energy cascade in 3D isotropic flows?

THE END

# 2D TURBULENCE

review numerics

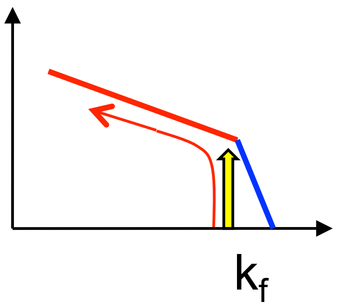
# 2D Turbulence: Numerical simulations

Boundary conditions: **<u>periodic</u>** (square domain L x L)
free slip
no slip

Forcing: **<u>random forcing</u>** (constant input rate)
time-correlated forcing
time-independent forcing

Spatial correlation of the forcing

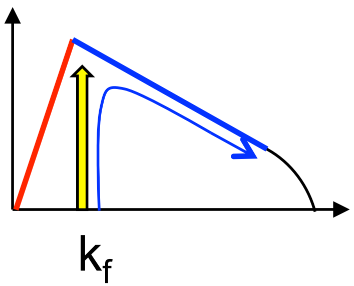
Inverse energy cascade:
Forcing at high wavenumber




Direct enstrophy cascade suppressed by viscosity (or hyper-viscosity)

$$\nu \Delta \boldsymbol{u} \to (-1)^{p+1} \nu_p \Delta^p \boldsymbol{u}$$

Direct enstrophy cascade:
Forcing at low wavenumber

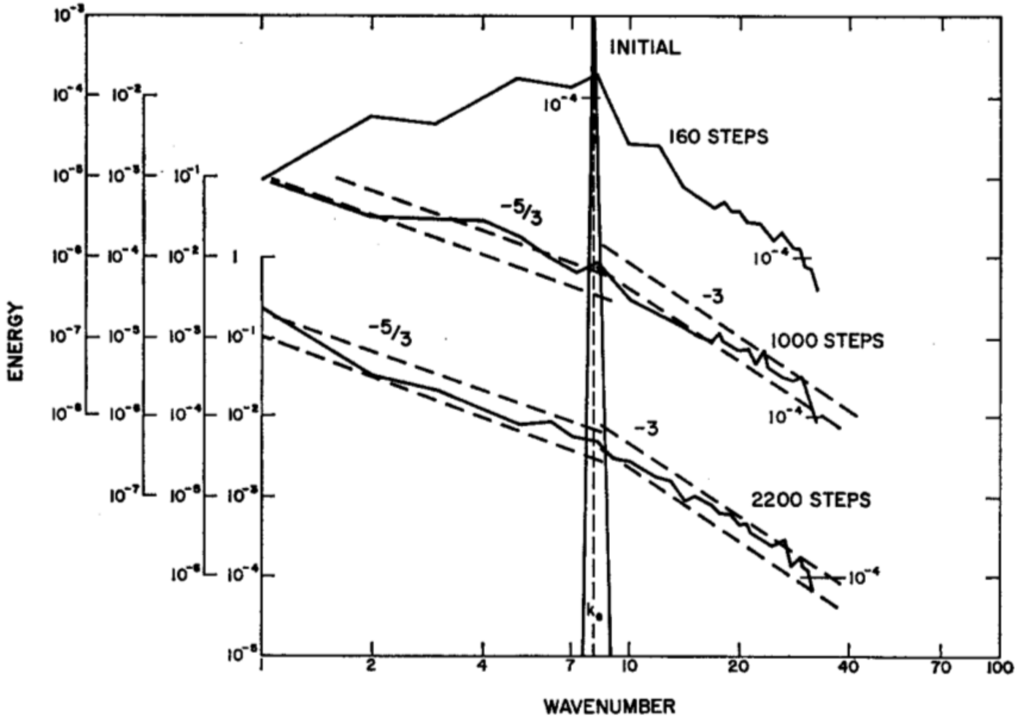



Inverse energy cascade suppressed by friction (or hypo-friction)

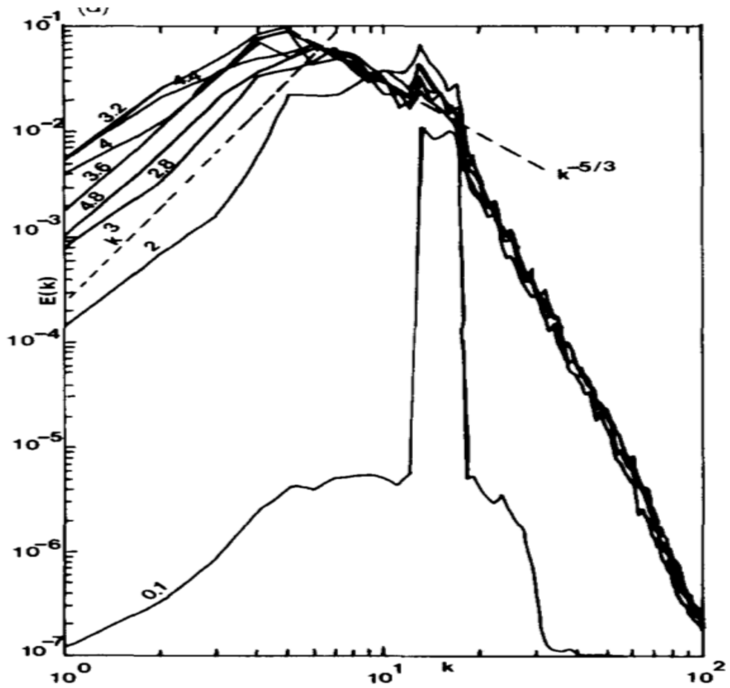
$$-\alpha \boldsymbol{u} \to (-1)^{1+q} \alpha_q \Delta^{-q} \boldsymbol{u}$$

# Inverse cascade: early numerical simulations

Lilly D. (1969) Phys. Fluids 12:II-240–49 $64^2$



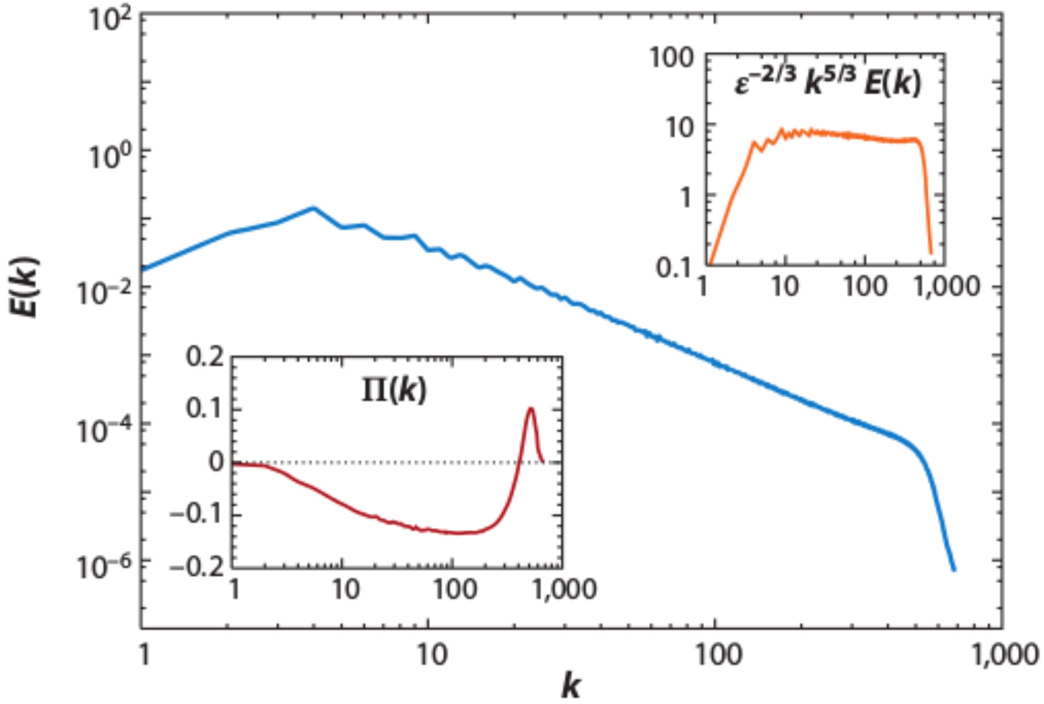

Frisch U, Sulem PL. (1984) Phys.Fluids 27:1921–23 $256^2$




Lilly (1969)
Siggia & Aref (1981)
Frisch & Sulem (1984)
Herring & McWilliams (1985)

# Inverse cascade: numerical simulations

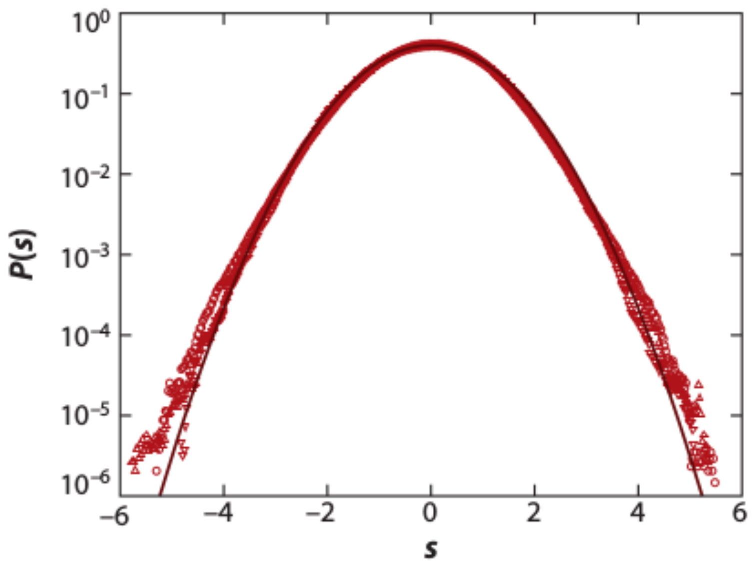
Boffetta G, Celani A, Vergassola M. (2000) Phys. Rev. E 61:R29–32




$2048^2$

Gaussian statistics for velocity fluctuations

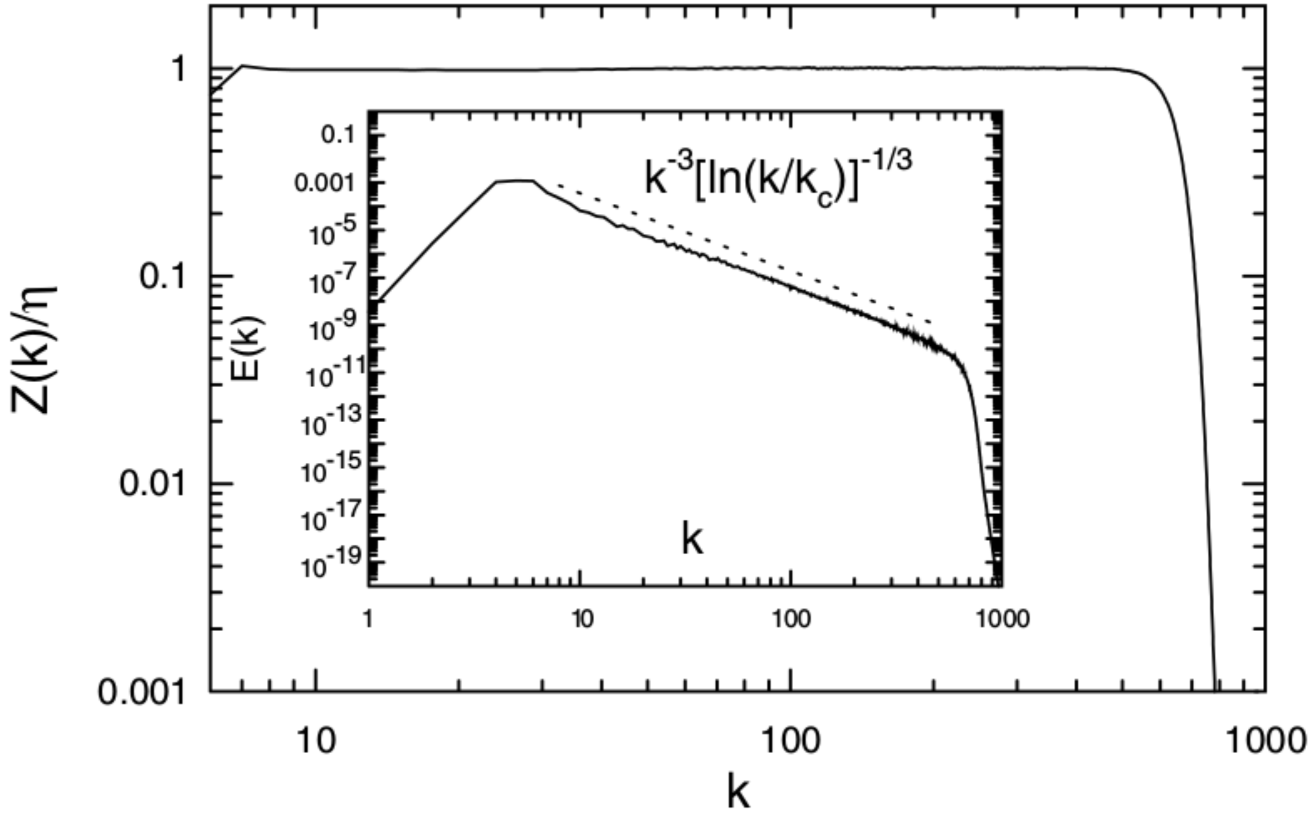
$$\delta u_{\parallel}(r) = [\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})] \cdot \frac{\boldsymbol{r}}{|\boldsymbol{r}|}$$



# Direct Cascade: numerical simulations

Chen S, Ecke R, Eyink G, Wang X, Xiao Z. (2003) Phys. Rev. Lett. 91:214501

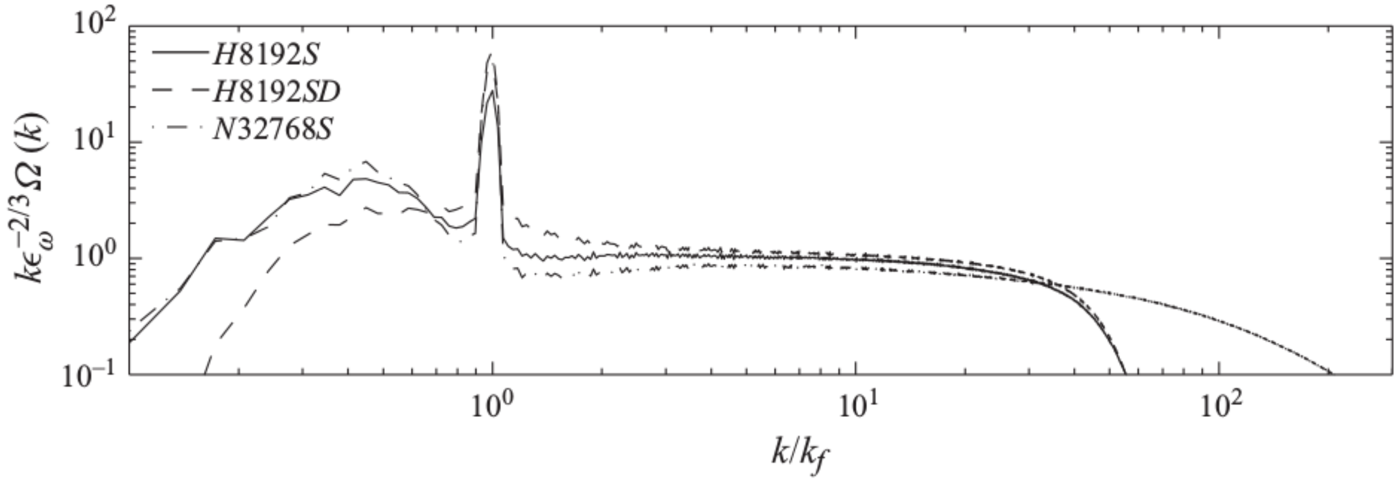
$2048^2$

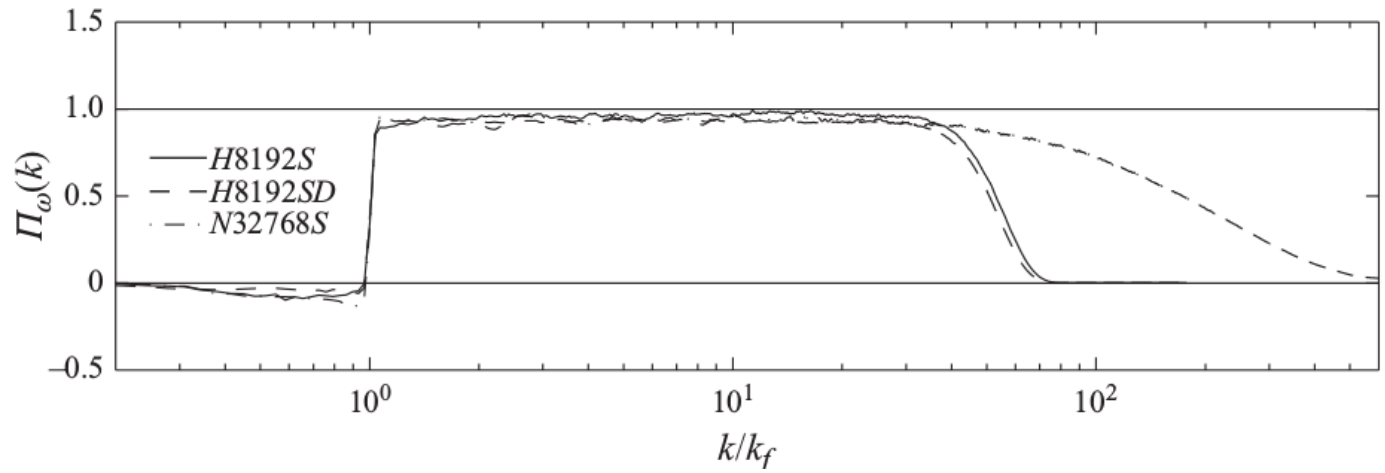


# Direct Cascade: numerical simulations

A Vallgren, E. Lindborg (2011) J. Fluid Mech. 671, 168

$32768^2$



# Double energy - enstrophy cascade

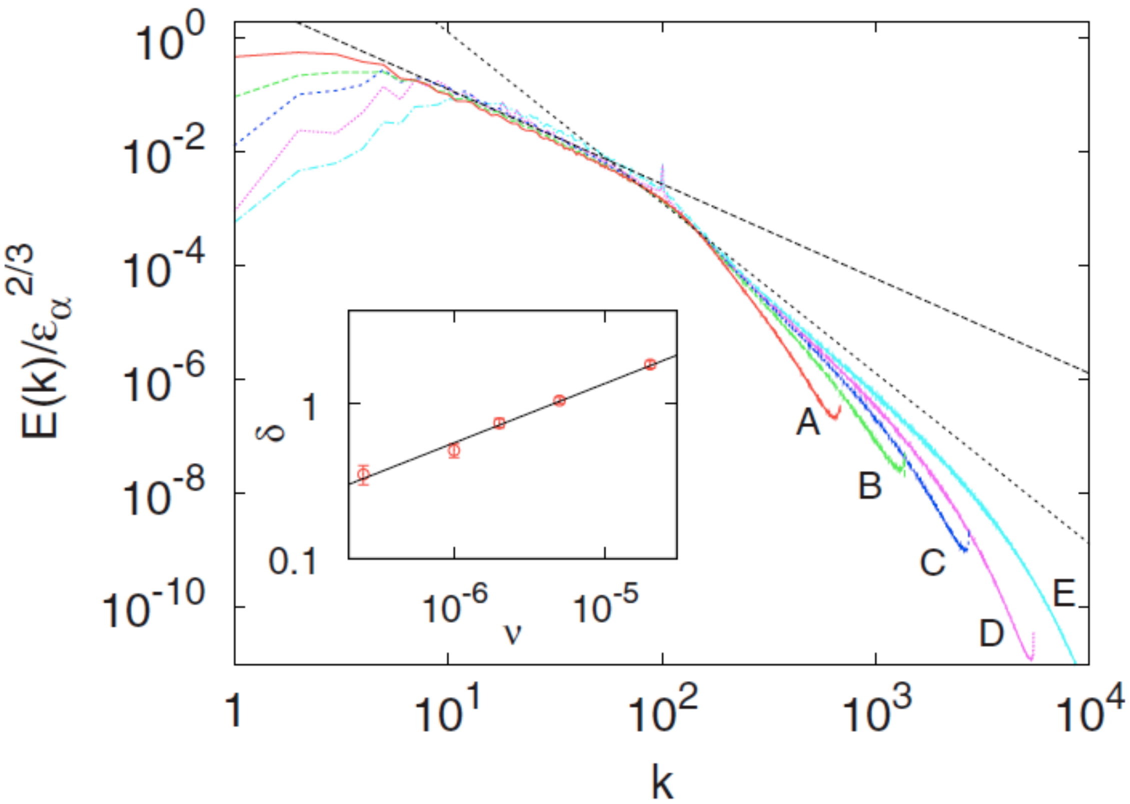
Simultaneous observation of the double cascade scenario
Numerical simulations of 2D NS at resolution 32768 x 32768

Large scale:
Inverse energy cascade

$$E(k) = C\varepsilon^{2/3}k^{-5/3}$$

Small scale:
Direct enstrophy cascade

$$E(k) \sim \eta^{2/3}k^{-(3+\delta)}$$




G. Boffetta, SM, Phys. Rev. E 82, 016307 (2010)

# Third order structure function of 2D turbulence

D. Bernard, Phys. Rev. E 60, 6184 (1999)
V. Yakhot Phys. Rev. E 60, 5544 (1999)

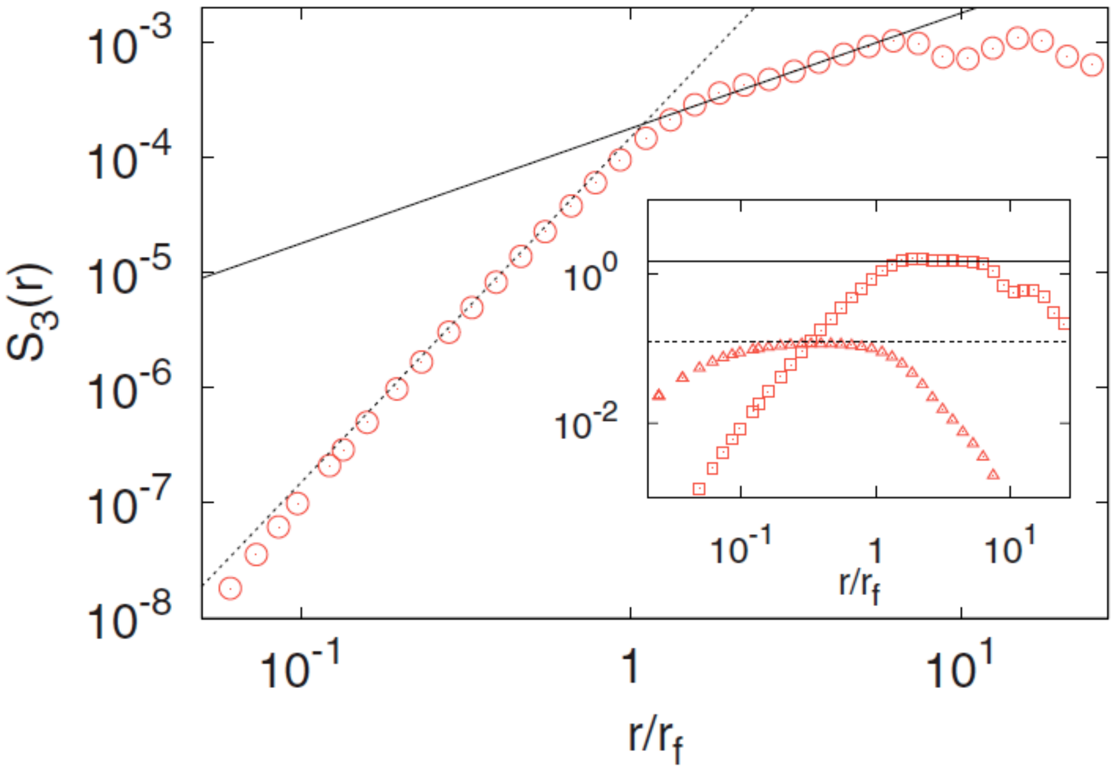
Direct enstrophy cascade

$$S_3(r) = \frac{1}{8}\eta r^3 \qquad r \ll \ell_f$$

Inverse energy cascade

$$S_3(r) = \frac{3}{2}\varepsilon r \qquad r \gg \ell_f$$

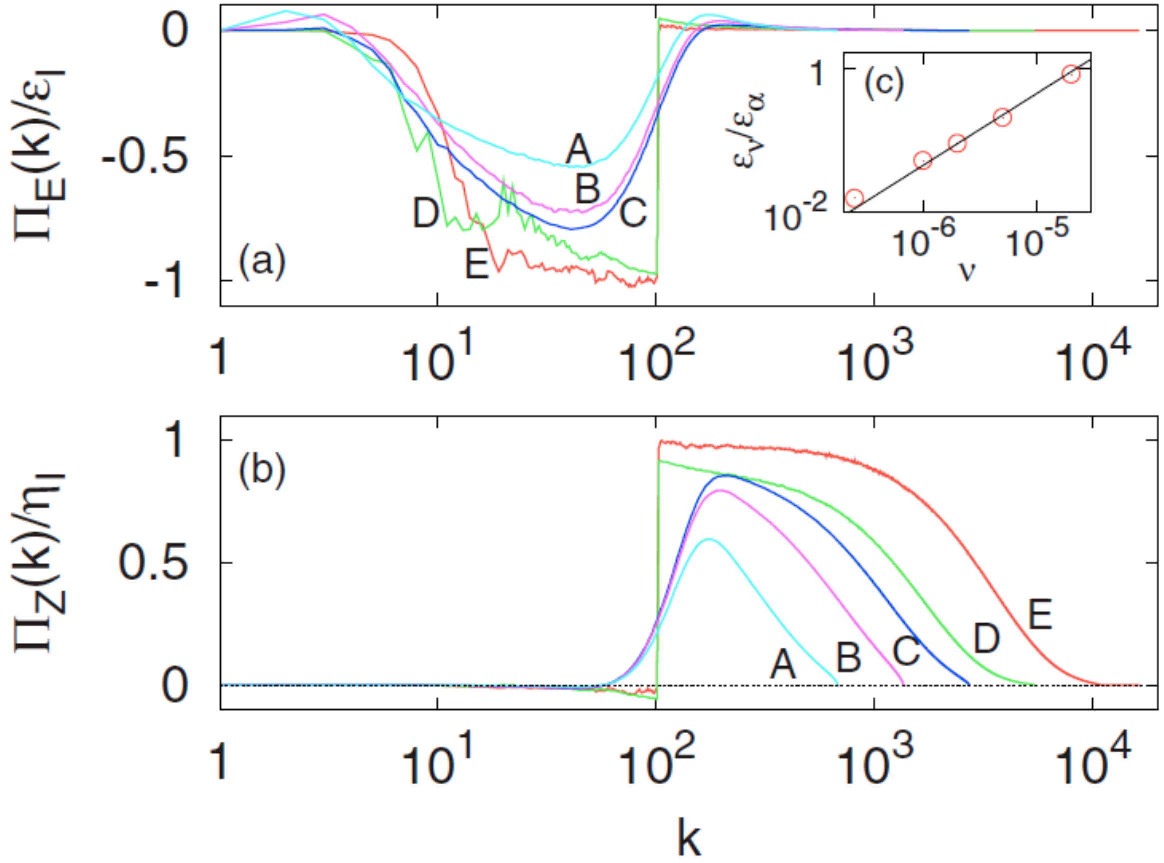
G. Boffetta, SM, Phys. Rev. E 82, 016307 (2010)



# Energy and enstrophy fluxes

G. Boffetta, SM, Phys. Rev. E 82, 016307 (2010)

Inverse cascade:
negative flux of energy




Direct cascade:
positive flux of enstrophy

Prediction for fluxes ratio

G. L. Eyink, Physica D 91, 97-142 (1996)

$$\frac{\varepsilon_\nu}{\varepsilon_\alpha} = \left(\frac{\ell_\nu}{\ell_f}\right)^2 \left(\frac{\ell_f}{\ell_\alpha}\right)^2 \frac{(\ell_\alpha/\ell_f)^2 - 1}{1 - (\ell_\nu/\ell_f)^2} \qquad \frac{\eta_\nu}{\eta_\alpha} = \frac{(\ell_\alpha/\ell_f)^2 - 1}{1 - (\ell_\nu/\ell_f)^2}$$

# 2D TURBULENCE

review experiments

# 2D Turbulence: Experiments

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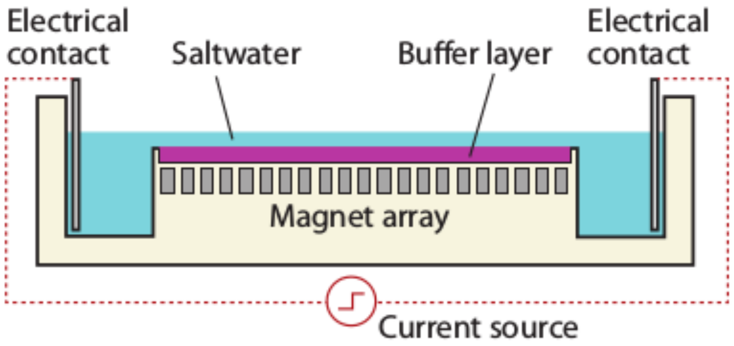


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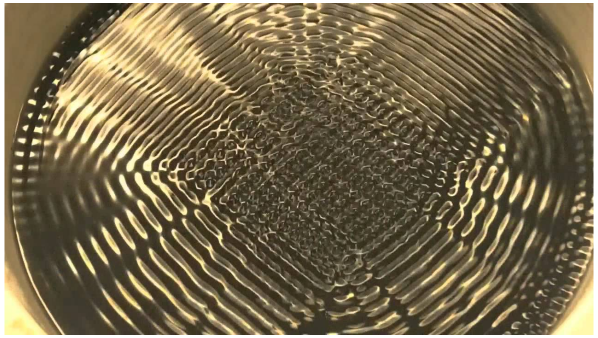

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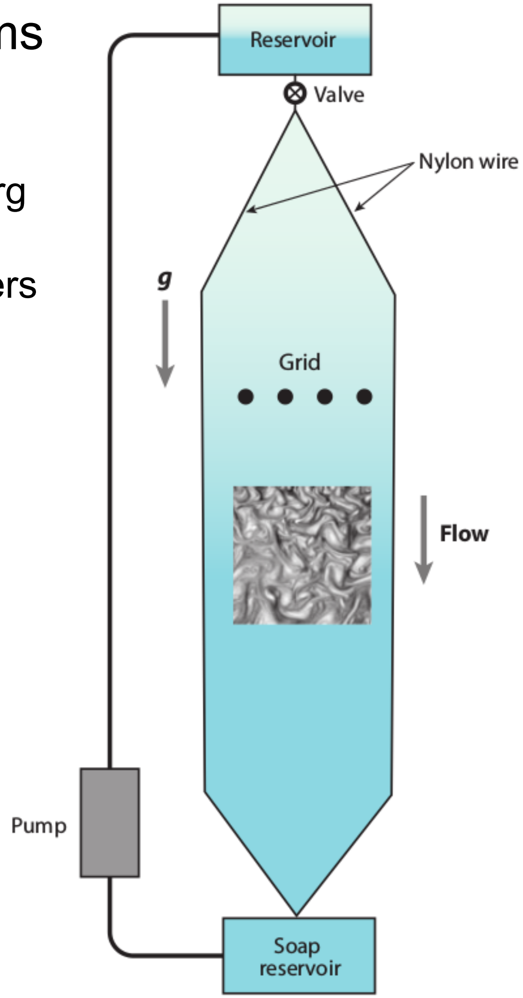


# Inverse cascade: early experiments

Sommeria J. (1986) J. Fluid Mech. 170:139–68

Thin layer of mercury with electrical forcing in a uniform magnetic field

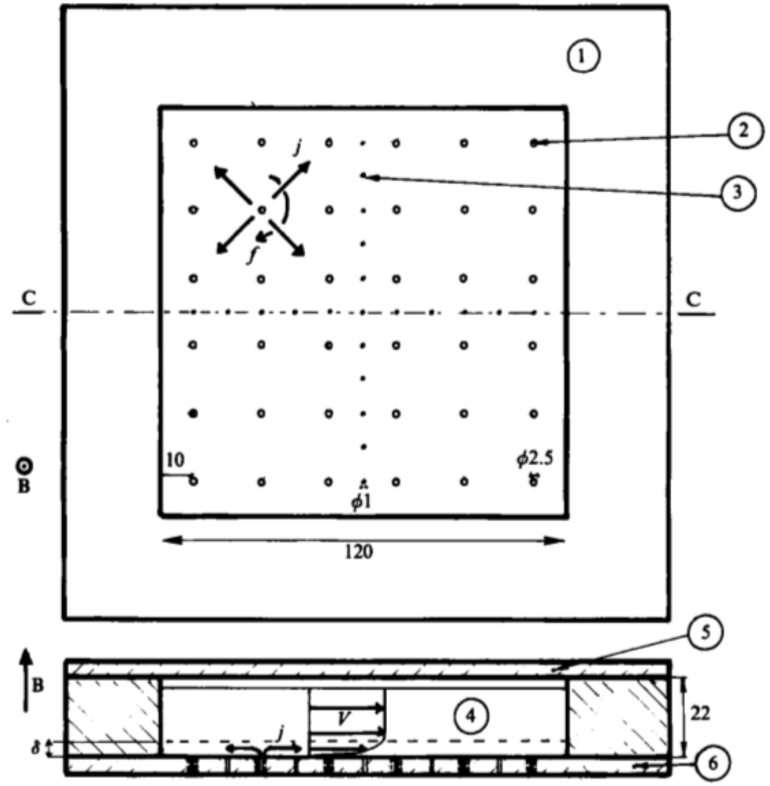
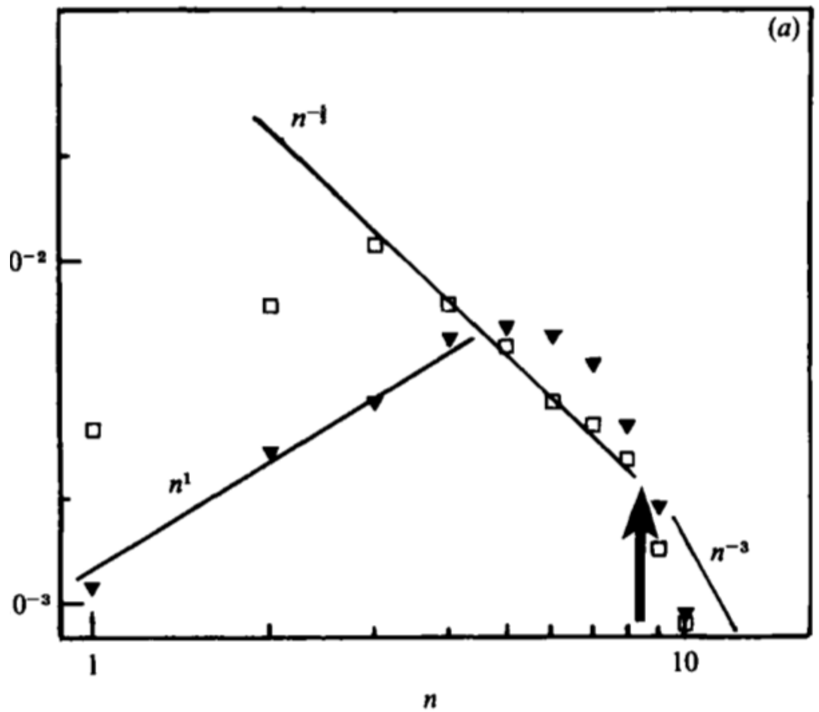



FIGURE 1. The apparatus; the current distribution near one electrode and the velocity profile are schematized. The Hartmann-layer depth is denoted by δ. (1) Copper frame. (2) Electrodes for current injection and electric potential measurements. (3) Electrodes for electric potential measurements only. (4) Mercury. (5) Glass cover. (6) Electrically insulating bottom plate in which electrodes are embedded.

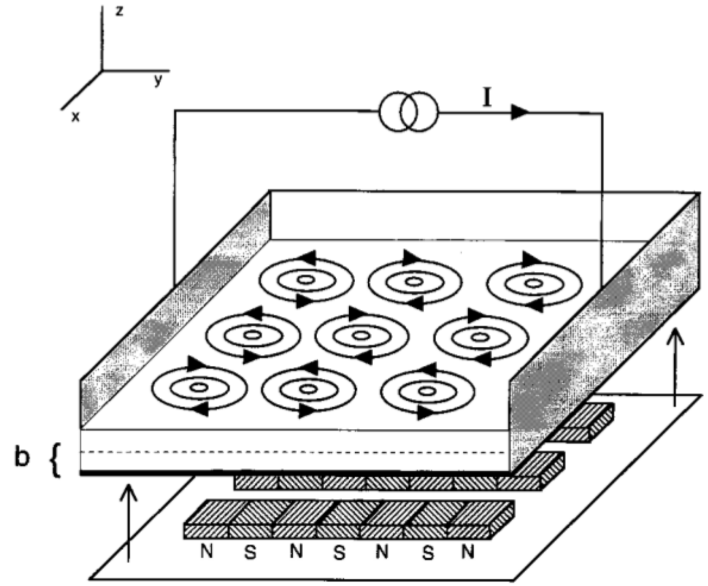
Energy spectrum

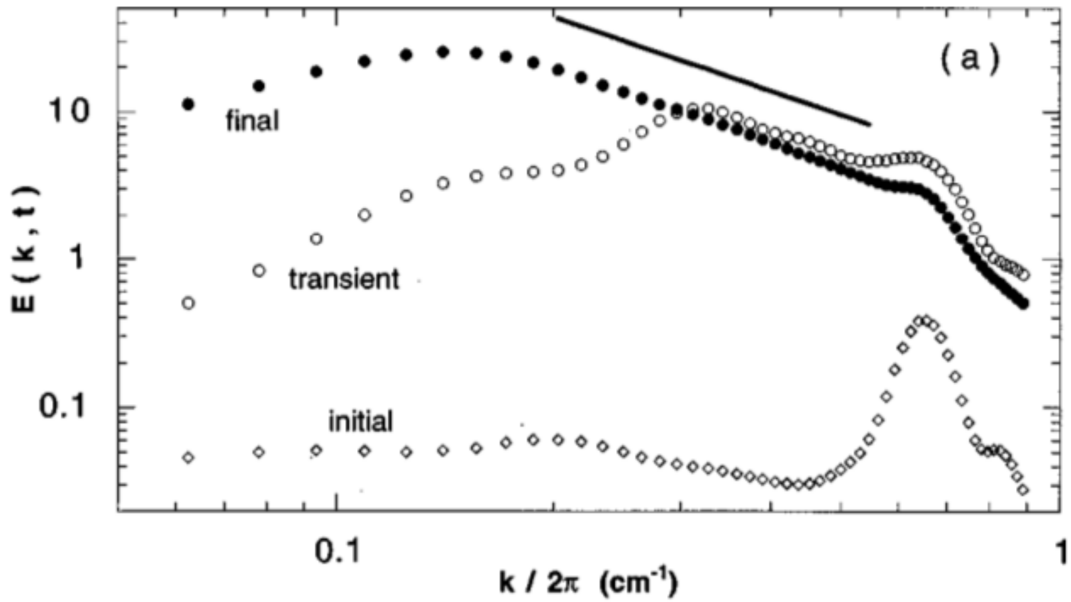


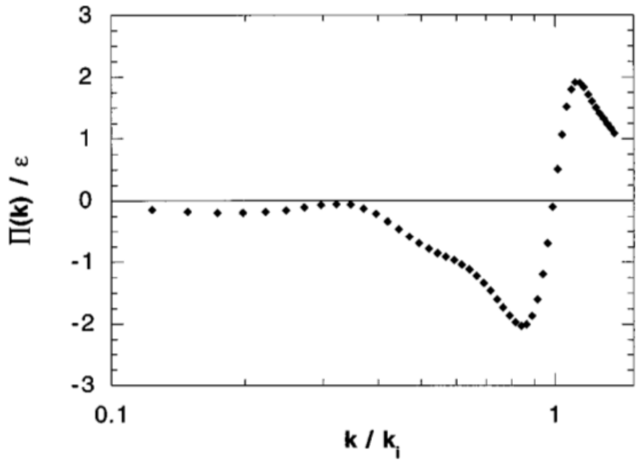
# Inverse cascade: EML experiments

Paret J, Tabeling P. (1997) Phys.Rev. Lett. 79:4162–65

Two layers of NaCl solution stably stratified electromagnetically forced (15x15 cm)



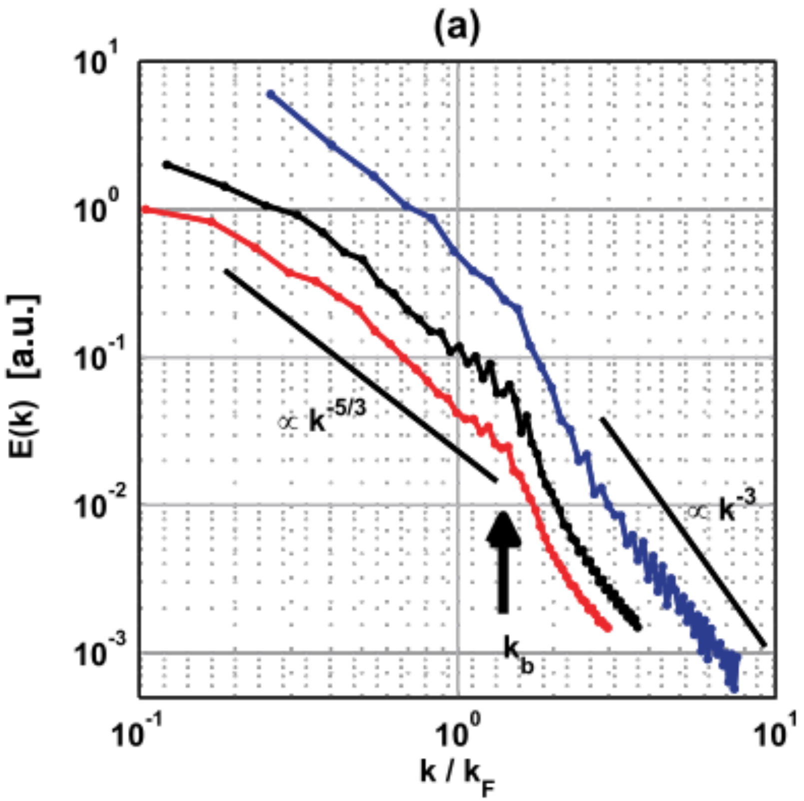



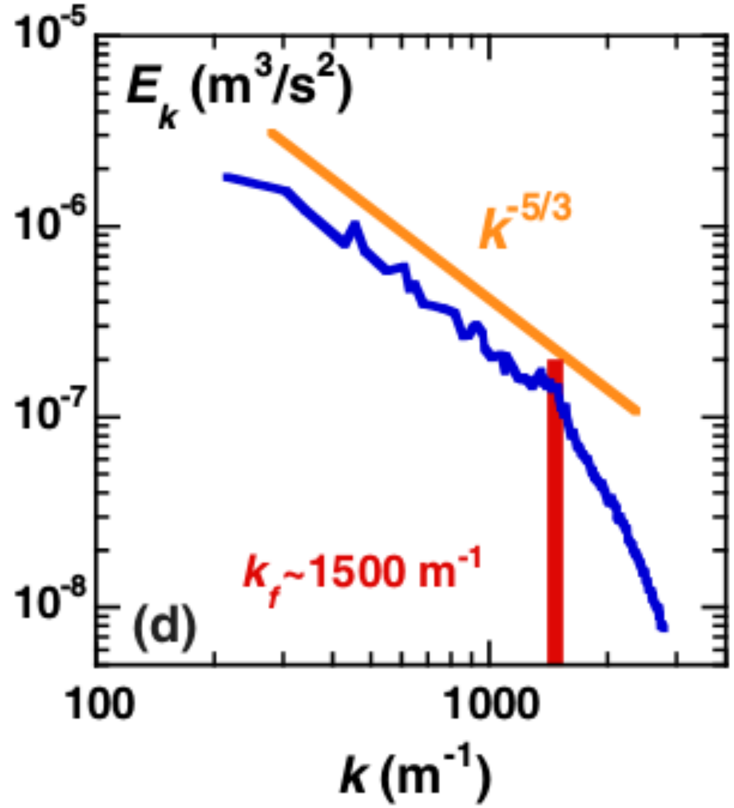



# Inverse cascade Faraday waves experiments

A. von Kameke, F. Huhn, G. Fernandez-Garcia, A.P. Munuzuri, and V. Perez-Munuzuri, Phys. Rev. Lett. 107,074502 (2011)

N. Francois, H. Xia, H. Punzmann, and M. Shats, Phys. Rev. Lett. 110, 194501 (2013)
N. Francois, H. Xia, H. Punzmann, S. Ramsden, and M. Shats, Phys. Rev. X 4, 021021 (2014)



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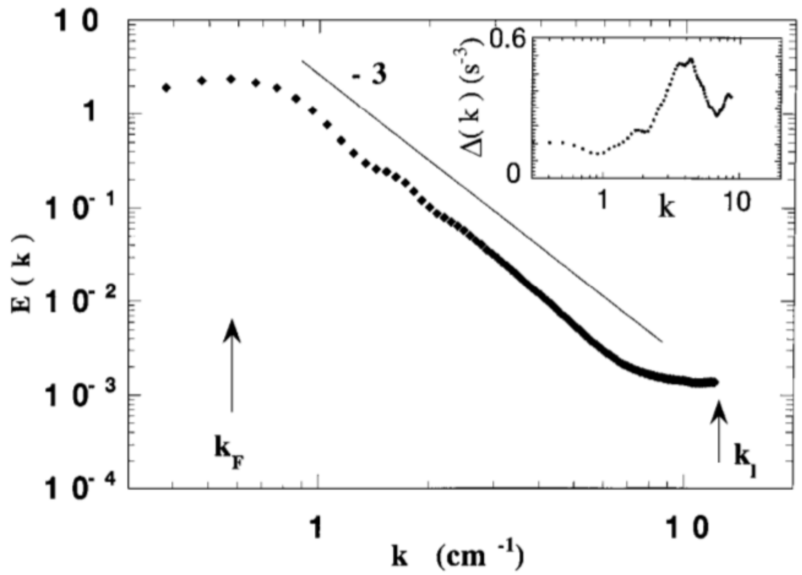
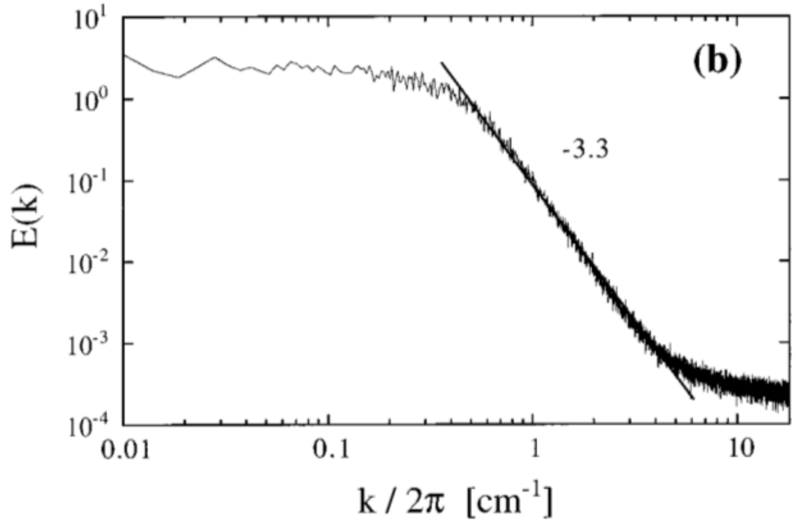
Two layers of NaCl solution stably stratified electromagnetically forced (15x15 cm)

J. Paret, M.C. Jullien, P. Tabeling, Phys.Rev.Lett. 83, 3418 (1999)



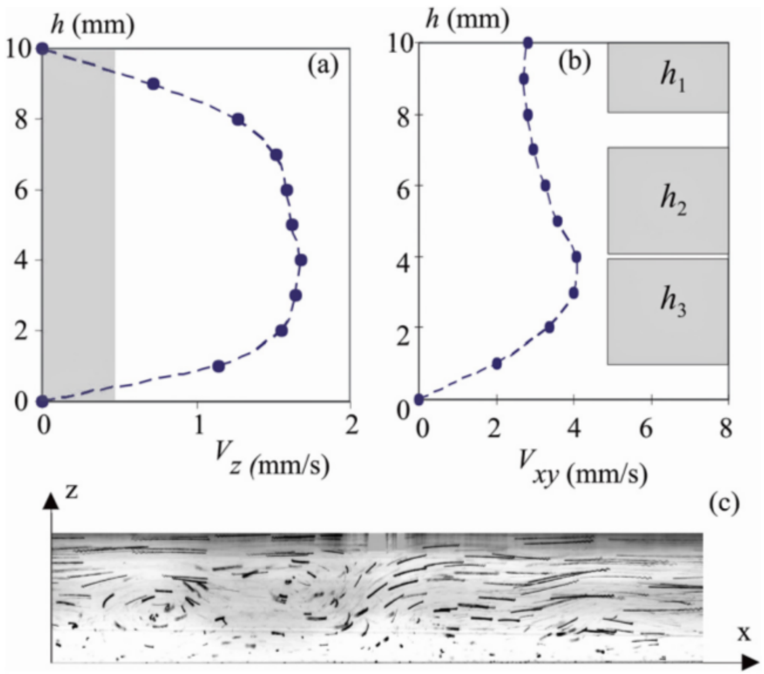

Laser-Doppler velocimetry in a flowing soap-film experiment

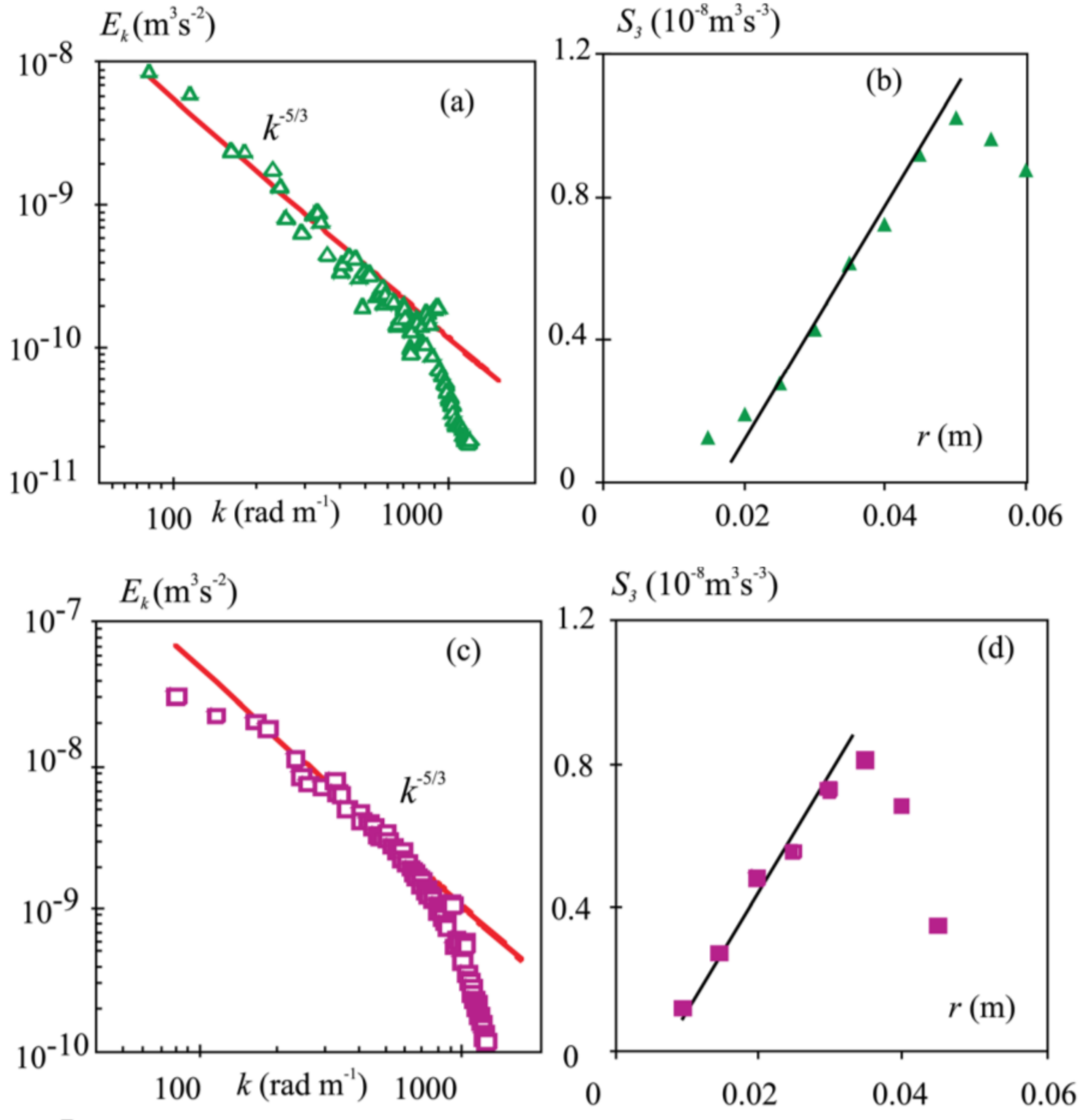
Belmonte A, Goldburg WI, Kellay H, Rutgers MA, Martin B, Wu XL. (1999) Phys. Fluids 11:1196–200



# Experiments of thick fluid layers

Single layer electrolitic cell



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M. Shats, D. Byrne, and H. Xia Phys. Rev. Lett. 105, 264501 (2010)

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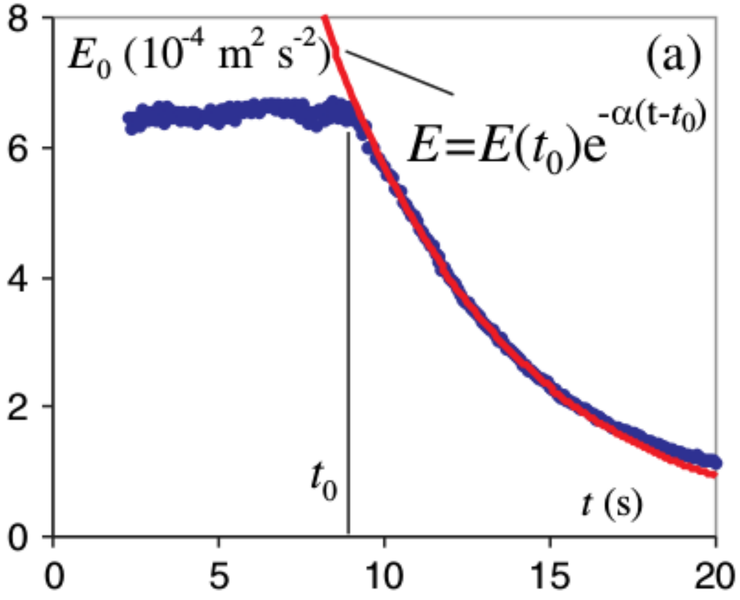
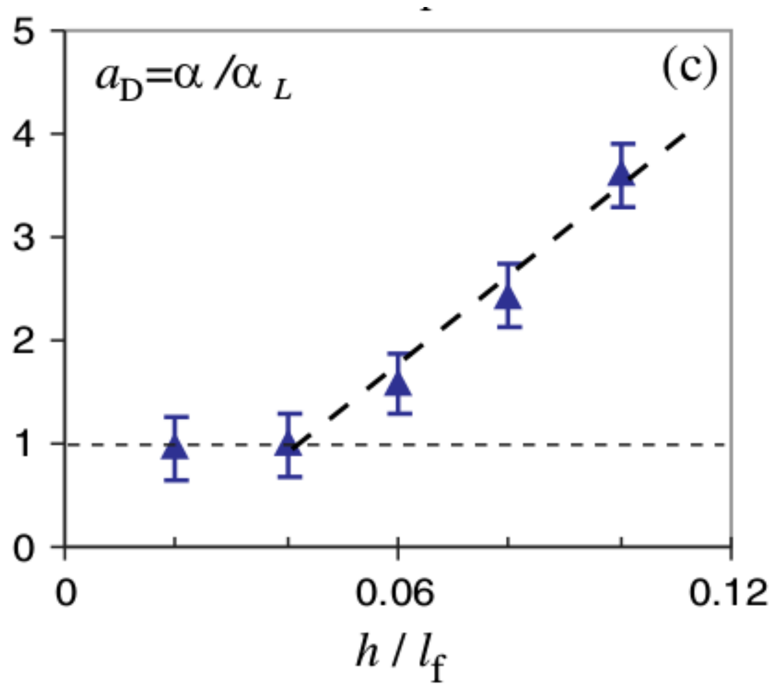


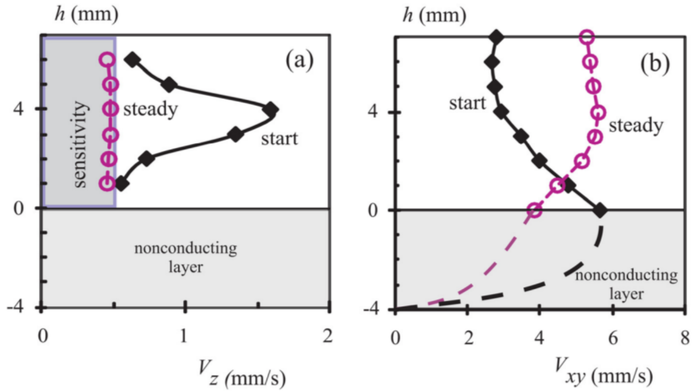



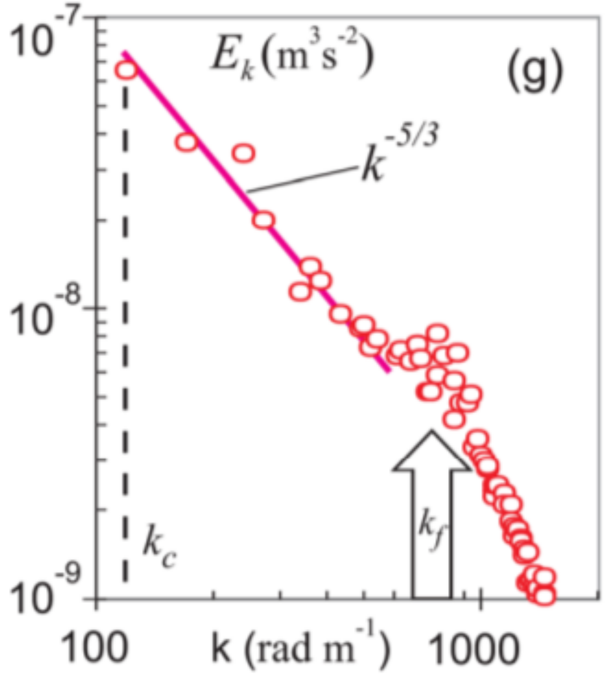
# Thick fluid layers: Experiments

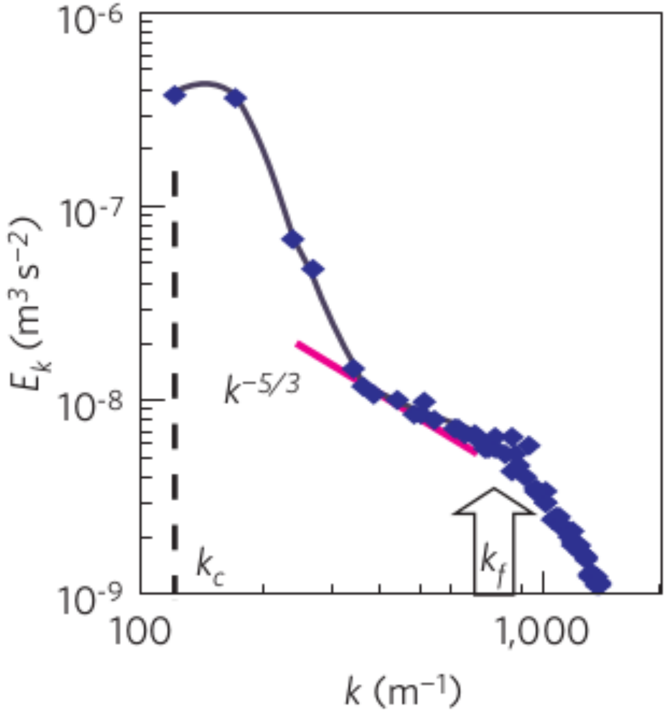
nature physics | LETTERS
PUBLISHED ONLINE: 6 FEBRUARY 2011 | DOI: 10.1038/NPHYS1910

## Upscale energy transfer in thick turbulent fluid layers

H. Xia[1], D. Byrne[1], G. Falkovich[2] and M. Shats[1]*



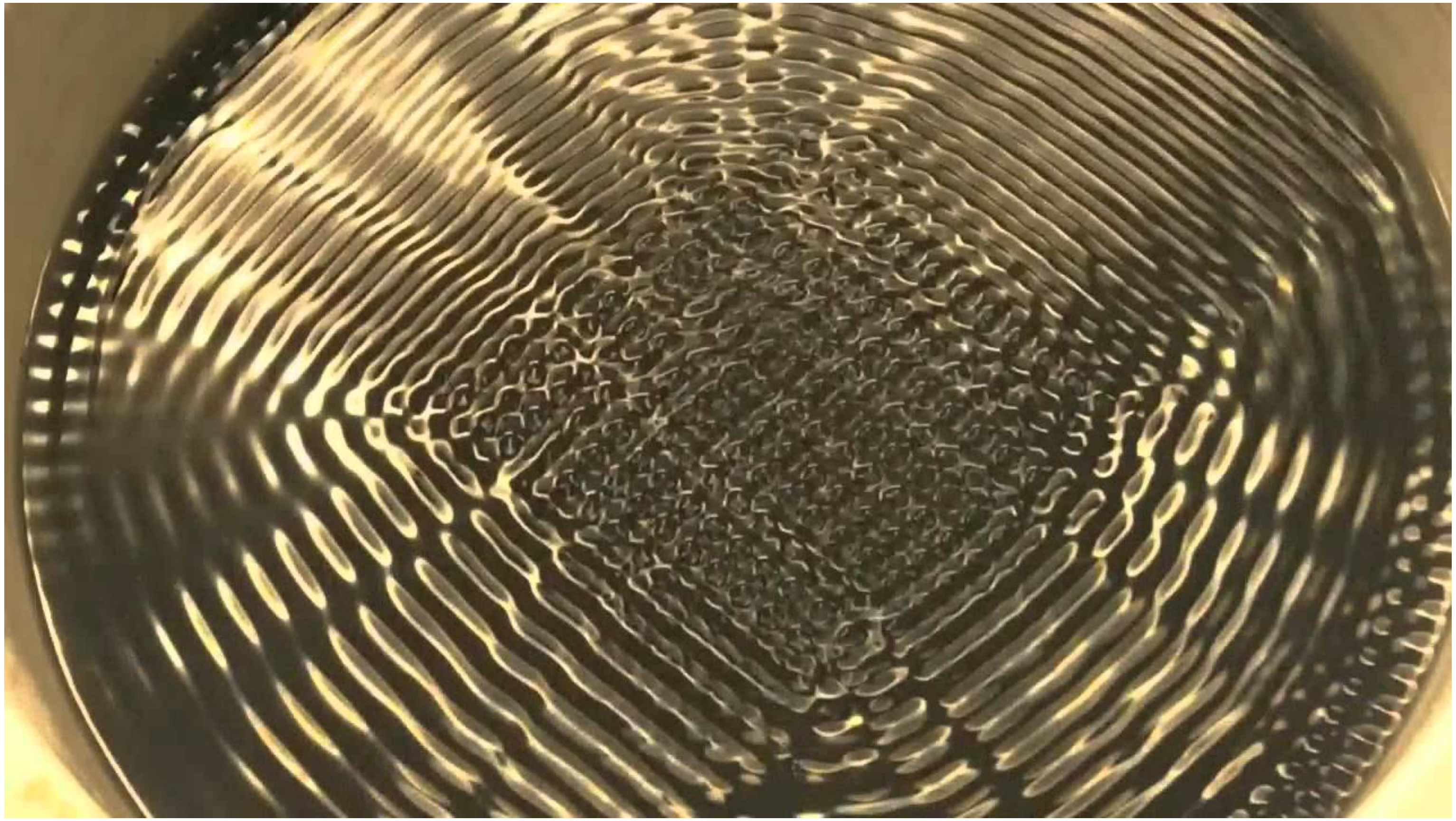

Inverse cascade

Condensate

H. Xia, D. Byrne, G. Falkovich, and M. Shats, Nat. Phys. 7, 321 (2011)
D. Byrne, H. Xia and M. Shats Phys. Fluids, 23, 095109 (2011)

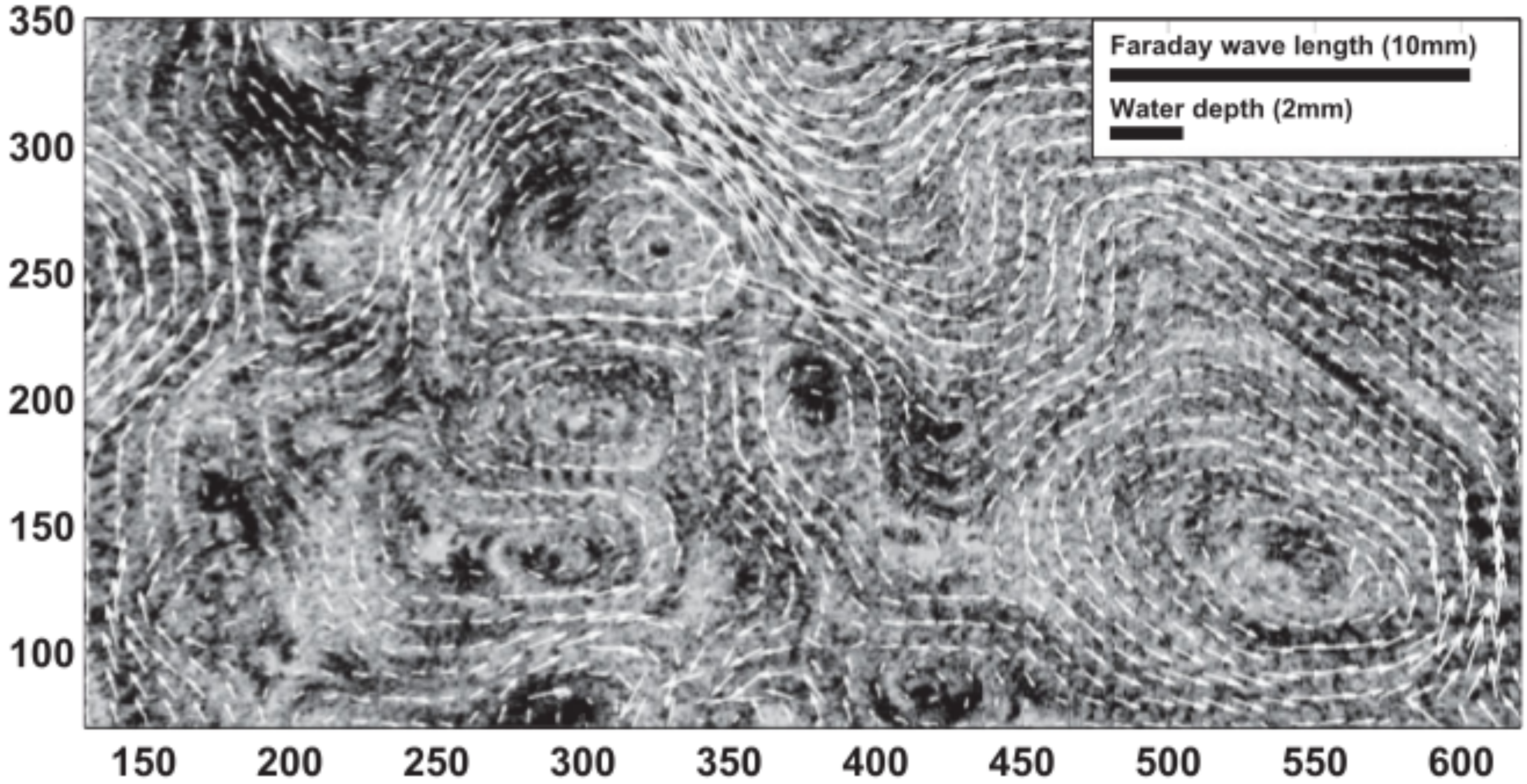
# Faraday waves

Faraday 1831:
Nonlinear standing waves on the surface of a liquid in a vibrating box



# Faraday-waves forced fluid layers

A. von Kameke, F. Huhn, G. Fernandez-Garcia, A.P. Munuzuri, and V. Perez-Munuzuri, Phys. Rev. Lett. 107,074502 (2011).



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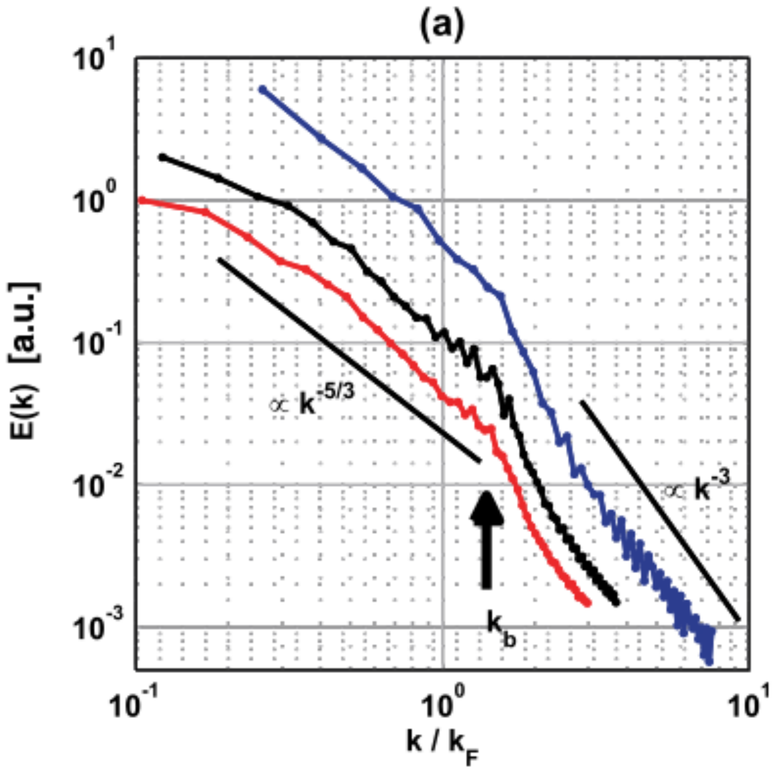


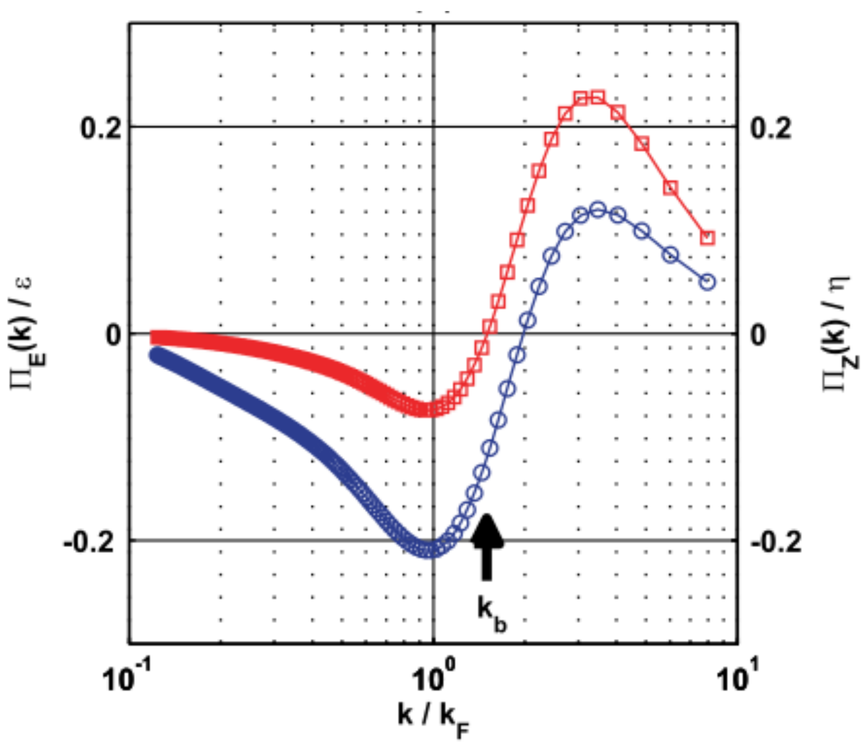


Double energy-enstrophy cascade

# Faraday-waves forced fluid layers

N. Francois, H. Xia, H. Punzmann, and M. Shats,
Phys. Rev. Lett. 110, 194501 (2013)

N. Francois, H. Xia, H. Punzmann, S. Ramsden, and M. Shats,
Phys. Rev. X 4, 021021 (2014)