# A depth-averaged model for droplets in thin microfluidic channels

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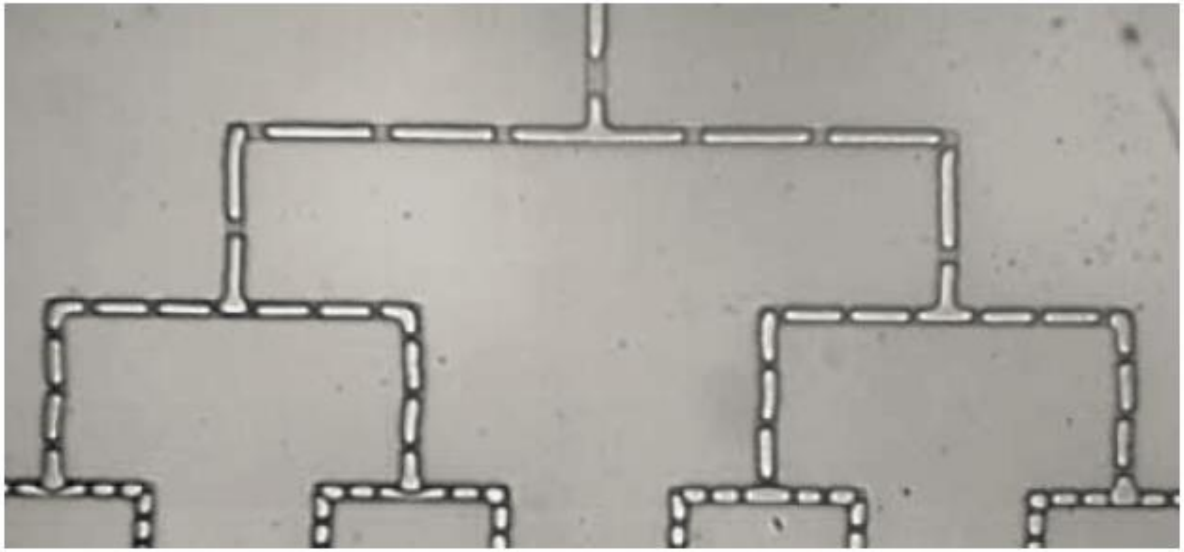
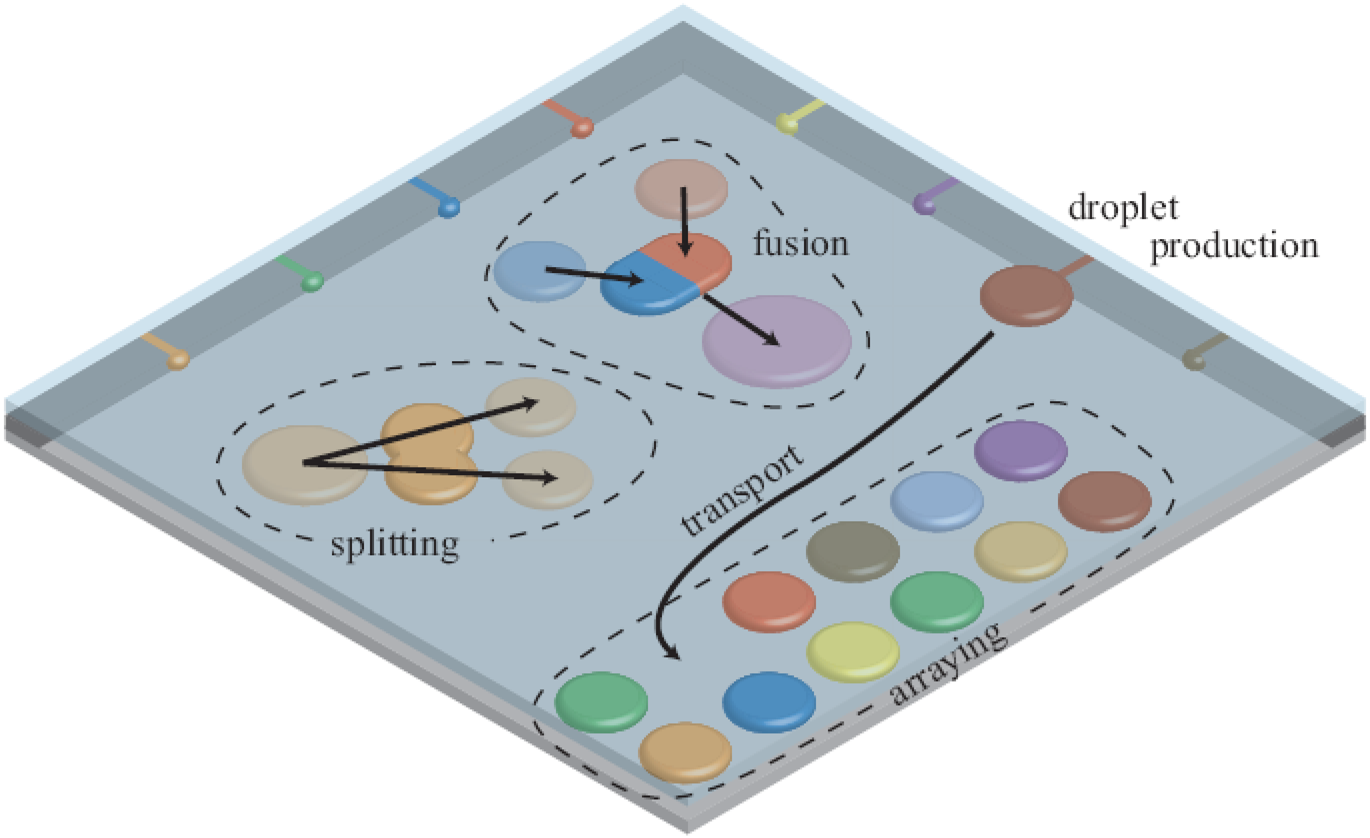
# Droplet microfluidics



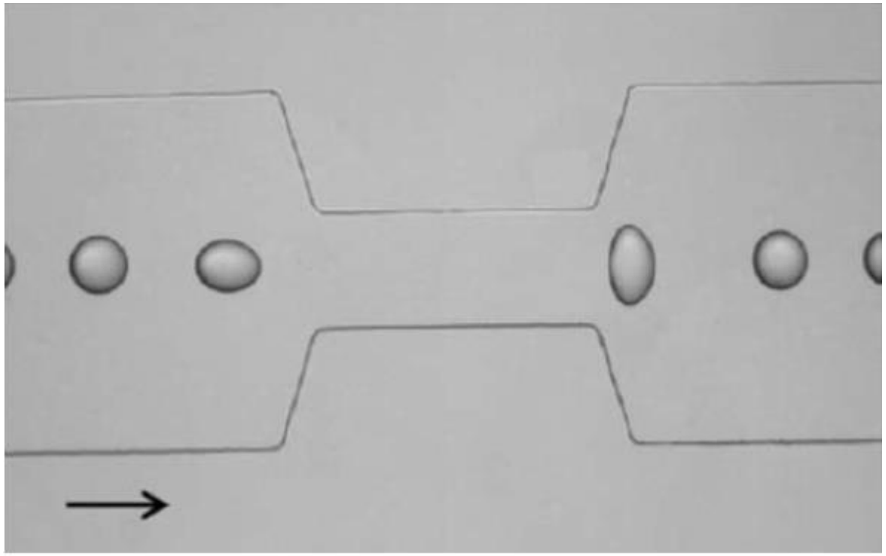
Link *et al.*, Phys. Rev. Lett. (2004)

# 2D Droplet microfluidics

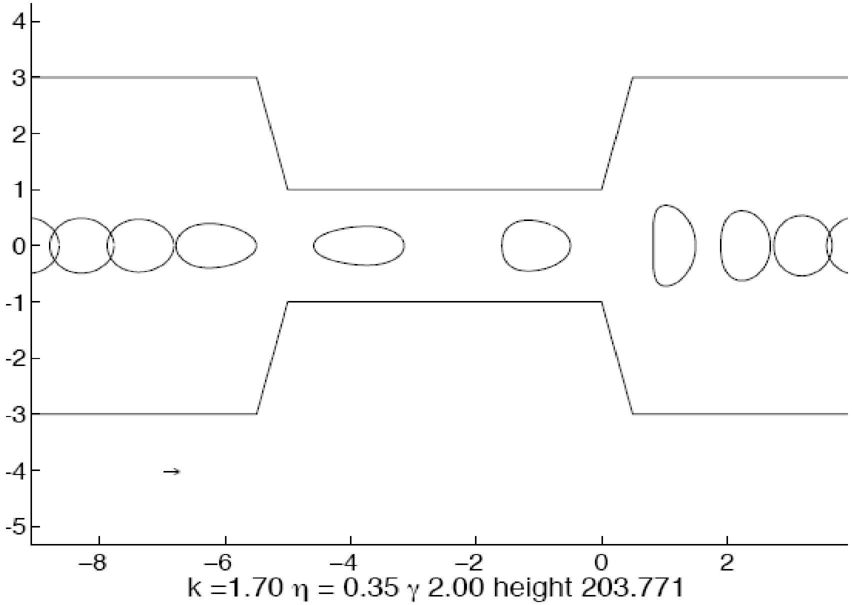



Dangla et al. (2012)

# Need for accurate simulations of moving droplets in thin microchannels

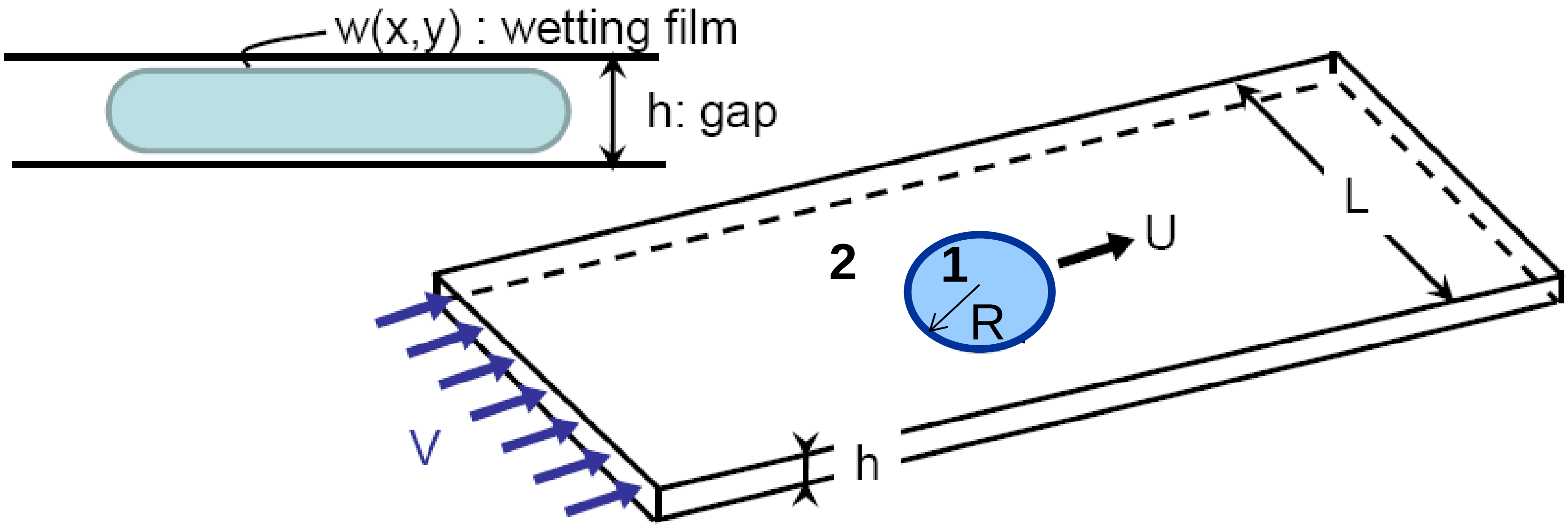


Experimental (by Cabral and Hudson)




Numerical

# Thin microchannels look like Hele-Shaw cells




- Stokes flow (low Re)
- Thin channel h/R<<1
- Unbounded outer flow (L/R>>1)

- Capillary number $$Ca = \frac{\mu U}{\gamma}$$

# Hele Shaw flows : Darcy approximation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\mu\frac{\partial^2 u}{\partial z^2}=\frac{\partial p}{\partial x}$$

$$\mu\frac{\partial^2 v}{\partial z^2}=\frac{\partial p}{\partial y}$$

$$\frac{\partial p}{\partial z}=0$$

$\varepsilon << 1$ $\quad$ $\mathrm{Re}\,\varepsilon << 1$

# Hele Shaw flows : Darcy approximation

$$u(x,y,z) = -\frac{1}{2\mu}\frac{\partial p}{\partial x} z(h-z)$$

$$v(x,y,z) = -\frac{1}{2\mu}\frac{\partial p}{\partial y} z(h-z)$$

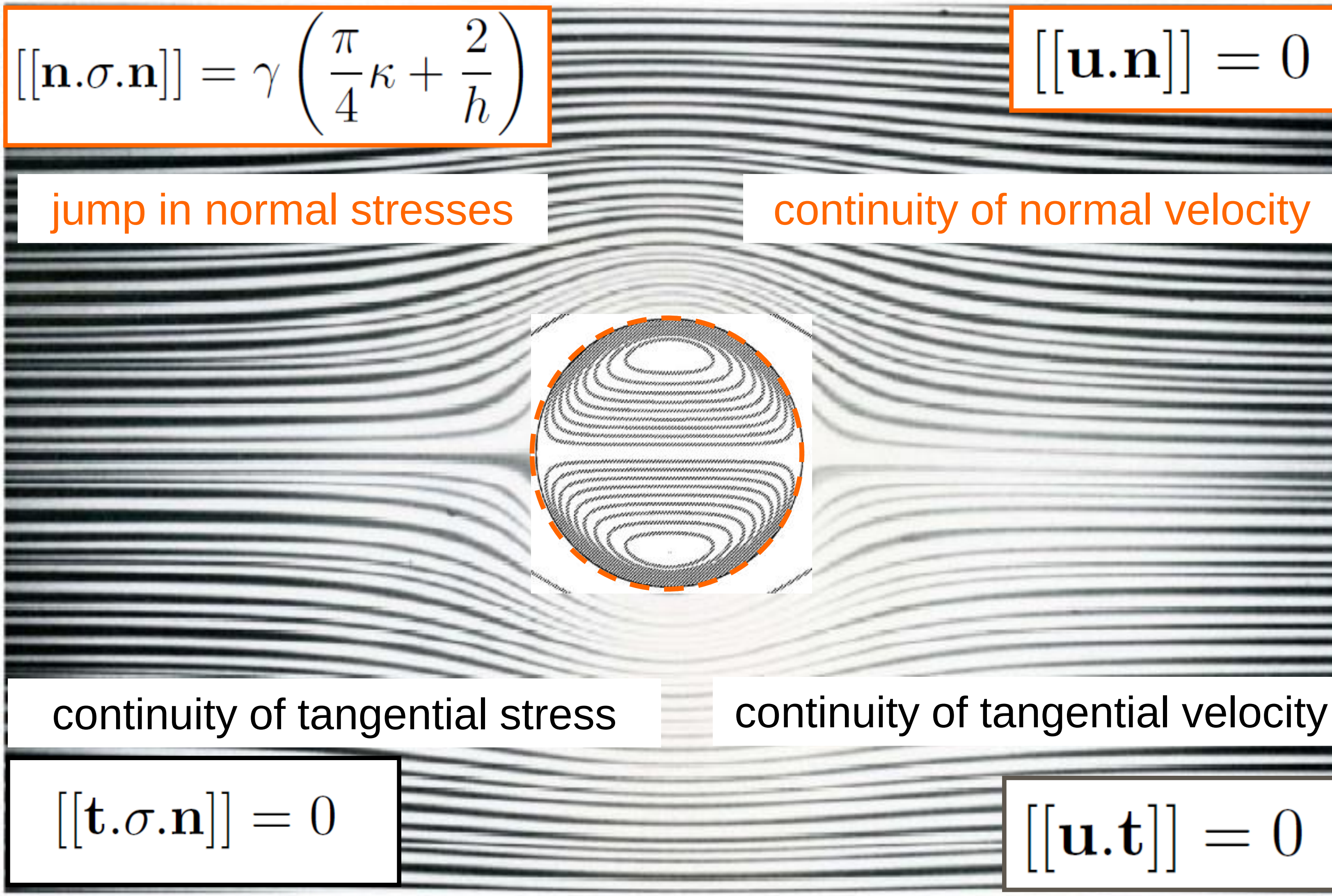
$$w(x,y,z) = 0 \; !$$

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \, !$$

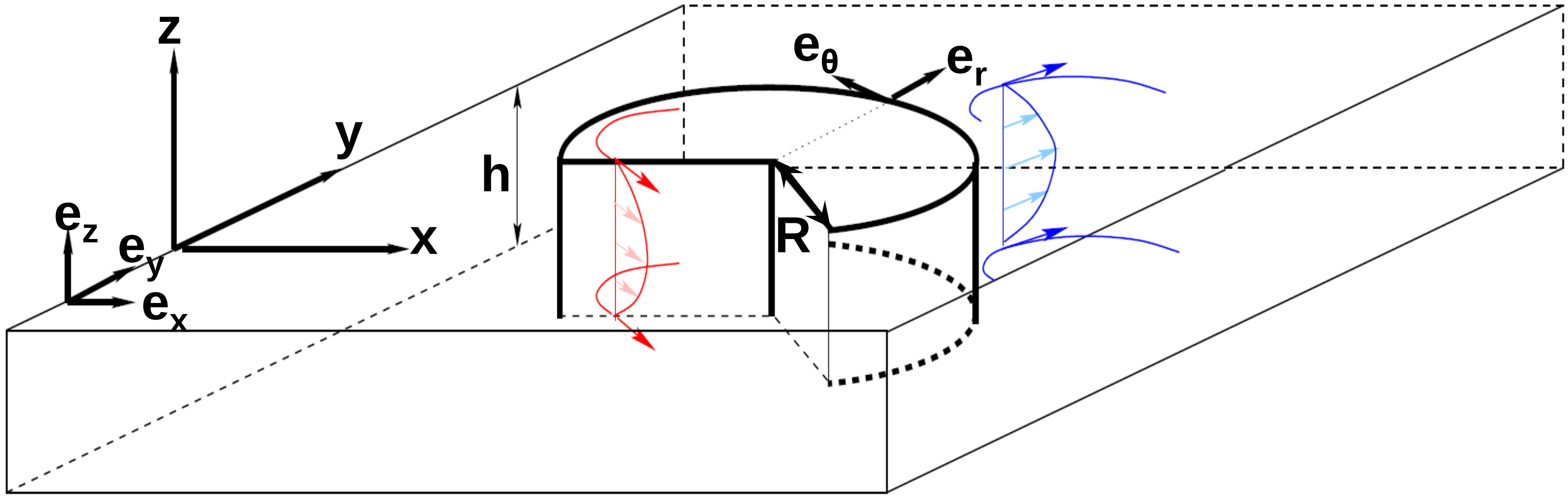
Potential flow
in the (x-y) plane

$$\nabla p = -k^2 \mathbf{u}$$

# Singular perturbation



# 2D Brinkman equations




- Aspect ratio $k=\sqrt{12}\, R/h \gg 1$
- w=0
- Parabolic profiles

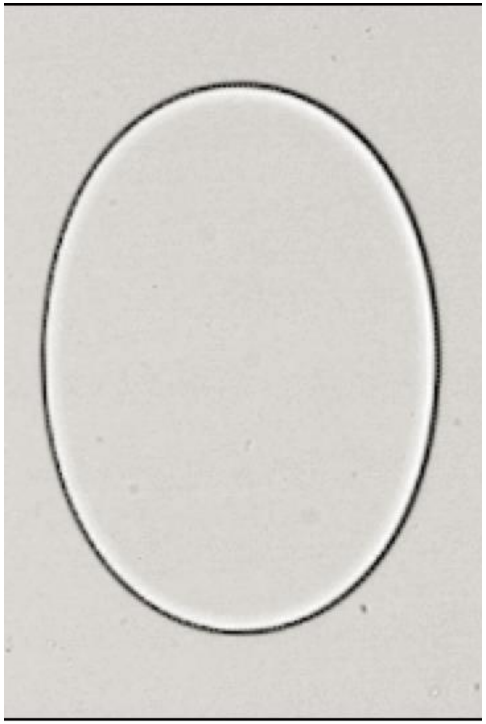
$$\mathbf{v}(x,y,z) = \mathbf{u}(x,y)\frac{6(h-z)}{h^2}$$

$$\nabla p = \Delta \mathbf{u} - k^2 \mathbf{u}$$
$$\mathrm{div}\,\mathbf{u} = 0$$

**Stokes eq.** ($\Delta \mathbf{u}$)

**Darcy eq.** ($k^2 \mathbf{u}$)

# 2D Brinkman equations



$$[[\mathbf{n}.\sigma.\mathbf{n}]] = \gamma\left(\frac{\pi}{4}\kappa + \frac{2}{h}\right)$$

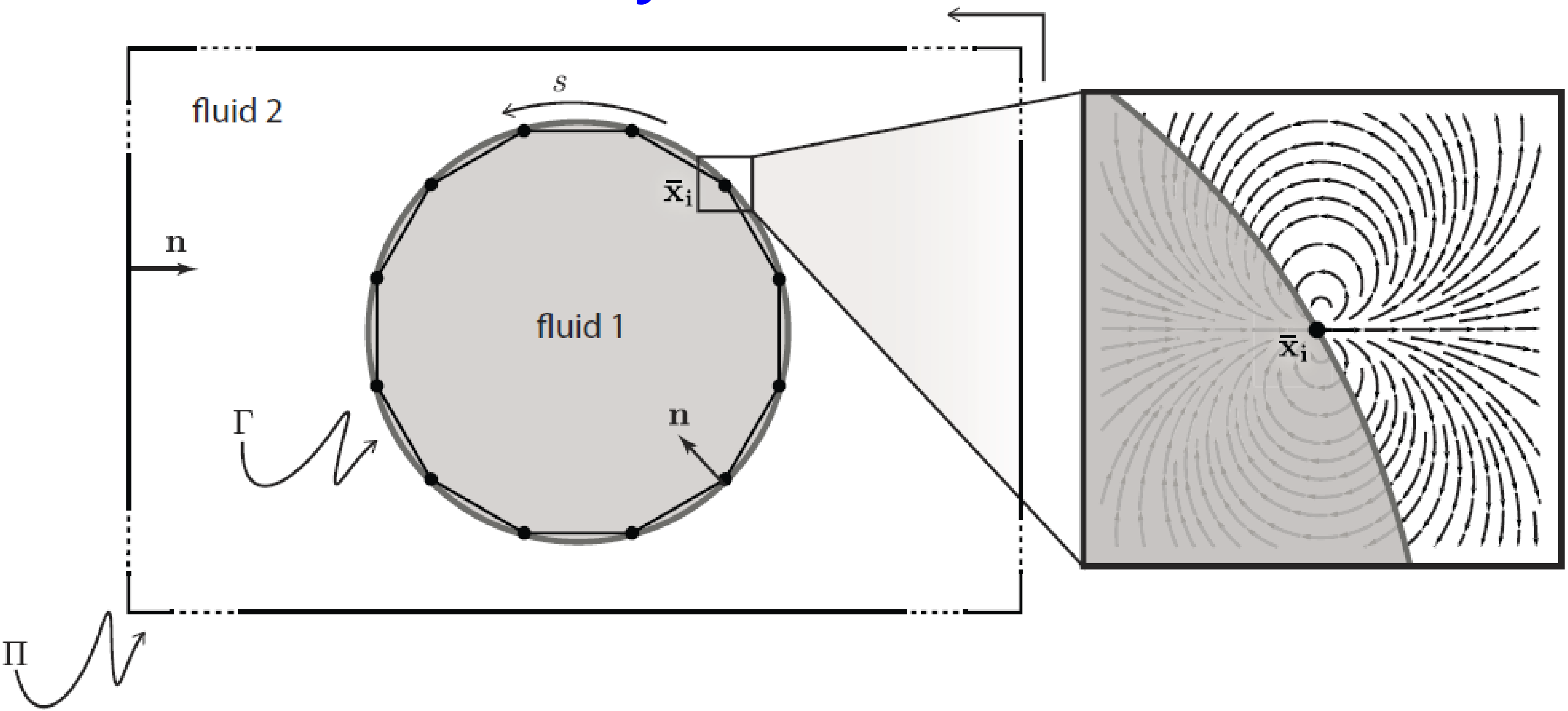
$$[[\mathbf{t}.\sigma.\mathbf{n}]] = 0$$

$$[[\mathbf{u}.\mathbf{n}]] = 0$$

$$[[\mathbf{u}.\mathbf{t}]] = 0$$

$$\nabla p = \Delta \mathbf{u} - k^2 \mathbf{u}$$
$$\mathrm{div}\mathbf{u} = 0$$

# Boundary Element Method



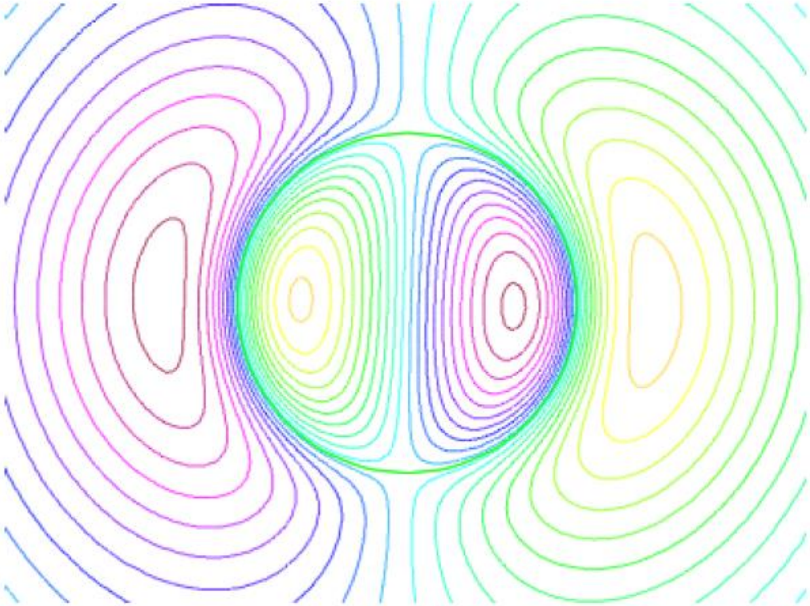

**Transformation to boundary integrals**

$$\oint_\Gamma \left( [\![\bar{\bar{\sigma}}\mathbf{n}]\!] \cdot \mathbf{G} - \mathbf{T}\,\mathbf{n} \cdot [\![\mu]\!]\mathbf{u} \right) ds + \ldots = 0$$

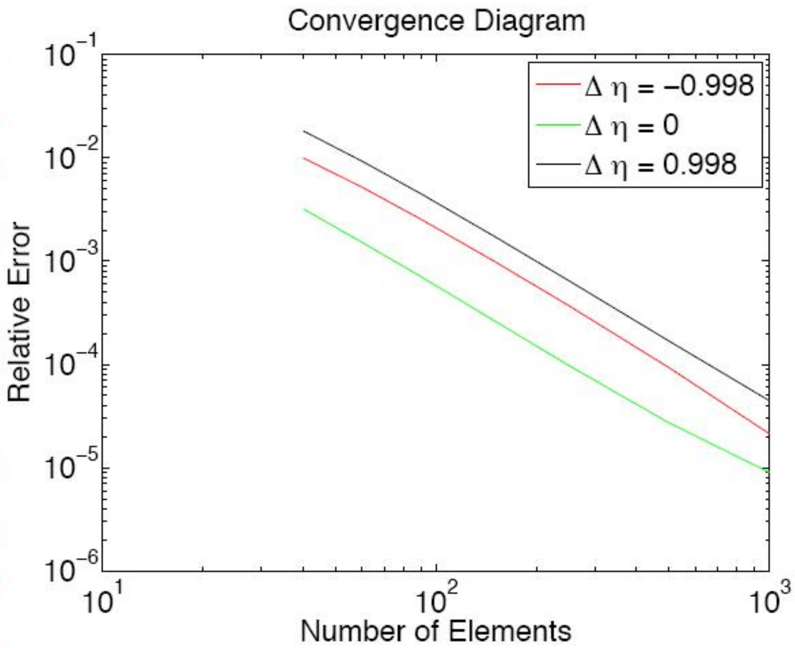
**jump in surface stress** **difference in viscosity**

Pozrikidis – Boundary Integral and Singularity Methods (1992)

# Validation of boundary element algorithm for Brinkman equations

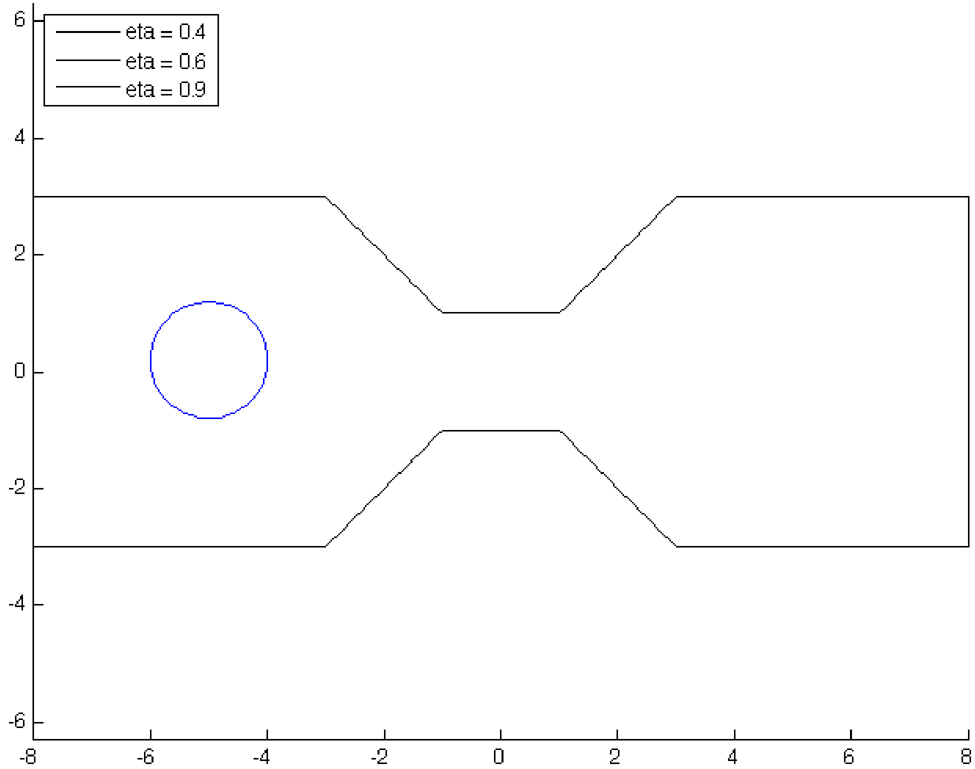


(e) Streamlines: thermo capillary flow




(f) Convergence: thermo capillary flow

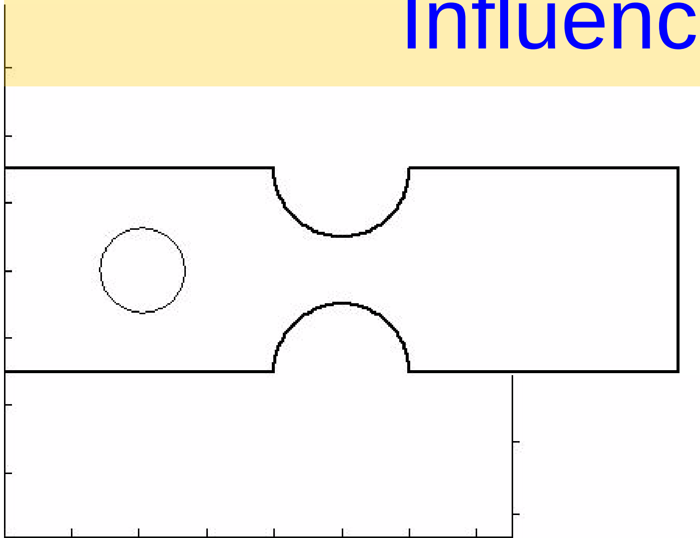
# Influence of viscosity ratio



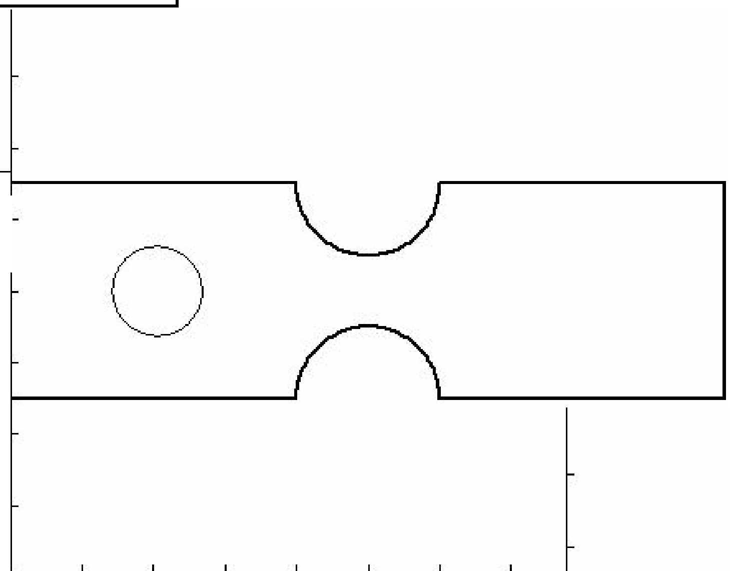

k=12, Ca=0.1

# Influence of capillary number

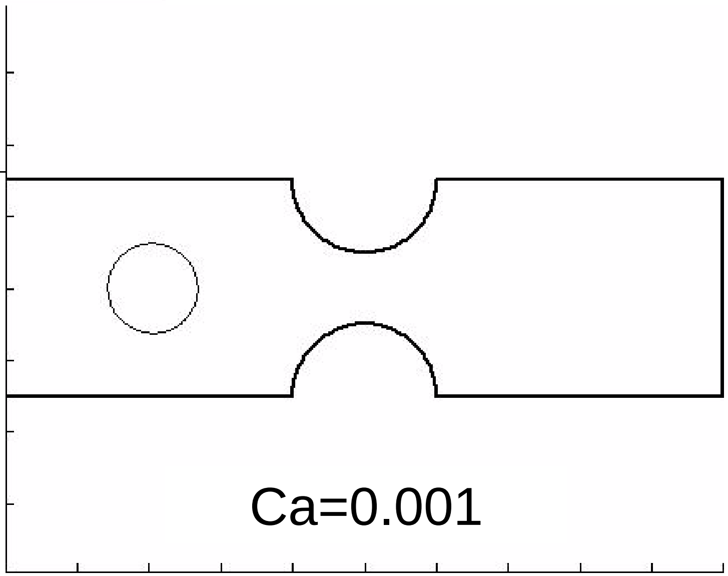
k=12, η=1



Ca=0.1

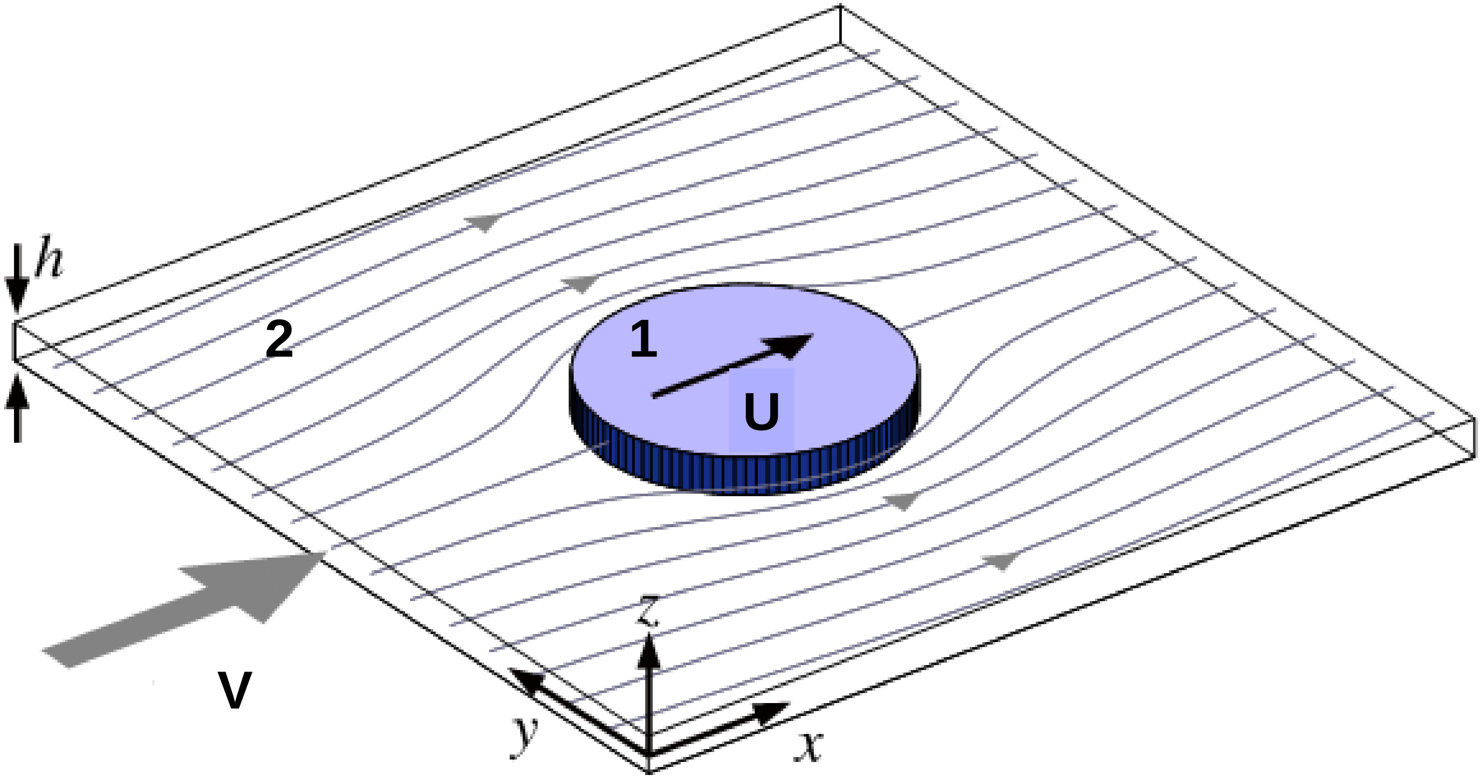


Ca=0.01



Ca=0.001

# Migration velocity for a rigid pancake droplet

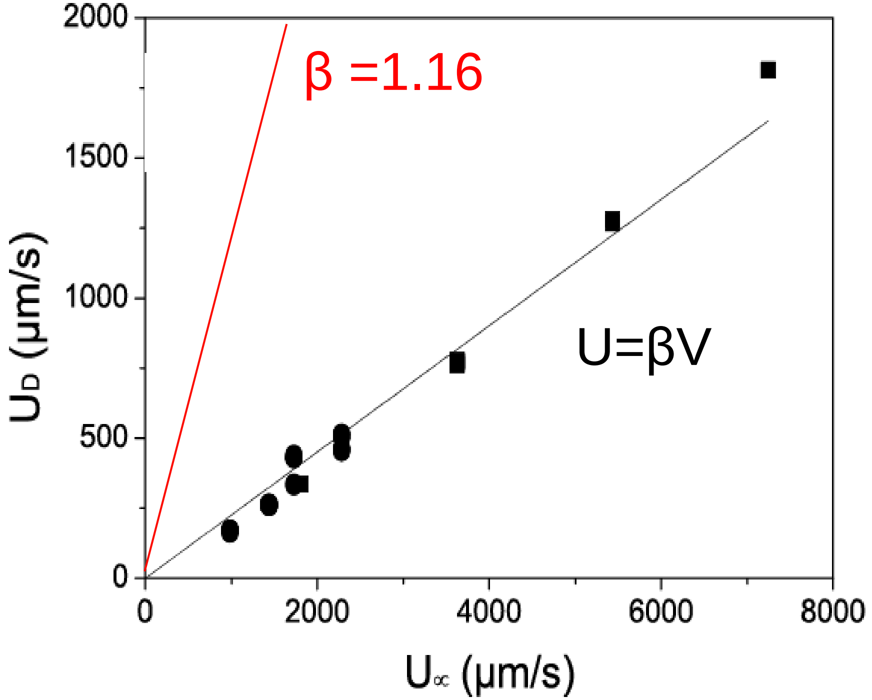



Ca<<1 ⇨ Freeze the droplet interface

$$U = \frac{2\mu_2}{\mu_1 + \mu_2} V + O(1/k)\, V$$

mobility β    Brinkman

# Recent drop velocity measurements (Leman and Tabeling)

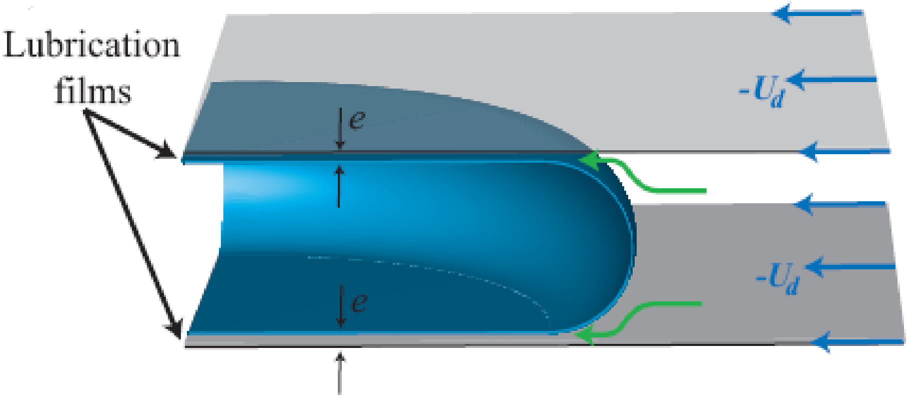


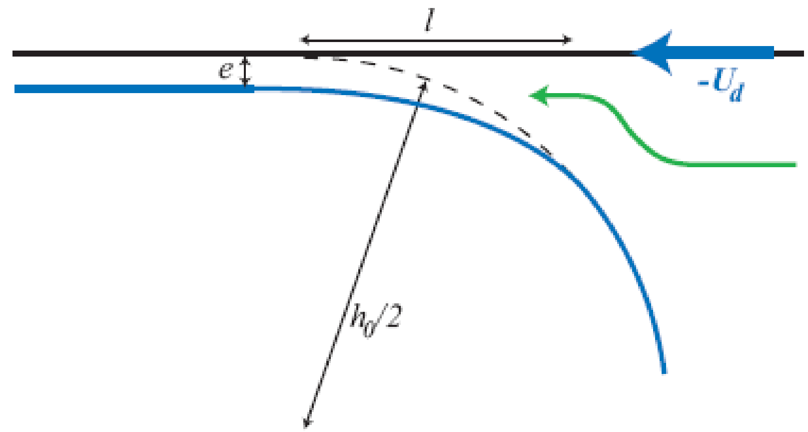

DI water with 1% w/w (SDS) surfactant droplets in fluorinated oil

(■) PDMS system, h=41μm, w2=500μm;

(●) NOA system, h=37μm, w2=3000μm;

# Landau-Levich-Bretherton films



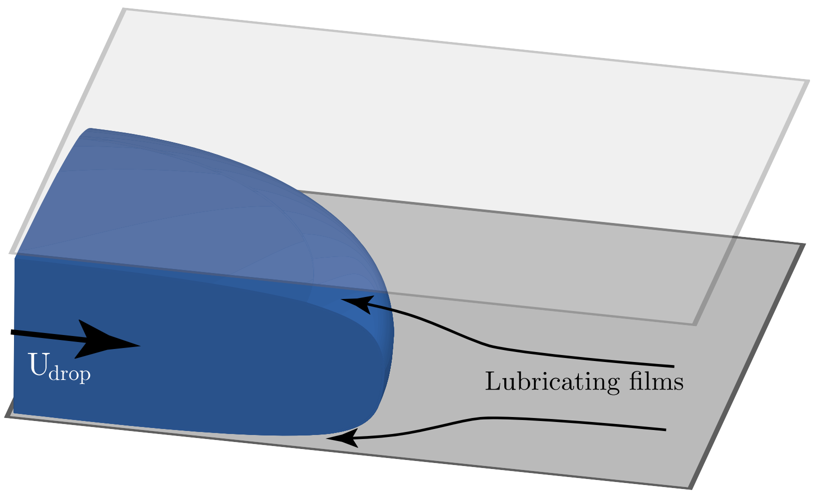




Kreutzer

# Asymptotic correction due to dynamic films

$$[[\mathbf{n}.\sigma.\mathbf{n}]] = \gamma \left( \frac{\pi}{4}\kappa + \frac{2}{h}\left(1 + \alpha \;\; \mathrm{ca}(\mathbf{x})^{2/3}\right)\right)$$

$$\mathrm{ca}(\mathbf{x}) = \frac{\mu_2\, \mathbf{u}(\mathbf{x}).\mathbf{n}(\mathbf{x})}{\gamma}$$



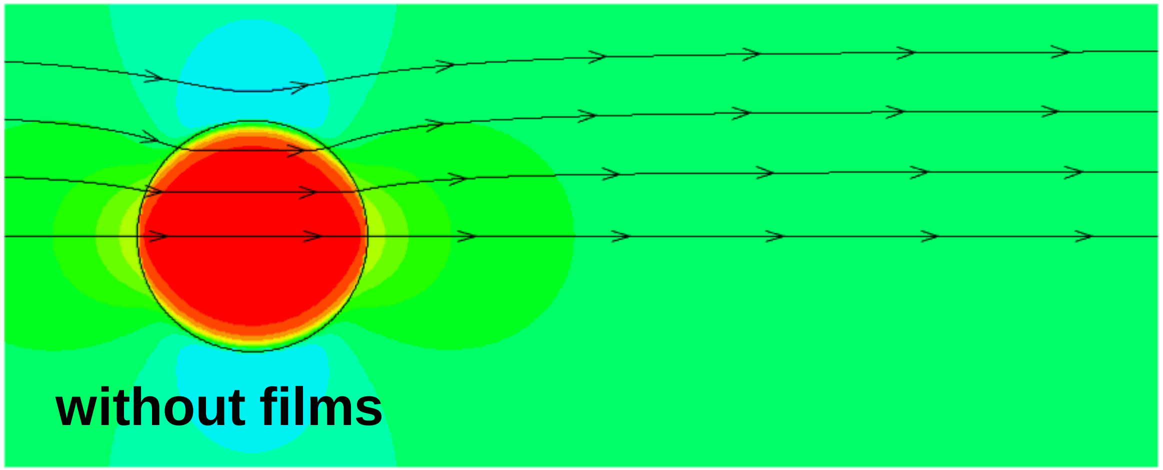

***Local capillary number at point*** $\mathbf{x}$

***Advancing meniscus :*** $\alpha = 3.8$

***Receding meniscus :*** $\alpha = -1.13$

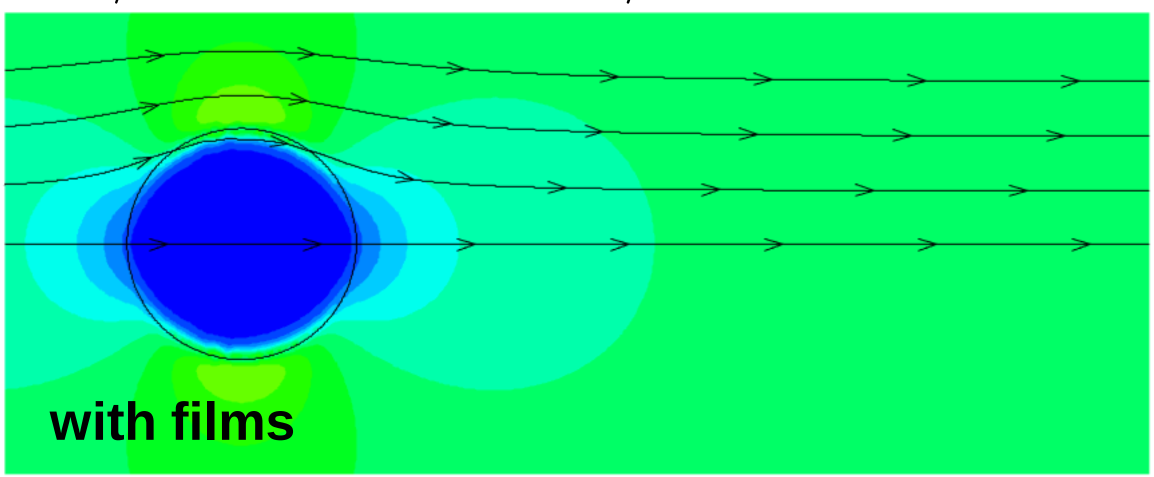
Park and Homsy – J. Fluid Mech. (1984)
Burgess and Foster – Phys Fluids (1984)
Meiburg – Phys Fluids (1984)

# Dynamics of deformable droplets



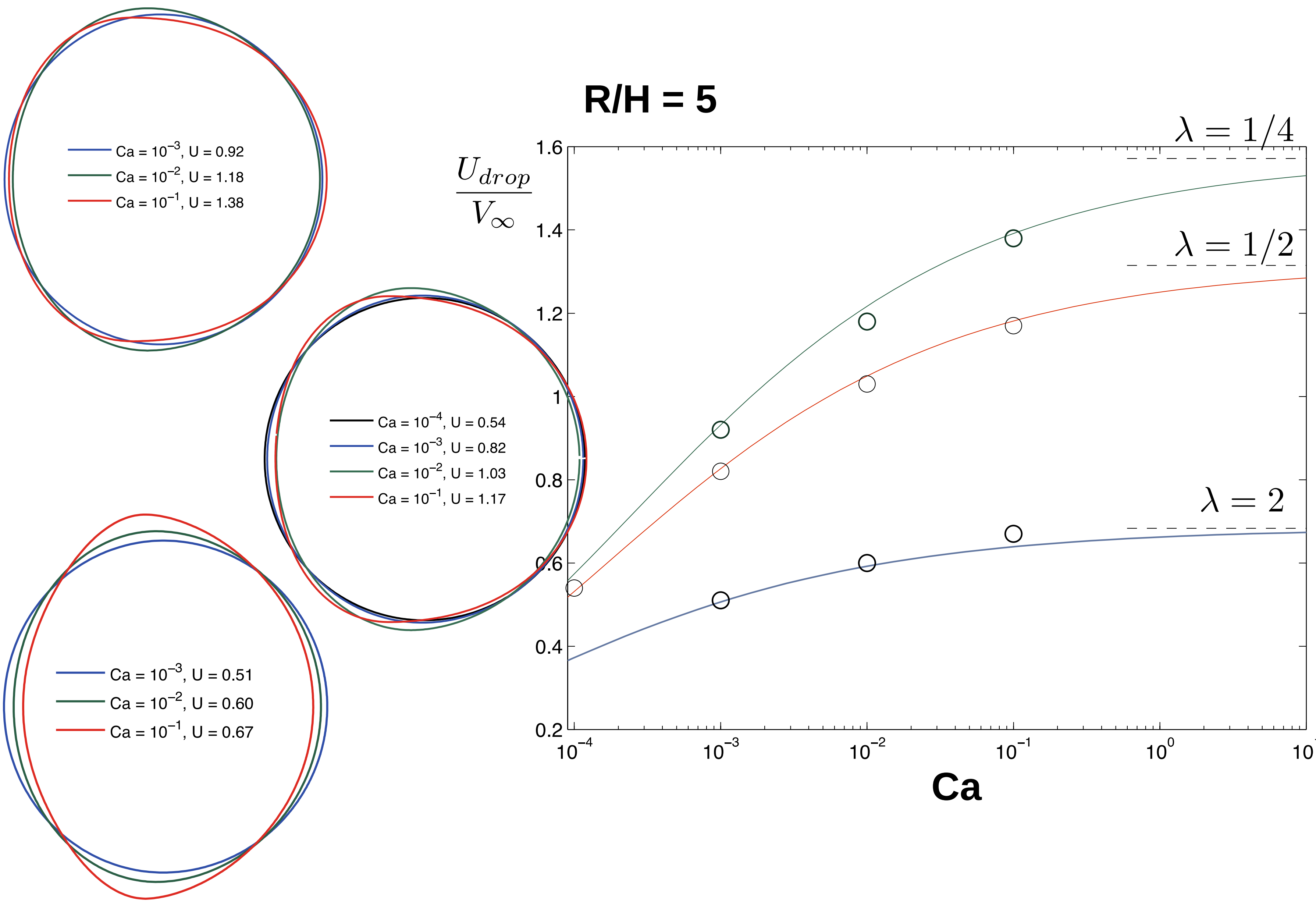

$$U_{drop} = 1.475$$

$R/H = 5$ $\lambda = 1/3$ $Ca = 10^{-4}$

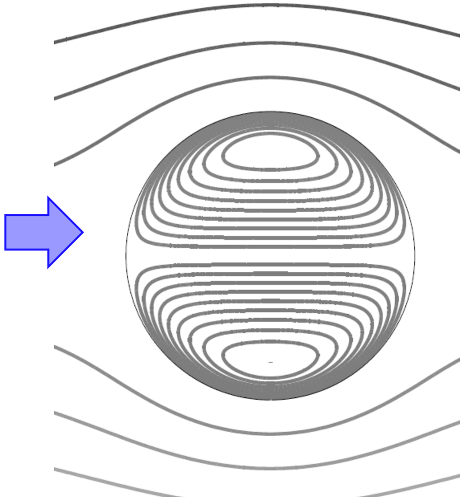



$$U_{drop} = 0.559$$

# Deformable droplets

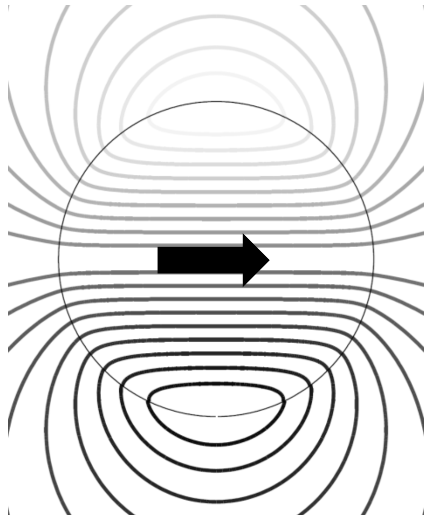


# Assuming that the droplet does not deform



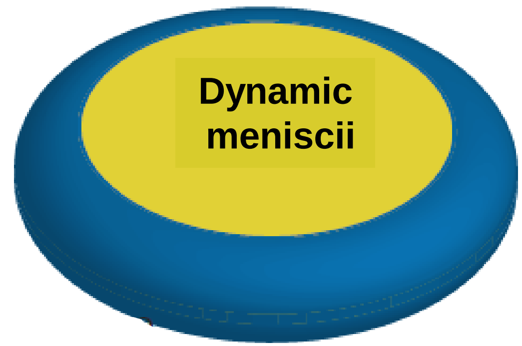
$$F_{flow} = 24\pi\mu_2 V \frac{R^2}{h}$$

+



$$F_{drop} = 12\pi(\mu_1 + \mu_2) U \frac{R^2}{h}$$

+




$$F_{\text{film}} = (3.8 + 1.13)\frac{2\Gamma(\frac{1}{3})}{5\sqrt{3\pi}\Gamma(\frac{5}{6})} k\pi \frac{U^{2/3}V^{1/3}}{Ca^{1/3}}$$

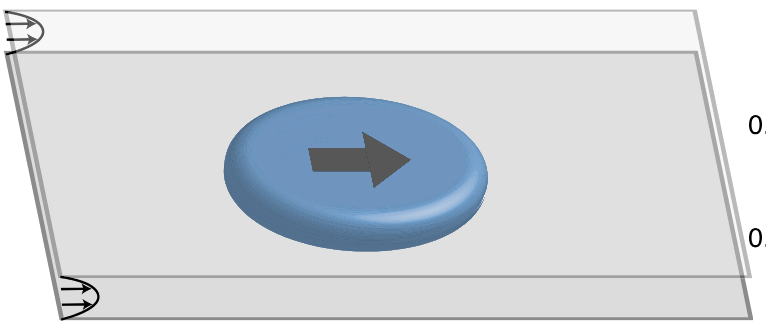
ΣF=0

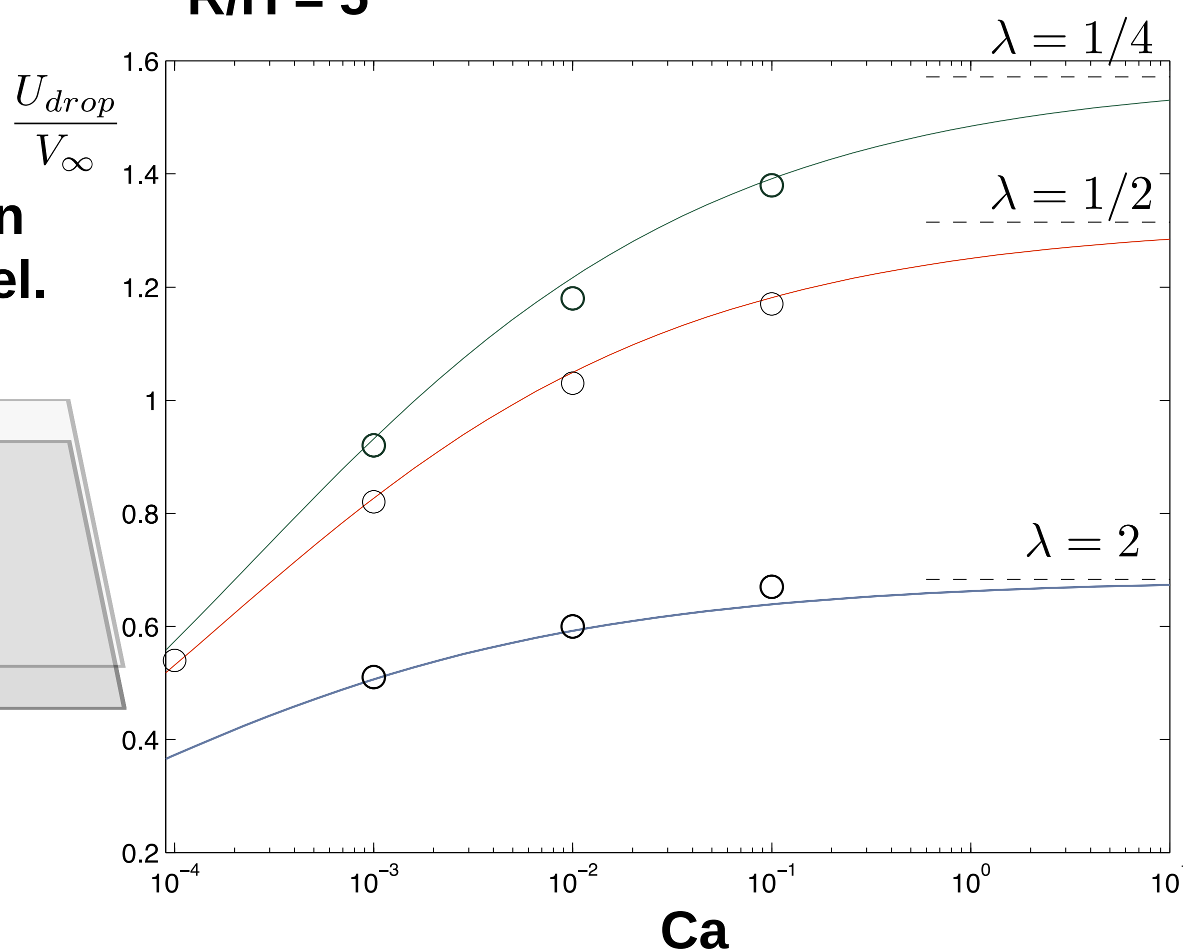
⇨U is a root of a cubic function

# Undeformable droplets

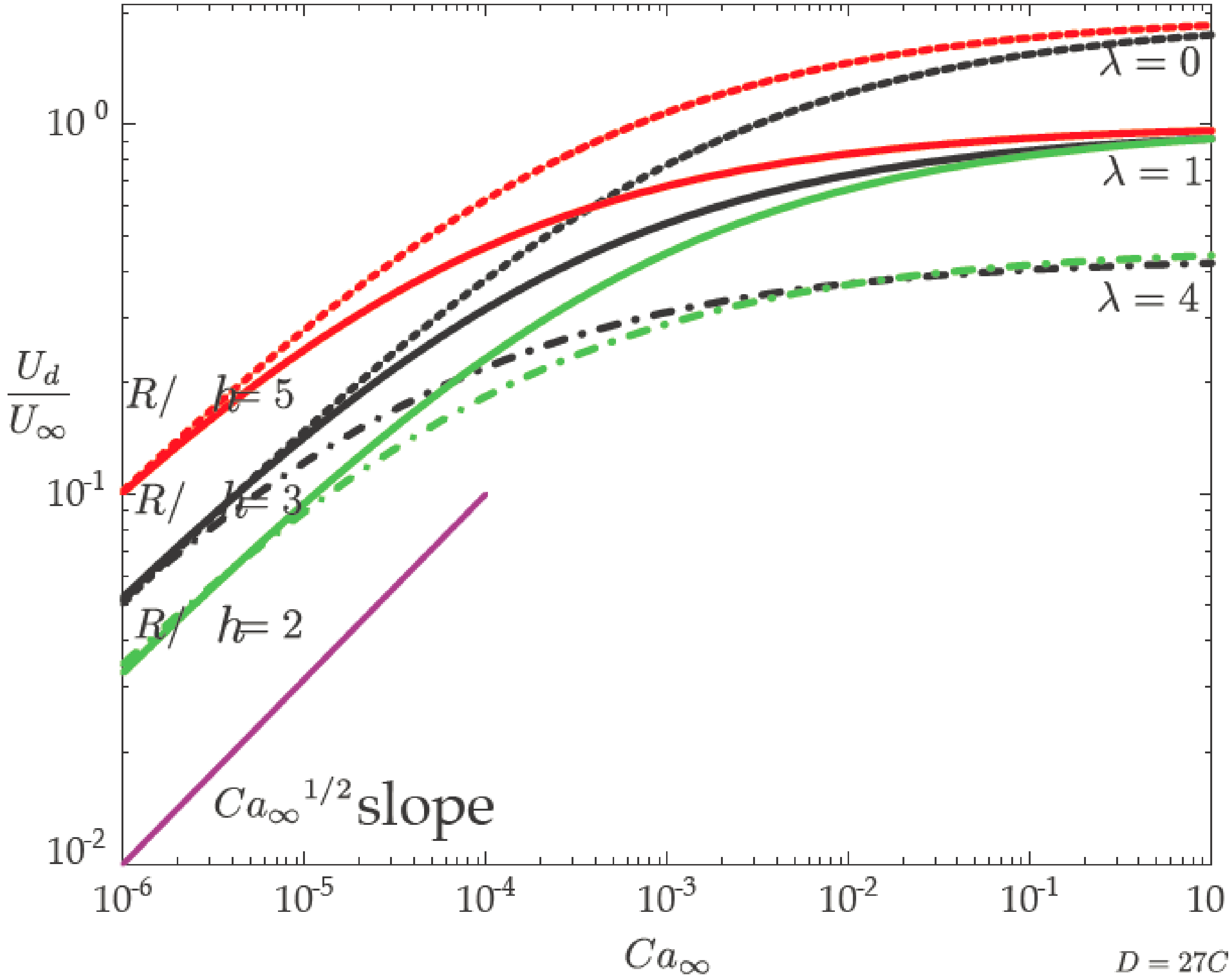
**R/H = 5**

**Droplet velocity in an infinitely wide channel.**



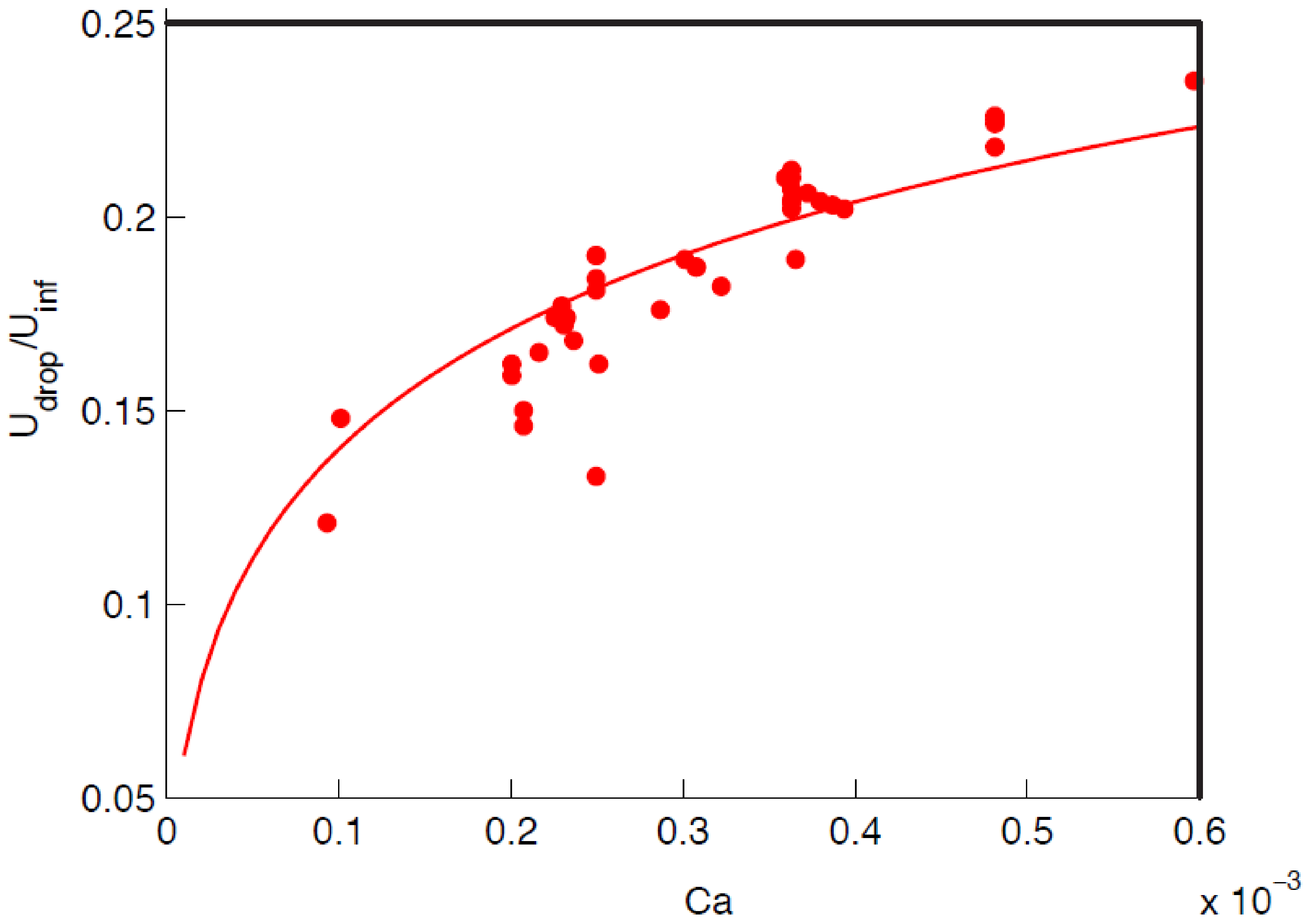


# Analytical expression

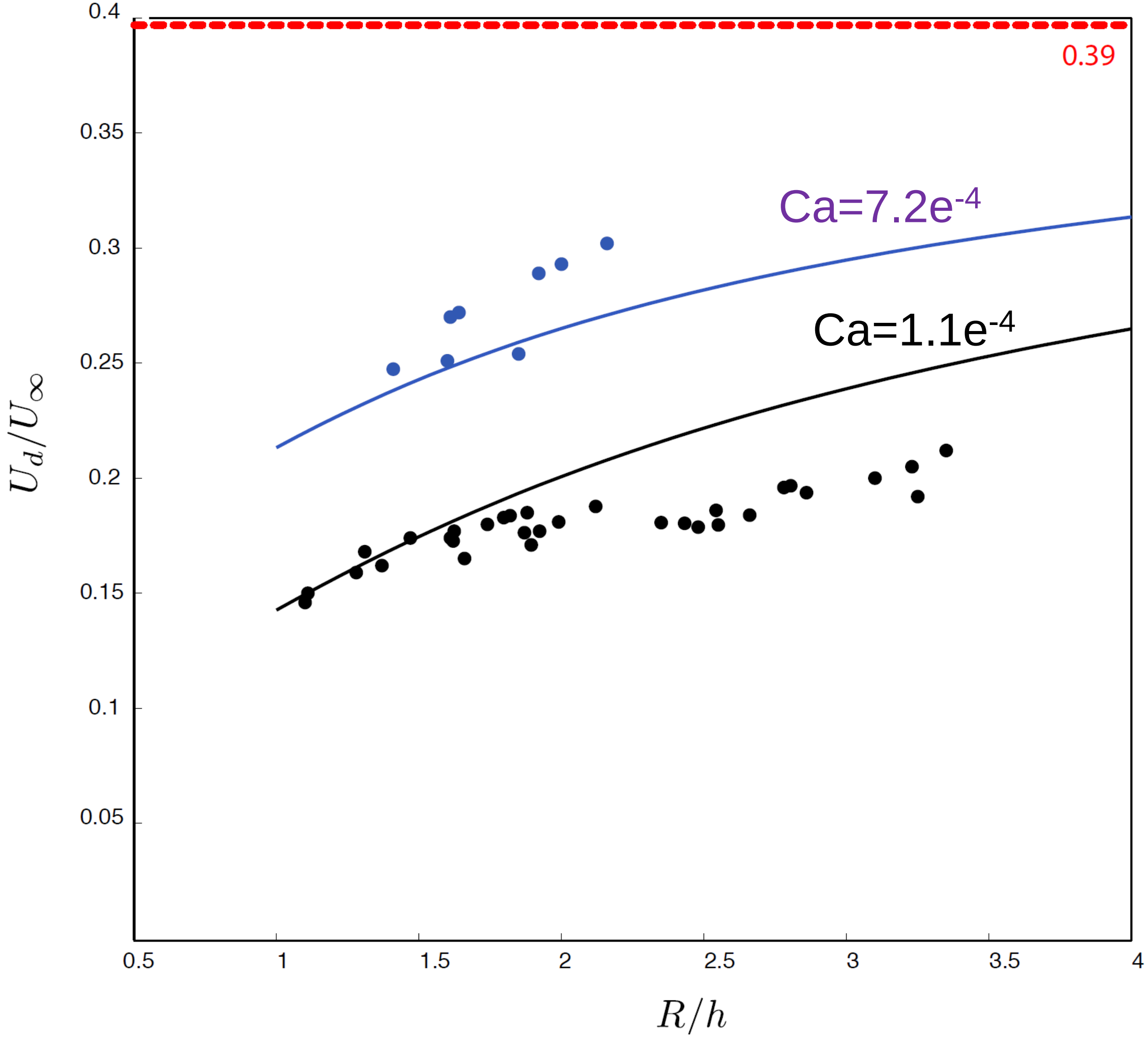


# Experimental comparison
# FC40 droplets in water

$\gamma$=53mPa.m, $\mu_1$=4.1cSt , R/h = 1.3



# Experimental comparison



# Conclusion

In absence of surfactants, the droplet of a velocity can be accurately captured at low Ca