Vortices and waves in the energy cascade of geophysical turbulence

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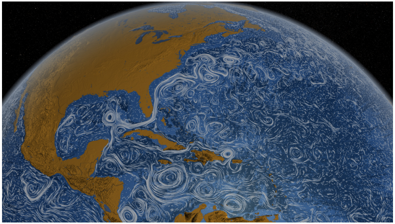
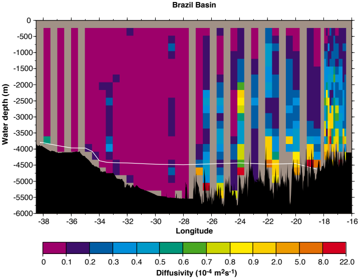
Flowing Matter Across the Scales, Rome

Motivations



NASA Visualization



K. L. Polzin et al. (1997). *Science*

- Geophysical Flows: Vortices and waves coexist and prevail over different scales in the atmosphere and ocean.
- Theoretical: rotation and stratification break Kolmogorov theory. Dimensional analysis does not yield a unique result because of the introduction of new timescales, corresponding to the propagation of internal waves:

$$\tau_{NL} = \frac{L}{U}, \qquad \tau_W = \frac{1}{f}, \frac{1}{N}, \frac{1}{\sigma(\mathbf{k})}$$

Role of vortices and waves in the idealized framework of rotating-stratified turbulence?

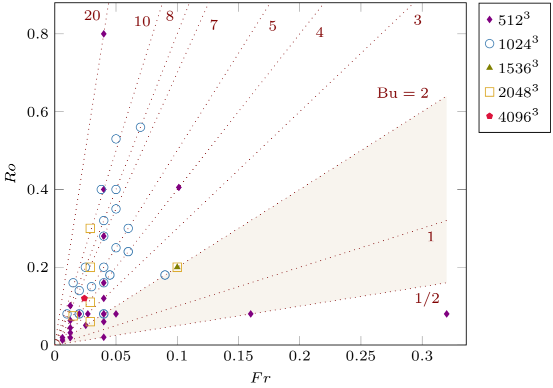
Idealized Rotating-Stratified Turbulence

Boussinesq Equations, Cubic domain, periodic BC:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P - 2\Omega \mathbf{e_z} \times \mathbf{u} - N\theta \mathbf{e_z} + \nu \Delta \mathbf{u} + \mathbf{F},$$

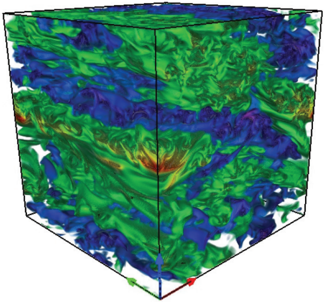
$$\partial \theta + \mathbf{u} \cdot \nabla \theta = N u_z + \kappa \Delta \theta,$$

$$\nabla \cdot \mathbf{u} = 0.$$



DNS Database (R. Marino, D. Rosenberg)

DNS[1] (512^3, Re $\sim 10^4$), θ:



Fr $= 0.1$, Ro $= 0.4$

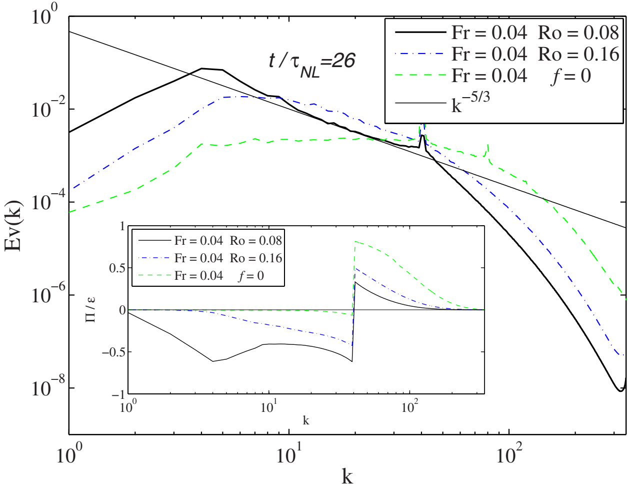
Non-dimensional numbers

- Stratification: $\mathrm{Fr} = \frac{U}{NL}$
- Rotation: $\mathrm{Ro} = \frac{U}{fL}$, $\quad (f = 2\Omega)$

[1] R. Marino et al. (2013a). *Phys. Rev. E*

Idealized Rotating-Stratified Turbulence

Kinetic energy spectrum and fluxes, DNS (1024^3, Re $\approx 10^3$, $k_f = 40$) of stratified flows with or without rotation[1]:



Transition from upscale to downscale energy cascade as rotation weakens (Ro increases).

[1] R. Marino et al. (2013b). *Europhys. Lett.*

Normal modes of the linearized equations

Linearized Boussinesq dynamics in Fourier space[2]:

$$\dot{\mathbf{X}}(\mathbf{k}) = \mathbf{L}(\mathbf{k})\mathbf{X}(\mathbf{k}), \quad \text{with } \mathbf{X}(\mathbf{k}) = \left(\hat{u}_x(\mathbf{k}), \hat{u}_y(\mathbf{k}), \hat{u}_z(\mathbf{k}), \hat{\theta}(\mathbf{k})\right)^T$$

$\mathrm{Sp}\,\mathbf{L}(\mathbf{k}) = \{0, i\sigma(\mathbf{k}), -i\sigma(\mathbf{k})\}$, with $\sigma(\mathbf{k}) = k^{-1}\sqrt{f^2 k_\parallel^2 + N^2 k_\perp^2}$.

Eigenmodes

- Two *inertia-gravity wave modes* $\mathbf{X}_\pm(\mathbf{k})$:

$$\mathbf{L}(\mathbf{k})\mathbf{X}_\pm(\mathbf{k}) = \pm i\sigma(\mathbf{k})\mathbf{X}_\pm(\mathbf{k}).$$

- One *slow mode* $\mathbf{X}_0(\mathbf{k})$ with zero linear frequency:

$$\mathbf{L}(\mathbf{k})\mathbf{X}_0(\mathbf{k}) = 0.$$

Orthonormal basis:

$$\mathbf{X}(\mathbf{k}) = A_0(\mathbf{k})\mathbf{X}_0(\mathbf{k}) + A_-(\mathbf{k})\mathbf{X}_-(\mathbf{k}) + A_+(\mathbf{k})\mathbf{X}_+(\mathbf{k}),$$

$$\mathbf{X}_r(\mathbf{k})^\dagger \mathbf{X}_s(\mathbf{k}) = \delta_{rs}.$$

[2] C. E. Leith (1980). *J. Atmos. Sci.* P. Bartello (1995). *J. Atmos. Sci.*

Properties of the normal modes

Slow modes and balanced motion

- For rotating-stratified flows: The slow modes are in *hydrostatic balance*:
  $\partial_z P = -\rho g$,
  and *geostrophic balance*: $\nabla_\perp P = -2\boldsymbol{\Omega} \times \mathbf{u}$.
- For stratified flows, the slow modes are *not in hydrostatic balance*, unless $k_\perp = 0$ (*vertically sheared horizontal flow (VSHF)* modes).

Slow modes and potential enstrophy

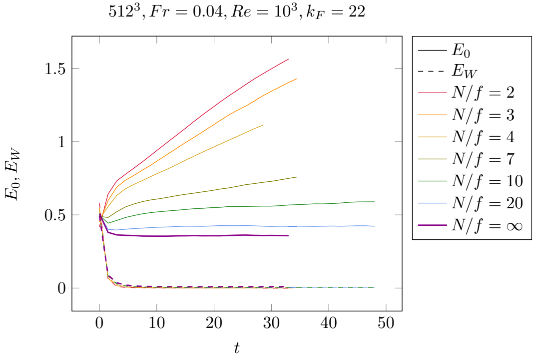
Potential vorticity $\Pi = f\partial_z\theta - N\omega_z + \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\theta$ is a Lagrangian invariant: $\partial_t\Pi + \mathbf{u}\cdot\boldsymbol{\nabla}\Pi = 0$.
Potential enstrophy $\int \Pi^2$ is a global invariant. Quadratic part Γ_2:

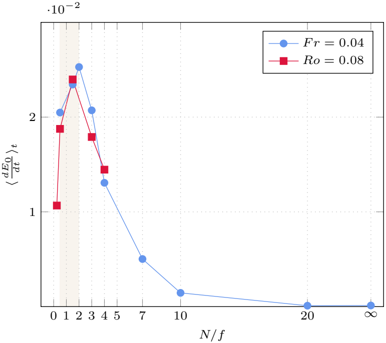
$$\Gamma_2 = \frac{1}{2}\int (f\partial_z\theta - N\omega_z)^2 = \frac{1}{2}\sum_{\mathbf{k}\in\mathcal{B}} k^2\sigma(\mathbf{k})^2|A_0(\mathbf{k})|^2$$

For stratified flows, the only modes which carry PV have $k_\perp \neq 0$.

- 2D: Enstrophy is positive definite.
- 3D HIT: Helicity is not sign definite.
- Here: Potential enstrophy is positive, but degenerate.

Wave and vortices: global analysis



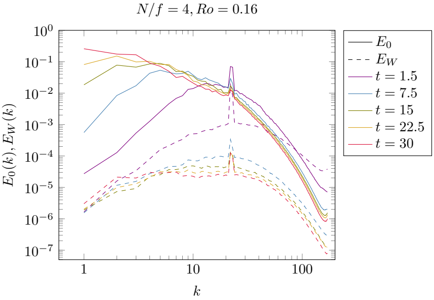


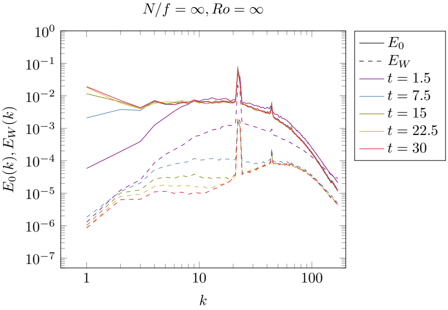
$$E_0 = \frac{1}{2} \sum_{\mathbf{k} \in \mathcal{B}} |A_0(\mathbf{k})|^2, \qquad E_W = \frac{1}{2} \sum_{\mathbf{k} \in \mathcal{B}} [|A_+(\mathbf{k})|^2 + |A_-(\mathbf{k})|^2], \qquad E = E_0 + E_W.$$

- Global energy is dominated by slow modes, regardless of N/f.
- Inverse cascade only when N/f small enough.
- Mostly kinetic energy (not shown)

Wave and vortices: spectral analysis

512^3, $Fr = 0.04$, $Re = 10^3$, $k_F = 22$





- Inverse cascade of the slow modes.
- Waves dominate at small scales before turbulence develops, then decay rapidly.

- No inverse cascade of the slow modes.
- Wave spectrum dominated by small scales.
- The larger N/f, the larger $E_W(k)/E_0(k)$ at small scales.

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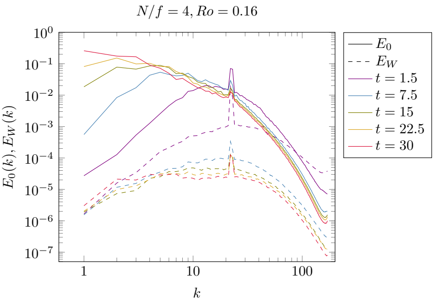

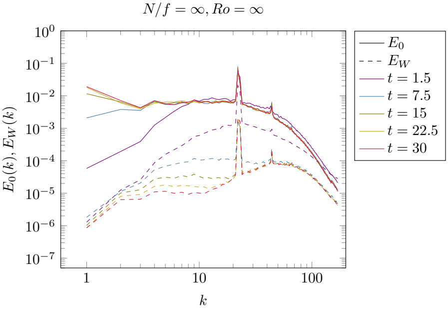

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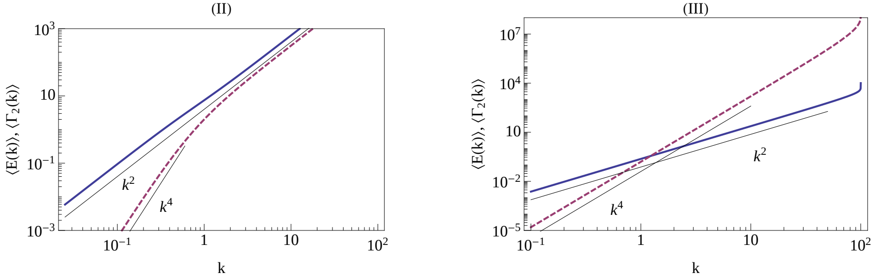
Absolute Equilibrium

Introduce canonical probability measure based on the invariants of the system[3]:
$\rho = \mathcal{Z}^{-1} \exp(-\beta E - \alpha \Gamma_2)$

- 2D Turbulence: negative temperature β, infrared divergence, inverse cascade.
- 3D Turbulence: only positive temperatures β, ultraviolet divergence, direct cascade.

Rotating-Stratified flows at absolute equilibrium[4]:

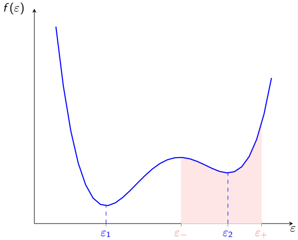


- $\beta > 0$, energy equipartition (ultraviolet catastrophe), like 3D turbulence, which points at a *downscale cascade of energy*.
- In fact, the dynamics remains in the vicinity of a *slow manifold*.

[3] T. D. Lee (1952). *Q. Appl. Math.* R. H. Kraichnan (1967). *Phys. Fluids*

[4] P. Bartello (1995). *J. Atmos. Sci.* see M. L. Waite and P. Bartello (2004). *J. Fluid Mech.* for the purely stratified case.

Restricted partition function



Metastable states (local minima of the free energy $f(\varepsilon)$): restrict the integral defining the partition function to a subset Λ' of phase space[5].

Absolute equilibrium:

$$\begin{aligned}\mathcal{Z}(\beta) &= \int_\Lambda e^{-\beta N h(x)} \mu(dx), \\ &= \int_0^{+\infty} e^{-\beta N \varepsilon} \Omega(\varepsilon) d\varepsilon, \\ &\sim e^{-N\phi(\beta)},\end{aligned}$$

$$\phi(\beta) = \min_{\varepsilon \in \mathbb{R}_+} (\beta\varepsilon - s(\varepsilon)) = \beta\varepsilon_1 - s(\varepsilon_1).$$

Restricted equilibrium:

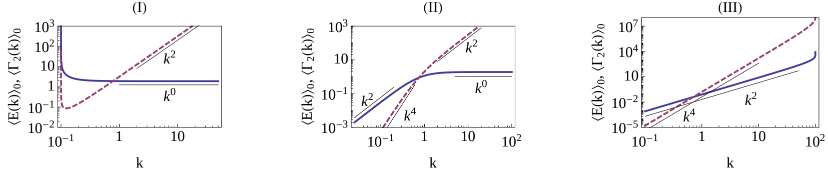
$$\begin{aligned}\mathcal{Z}'(\beta) &= \int_{\Lambda'} e^{-\beta N h(x)} \mu(dx), \\ &= \int_{\varepsilon_-}^{\varepsilon_+} e^{-\beta N \varepsilon} \Omega(\varepsilon) d\varepsilon, \\ &\sim e^{-N\phi'(\beta)},\end{aligned}$$

$$\phi'(\beta) = \min_{\varepsilon \in [\varepsilon_-, \varepsilon_+]} (\beta\varepsilon - s(\varepsilon)) = \beta\varepsilon_2 - s(\varepsilon_2).$$

[5] O. Penrose and J. L. Lebowitz (1971). *J. Stat. Phys.* O. Penrose and J. L. Lebowitz (1979). In: *Fluctuation Phenomena*. Ed. by E. W. Montroll and J. L. Lebowitz. Amsterdam: North-Holland

Restricted partition function[6]

- Rotating-Stratified flows at restricted equilibrium (slow manifold only)
  Convergence condition: $\beta + \alpha(f^2 k_\parallel^2 + N^2 k_\perp^2) > 0$.



$\beta < 0$ regime (I): infrared divergence of the restricted equilibrium energy spectrum, like in 2D turbulence, which points at the existence of an inverse cascade.

[6] C. Herbert et al. (2014). *J. Fluid Mech.*

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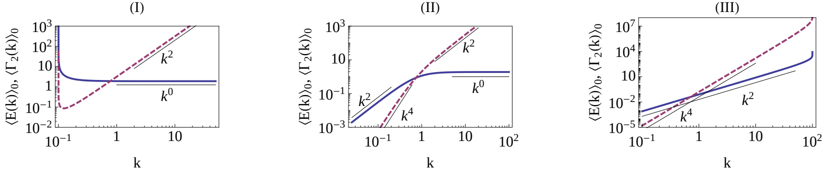


  $\beta < 0$ regime (I): infrared divergence of the restricted equilibrium energy spectrum, like in 2D turbulence, which points at the existence of an inverse cascade.

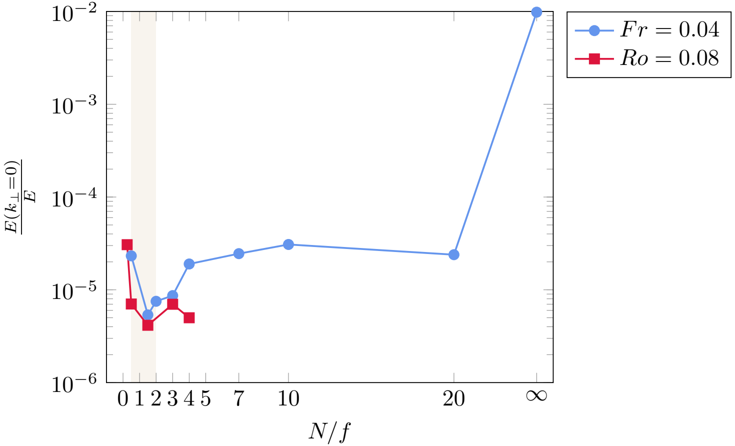
- Purely stratified flows at restricted equilibrium
  Convergence condition: $\beta + \alpha N^2 k_\perp^2 > 0$.
  $\beta > 0$ (regimes (II) and (III)): forward energy cascade. Because of the VSHF modes ($k_\perp = 0$).

[6] C. Herbert et al. (2014). *J. Fluid Mech.*

VSHF modes in the DNS

Vertically sheared horizontal flows: inertial waves ($\sigma(\mathbf{k}) = f$), $u_z = 0$. In the stratified case they become slow modes: characteristic time = eddy turnover time.

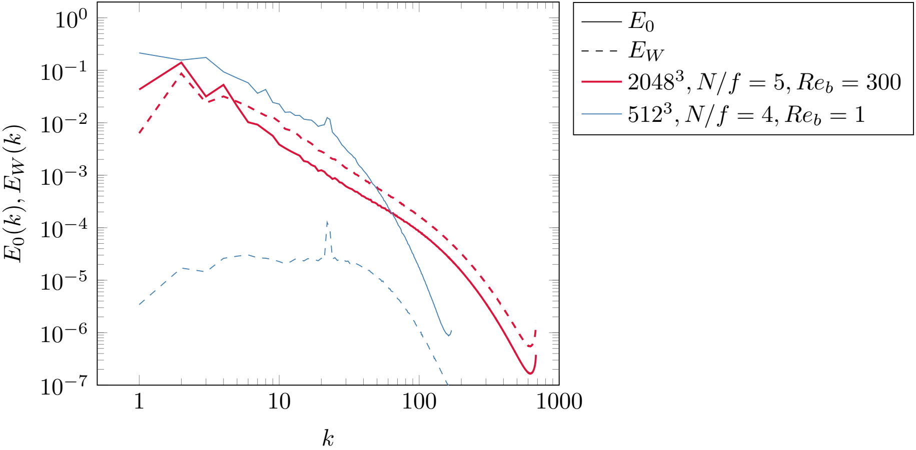
512^3, $Re = 10^3$, $k_F = 22$



In the stratified case, some energy is transferred to the large horizontal scales, but not by a cascade process.

Role of the buoyancy Reynolds number

All the runs above have $Re_b = ReFr^2 \sim 1$.
In the inverse cascade regime, Re_b is necessarily limited.



Summary

What is the role of vortices and waves in the energy cascade of rotating-stratified turbulence?

Conclusions

- Vortical modes dominate our DNS at low Re_b.
- They undergo an inverse cascade when rotation is strong enough.
- Statistical Mechanics in the restricted ensemble supports the idea that the inverse cascade is due to the slow modes.
- It also explains why there is not inverse cascade in the absence of rotation and points to the role of vertically sheared horizontal modes.

Degenerate inviscid invariant:

- Inverse cascade when there is timescale separation.
- Direct cascade otherwise.

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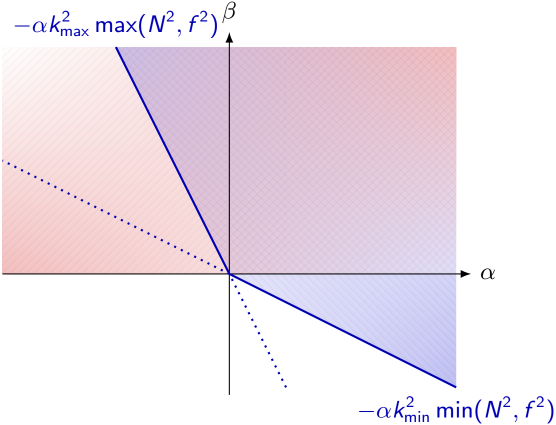

Accessible thermodynamic space for rotating-stratified flows, waves (red) and slow manifold (blue).