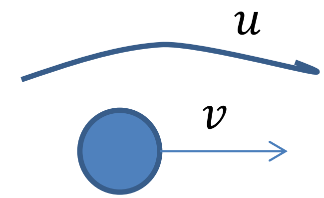
Particles in turbulence

Irreversibility in the statistics of inertial particles in a flow

Localization-delocalization phase transition

Twisted logic: particles first, turbulence second. Reason – integration.

One particle in a flow



$$\frac{d}{dt}\rho_0 V v = forces = \rho V \frac{du}{dt} + \frac{d}{dt}\rho\frac{V}{2}(u-v) + \cdots$$

Condensation

$$\rho_0 V \frac{d}{dt} v = forces = \rho V \frac{du}{dt} + \frac{d}{dt}\rho\frac{V}{2}(u-v) + \cdots$$

Evaporation

Assuming constant mass, $\frac{d}{dt}\rho_0 V = 0$,

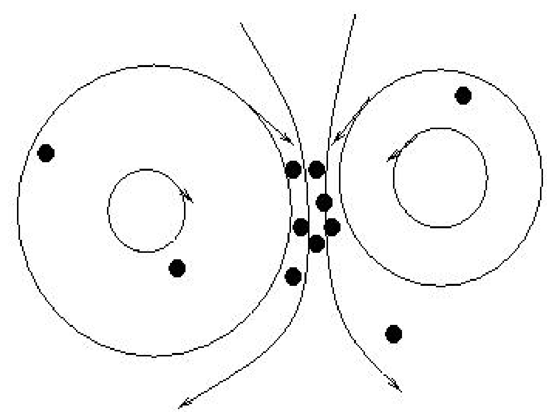
$$\beta = 3\,\rho/(\rho + 2\rho_0)$$

$$\frac{dv}{dt} - \beta\frac{du}{dt} = g + \frac{u-v}{\tau}.$$

$$\tau = \frac{2}{9}\frac{\rho_0}{\rho}\frac{a^2}{\nu}$$

$\tau \ll \tau_c, 1/\nabla u$

$$v = u + g\tau + (\beta - 1)\frac{du}{dt} = u + g\tau + (\beta - 1)\left[\frac{\partial u}{\partial t} + (u\nabla)u\right].$$



$$div\ v = (\beta - 1)div(u\nabla)u = (1-\beta)\Delta p.$$

Already in the limit of small inertia, one sees preferential concentration and sling effect

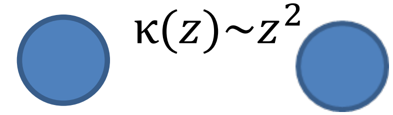
$$\frac{d\mathbf{R}}{dt} = \Delta\mathbf{v}(\mathbf{R},t), \quad \frac{d\Delta\mathbf{v}}{dt} + \frac{\Delta\mathbf{v}}{\tau} = \frac{\Delta\mathbf{u}(\mathbf{R},t)}{\tau}$$

One-dimensional projection $\quad R = z,\ u = \frac{\Delta \boldsymbol{u} \cdot \boldsymbol{R}}{R},$ looking for $P(z, v)$.

$$\tau_c \ll \tau \ \rightarrow \ < u(0,0)u(z,t) >= \kappa(z)\delta(t)$$

$\bar{v}(z) \sim z$

$\kappa(z) \sim z^2$



Two small particles at the distance below the viscous scale

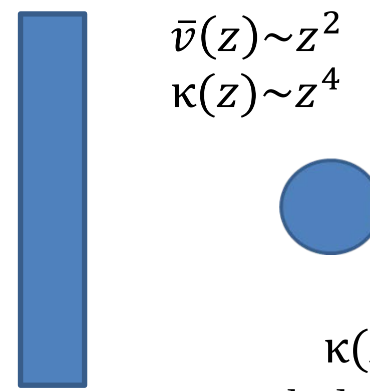
$$\frac{\partial P}{\partial t} = -v\frac{\partial P}{\partial z} + \frac{1}{\tau}\frac{\partial (vP)}{\partial v} + \frac{\kappa(z)}{\tau}\frac{\partial^2 P}{\partial v^2}$$

$$\bar{v}(z) = \sqrt{\kappa(z)/\tau}$$

Viscous BL

$\bar{v}(z) \sim z^2$

$\kappa(z) \sim z^4$



$\kappa(z) \sim z$

Turbulent BL

Particle near a wall

Third timescale

$$\tilde{\tau}(z) = \frac{\kappa/\kappa'}{\bar{v}(z)} \sim \frac{z}{\bar{v}(z)}$$

Spatially smooth flow

$$\frac{d\mathbf{q}}{dt} = \mathbf{v}(\mathbf{q}, t)\,, \quad \mathbf{q}(\mathbf{r}, 0) = \mathbf{r}$$
$$\frac{d\mathbf{v}}{dt} = \frac{u(\mathbf{q}, t) - \mathbf{v}}{\tau} + \mathbf{g}\,.$$

$$\mathbf{R} = \mathbf{q}(\mathbf{r}_1, t) - \mathbf{q}(\mathbf{r}_2, t)$$

$$\frac{d\mathbf{R}}{dt} = \Delta\mathbf{v}(\mathbf{R}, t), \quad \frac{d\Delta\mathbf{v}}{dt} + \frac{\Delta\mathbf{v}}{\tau} = \frac{\Delta\mathbf{u}(\mathbf{R}, t)}{\tau}$$

$$\Delta\mathbf{u}(\mathbf{R}, t) = \hat{s}(t)\mathbf{R}(t), \quad \hat{s} = \xi(t)\tau$$

$$\ddot{\mathbf{R}} + \dot{\mathbf{R}}/\tau = \xi\mathbf{R}$$

One-dimensional model

$$R = \Psi \exp(-t/2\tau), \; -\Psi'' + \xi\Psi = -\Psi/4\tau^2$$

Equivalent in 1d to Anderson localization:
localization length = Lyapunov exponent

$$\sigma = \Delta v/R$$

$$\frac{d\sigma}{dt} = -\sigma^2 - \sigma/\tau + \sqrt{D}\xi$$

$$\dot{x}(t) = -dU(x)/dx + \xi(t)$$

$$\langle \xi(0)\xi(t) \rangle \; = \; T\delta(t)$$

$$P(x) \; \propto \; \exp[-U(x)/T]$$

$$U(x) = \frac{x^3}{3} + \frac{x^2}{2\tau}$$

$$\int P(x)\, dx \text{ diverges}$$

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial \sigma}\left[\left(\sigma^2 + \frac{\sigma}{\tau}\right) P + D\frac{\partial P}{\partial \sigma}\right] \qquad D\frac{\partial P_0}{\partial \sigma} + \left(\sigma^2 + \frac{\sigma}{\tau}\right) P_0 = F$$

$$P = \frac{F}{D} \exp\left[-\frac{\sigma^3/3 + \sigma^2/(2\tau)}{D}\right] \int_{-\infty}^{\sigma} \exp\left[\frac{\sigma'^3/3 + \sigma'^2/(2\tau)}{D}\right] d\sigma'$$

$$P(\sigma) \to F\sigma^{-2} \quad \text{as} \quad \sigma \to \pm\infty$$

F – the flux of probability to $\sigma = -\infty$, the frequency of collisions.

The resurrecting solutions $\sigma(t)$ jumping instantaneously from $-\infty$ to $+\infty$ correspond here to the solutions for (δr, δu), where δr passes through zero with a non-vanishing speed, i.e. to the crossing of close particle trajectories with faster particles overcoming slower ones. Alternatively, every event $z \to 0, \sigma = \frac{v}{z} \to -\infty$ is the collision-reflection from a wall or another particle.

The solution with the flux - how non-equilibrium and irreversible it is?

Temporal correlation of fluctuations in equilibrium

$$\mathcal{H} = \mathcal{H}_0 - xf(t)$$

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial x}\frac{\partial \mathcal{H}}{\partial p} - \frac{\partial \rho}{\partial p}\frac{\partial \mathcal{H}}{\partial x} \equiv \{\rho, \mathcal{H}\}$$

$$\frac{\partial \rho_1}{\partial t} + \mathcal{L}\rho_1 = -f\beta\frac{\partial \mathcal{H}_0}{\partial p}\rho_0 \qquad \rho_0 = \exp(-\beta H_0)$$

$$\rho_1 = \beta\rho_0 \int_{t_0}^{t} e^{(\tau - t)\mathcal{L}}\dot{x}(\tau) f(\tau)\, d\tau = \beta\rho_0 \int_{t_0}^{t} \dot{x}(\tau - t) f(\tau)\, d\tau$$

$$\langle x(t)\rangle \equiv \int_{-\infty}^{t} \alpha(t, t') f(t')\, dt' = \int x dx \rho_1(x, t)$$

$$\alpha(t - t') \equiv \delta\langle x(t)\rangle / \delta f(t')$$

$$\frac{\partial}{\partial t'}\langle x(t) x(t')\rangle = T\alpha(t, t')\,, \quad t \geq t'$$

Fluctuation-dissipation theorem

Let $O^a(x)$ for $a = 1, \ldots, A$ be a collection of (classical) observables. With the shorthand notation O_t^a for the single-time functions $O^a(x_t)$ of the dynamical process x_t, the response function and the two-time correlation function in a steady state are, respectively,

$$\mathcal{R}^{ab}(t-s) = \frac{\delta}{\delta h_s}\bigg|_{h=0} \langle O_t^a \rangle_h \qquad \text{and} \qquad \mathcal{C}^{ab}(t-s) = \langle O_t^a O_s^b \rangle_0, \tag{1.1}$$

where $\langle - \rangle_h$ denotes the dynamical expectation obtained from the steady state by replacing the time-independent Hamiltonian $H(x)$ by a slightly perturbed time-dependent one $H(x) - h_t O^b(x)$. The FDT asserts that, when the unperturbed state is the equilibrium at inverse temperature β, then

$$\beta^{-1}\mathcal{R}^{ab}(t-s) = \partial_s \mathcal{C}^{ab}(t-s). \tag{1.2}$$

Modified fluctuation dissipation theorem

current j_0

$$B^b = \varrho_0^{-1} j_0 \cdot \nabla O^b$$

$$\theta(t-s)\langle O_t^a B_s^b \rangle_0 \equiv \mathcal{B}^{ab}(t-s)$$

$$\beta^{-1}\mathcal{R}^{ab}(t-s) = \partial_s \mathcal{C}^{ab}(t-s) - \mathcal{B}^{ab}(t-s)$$

In the Lagrangian reference frame with the velocity $\varrho_0^{-1} j_0 \equiv \nu_0$

$$\partial_t O^a(t,x) + \nu_0(x) \cdot \nabla O^a(t,x) = 0$$

$$\beta^{-1}\mathcal{R}_L^{ab}(t,s) = \partial_s \mathcal{C}_L^{ab}(t,s)$$

$$\dot{x} = -\partial_x H(x) + \zeta$$

$$\langle \zeta_t^i \zeta_s^j \rangle = 2\beta^{-1}\Gamma^{ij}\delta(t-s)$$

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial j}{\partial x}$$

$$j = -\beta^{-1}\frac{\partial \rho}{\partial x} - \rho\frac{\partial H}{\partial x}$$

$$\frac{1}{Z}\left(\int_{-\infty}^{x} \mathrm{e}^{\beta H(y)}\mathrm{d}y\right)\mathrm{e}^{-\beta H(x)}\,\mathrm{d}x \equiv \varrho_H(x)\,\mathrm{d}x$$

$$\text{constant current } j = -(\beta Z)^{-1}$$

All systems driven by a white noise are in equilibrium in the Lagrangian frame in the phase space

Let us return to particles and describe their distribution in space

When the dust settles

Path coalescence transition (two particles)
Localization-delocalization transition (particle near a minimum)

$\tau_c \ll \tau, \tilde{\tau}$

$\kappa(z) = \int_0^\infty \langle u_z(z,t) u_z(z,0) \rangle dt$ $\quad\quad$ $\rho(z,v,t) = \langle \delta(z - z(t)) \delta(v - v(t)) \rangle$

$$\partial_t \rho = -v \partial_z \rho + \gamma \partial_v (v\rho) + \gamma^2 \kappa(z) \partial_v^2 \rho$$

$I(z) = \left(\frac{\tau}{\tilde{\tau}(z)} \right)^{2/3}$ $\quad\quad$ $\kappa(z) = \mu z^2$ ⟹ $I = (\mu\tau)^{1/3}$

$I \ll 1$ ⟹ local equilibrium $\quad\quad$ $n(z,t) = \int_{-\infty}^{+\infty} dv \rho(z,v,t)$

$$\partial_t n = -\partial_z j$$

$$j = -\kappa(z) \partial_z n - n \frac{d\kappa(z)}{dz}$$

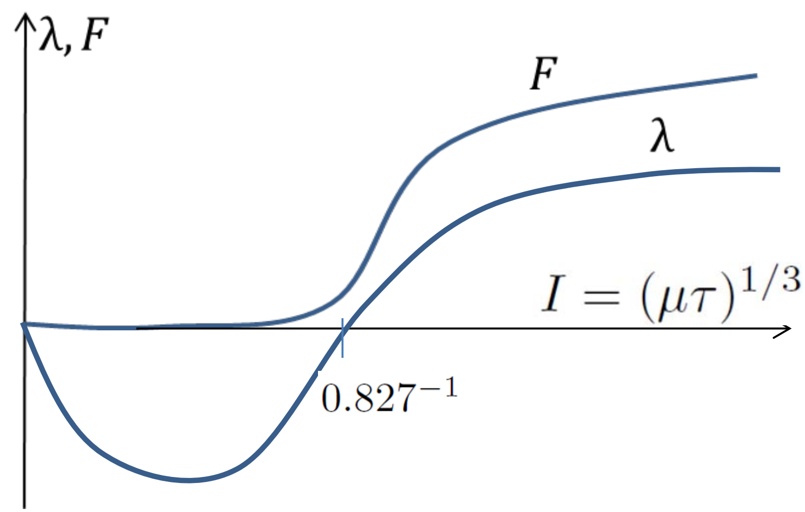
$$\sigma = v/z$$

$$P(\sigma) = \frac{F}{\mu\gamma^2} e^{-U(\sigma)/\mu\gamma^2} \int_{-\infty}^{\sigma} e^{U(\sigma')/\mu\gamma^2} d\sigma'$$

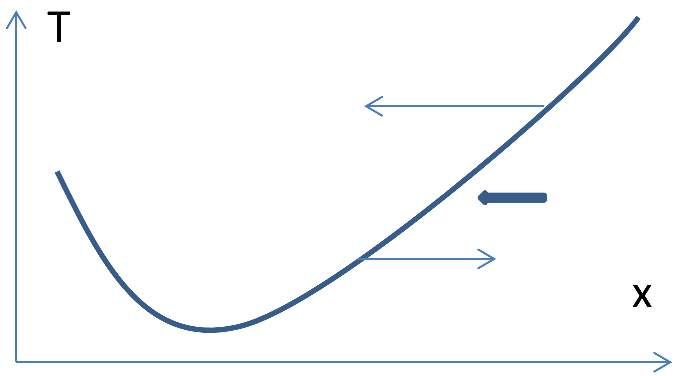
$$\lambda = \lim_{t\to+\infty} \frac{1}{t} \ln \frac{z(t)}{z(0)} = \lim_{t\to+\infty} \frac{1}{t} \text{p.v.} \int_0^t \sigma(t')dt' = \langle\sigma\rangle$$

F – the flux of probability to $\sigma = -\infty$, the frequency of collisions.

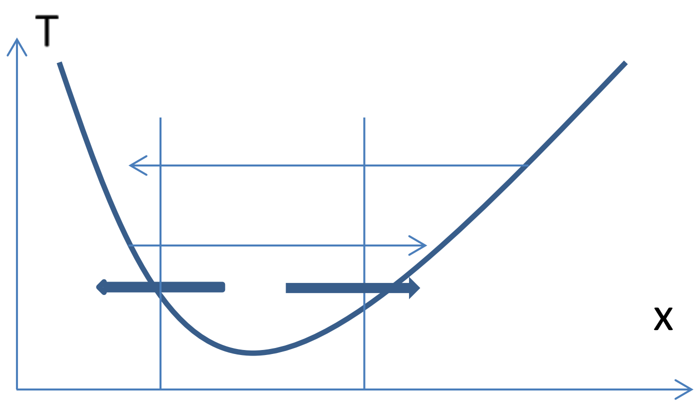
The rate of caustic creation.



Thermo- and turbo-phoresis



Maxwell: inertial particles in a temperature gradient move on average towards minimum **down** the gradient.



Belan & GF: very inertial particles fly through the maximum, so that the net flux is from the minimum **up** the gradient.

The best way out is always through.

Robert Frost

Transition under inelastic collisions

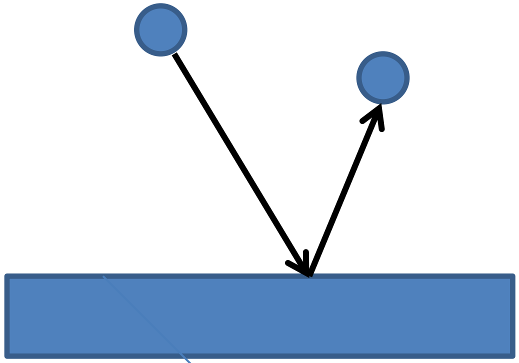
$$v(t_i+\delta t) = -\beta v(t_i-\delta t) \text{ and } z(t_i+\delta t) = \beta z(t_i-\delta t)$$

$$z(t_i) = 0 \text{ and } \delta t \to +0 \longrightarrow \sigma(t_i+\delta t) = -\sigma(t_i-\delta t)$$

$$z(t) = z(0)\beta^{N(t)} \exp(\text{p.v.} \int_0^t \sigma(t')dt')$$

$$\lambda(\beta, I) = \lim_{t\to+\infty} \frac{1}{t} \ln \frac{z(t)}{z(0)} = \lim_{t\to+\infty} \frac{1}{t} \text{p.v.} \int_0^t \sigma(t')dt' + \ln\beta \lim_{t\to+\infty} \frac{N(t)}{t}$$

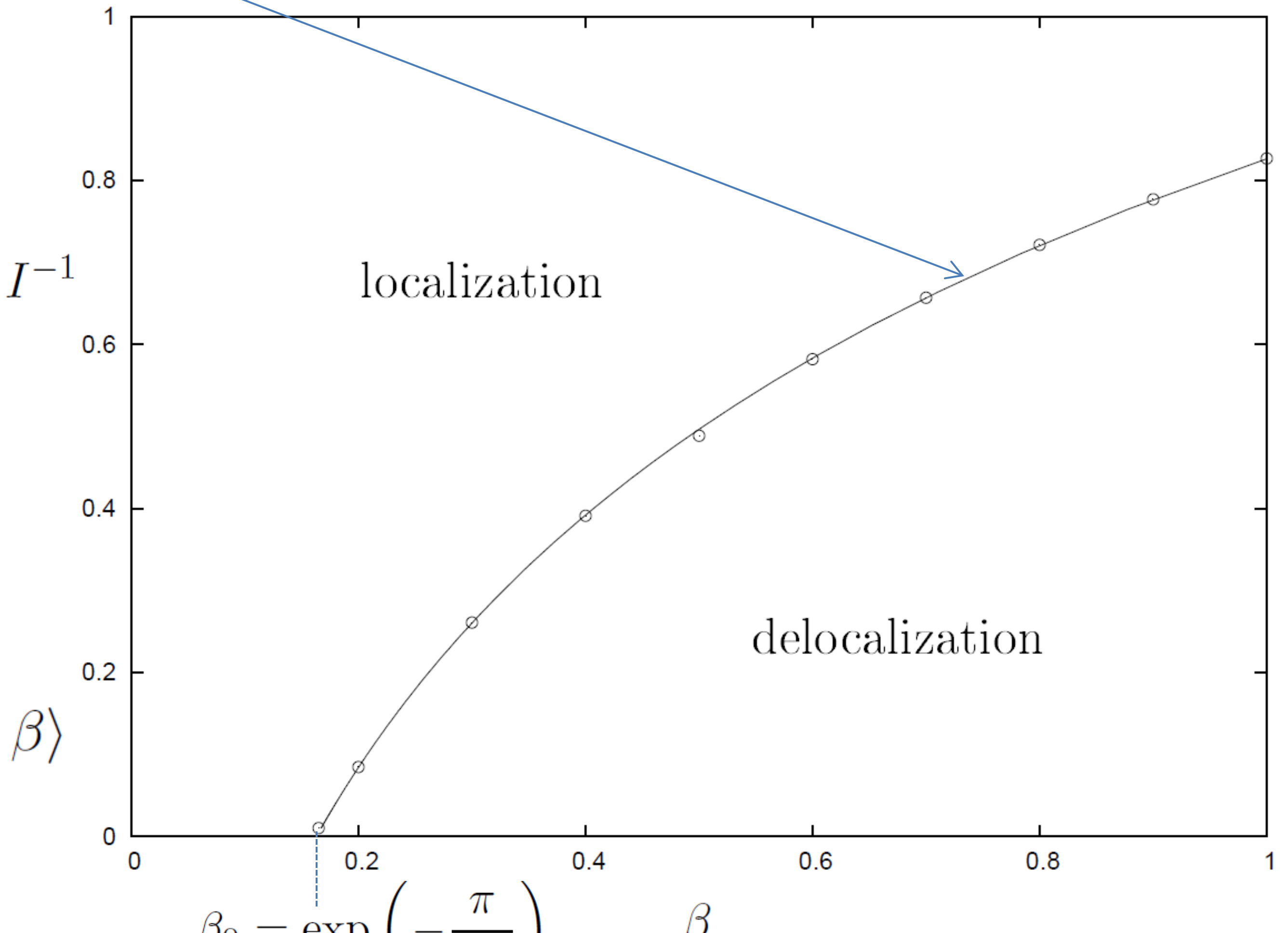
$$\lambda(\beta, I) = \lambda(1, I) + F(I)\ln\beta$$



Phase transition

$$\lambda(1, I) + \ln \beta F(I) = 0$$

$$\ln \beta_c(I) = -\frac{\lambda(I)}{F(I)} = -\frac{\sqrt{\pi}}{2} \int_0^{+\infty} dx \frac{x - I^{-1}}{\sqrt{x}} \exp\left(-\frac{x^3}{12} + \frac{x}{4I^2}\right)$$

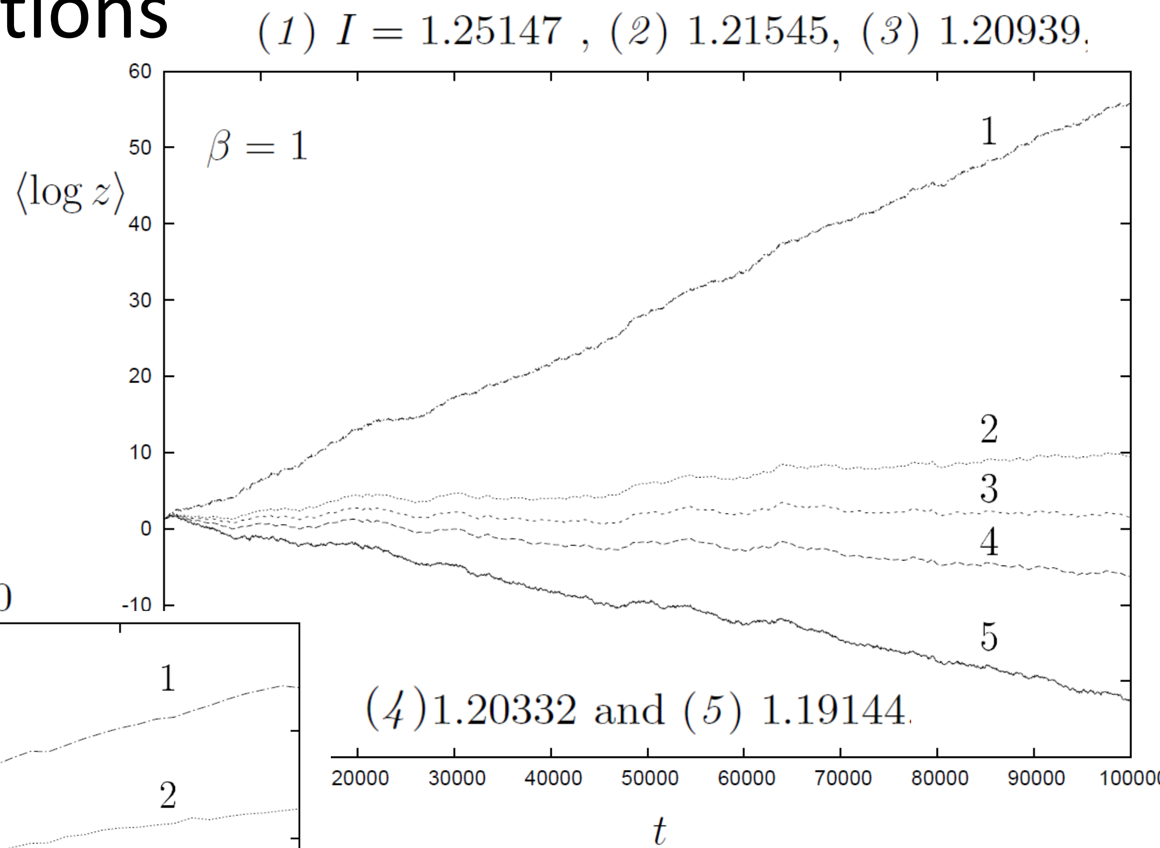


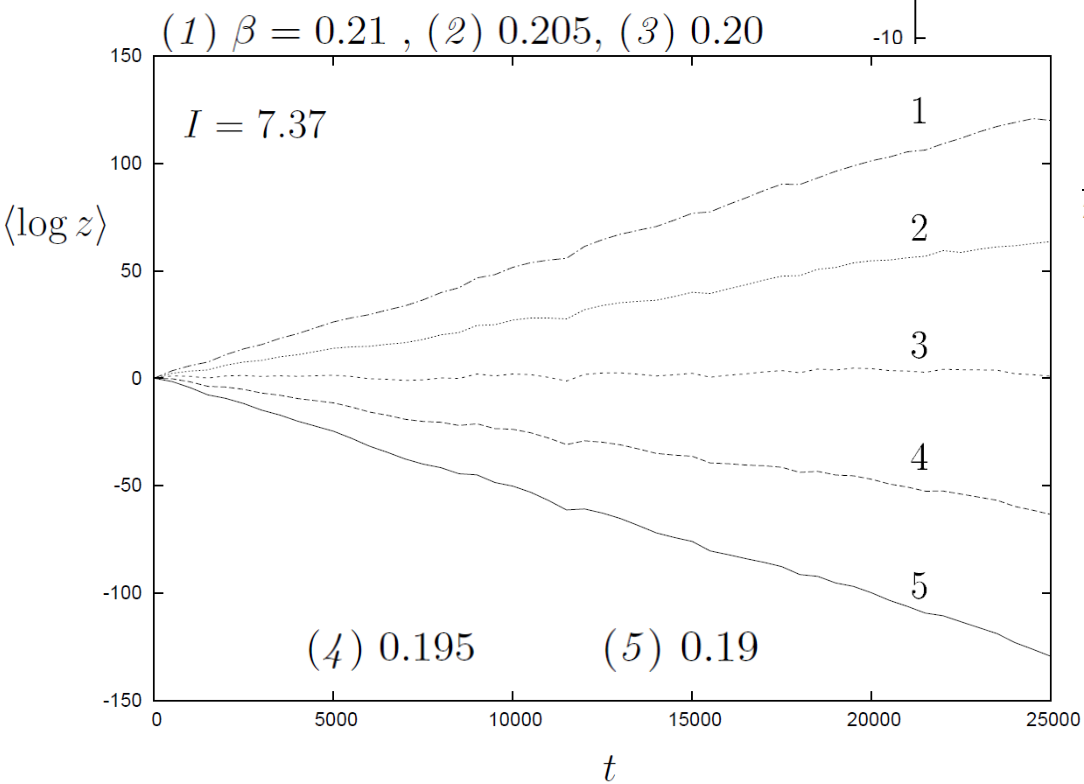
$$\beta_0 = \exp\left(-\frac{\pi}{\sqrt{3}}\right)$$

Rough wall p(β)

$$\tilde{\lambda} = \lambda(I) + F(I)\langle \ln \beta \rangle$$

Numerical simulations





Phase transition upon the change of $St = \frac{\tau}{\tau_c}$

$$\tau_c, \tau \ll \tilde{\tau}$$

$$\partial_t n = \frac{\mathrm{St}}{1+\mathrm{St}} \partial_z^2 [\kappa(z) n] + \frac{1}{1+\mathrm{St}} \partial_z [\kappa(z) \partial_z n]$$

$$\frac{\partial n}{\partial t} = -\frac{\partial J}{\partial z}$$

$$J = -\kappa(z) \frac{\partial n}{\partial z} - n \frac{\partial \kappa}{\partial z} \frac{St}{1+St}$$

$$\kappa(z) = z^m$$

$$n(z) = -\frac{J(1+St) z^{1-m}}{St + 1 - m}$$

$$\kappa(z) = \mu z^2$$

$$\langle \ln z(t) \rangle = \int n(z,t) \ln z \, dz$$

$$\partial_t \langle \ln z(t) \rangle = \lambda = \mu \frac{1 - \mathrm{St}}{1 + \mathrm{St}}$$

Conclusions

- Nonlinear systems driven by a thermal noise can be non-equilibrium i.e. have a current in phase space. In the Lagrangian frame, they are in equilibrium (to compare with turbulence tomorrow)

- Inertial particles undergo phase transitions upon the change of inertia parameters and restitution coefficient.

Dust does not always settle on the walls.

Fluid Mechanics

The multi-disciplinary field of fluid mechanics is one of the most actively developing fields of physics, mathematics and engineering. In this book, the fundamental ideas of fluid mechanics are presented from a physics perspective.

Using examples taken from everyday life, from hydraulic jumps in a kitchen sink to Kelvin–Helmholtz instabilities in clouds, the book provides readers with a better understanding of the world around them. It teaches the art of fluid-mechanical estimates and shows how the ideas and methods developed to study the mechanics of fluids are used to analyse other systems with many degrees of freedom in statistical physics and field theory.

Aimed at undergraduate and graduate students, the book assumes no prior knowledge of the subject and only a basic understanding of vector calculus and analysis. It contains 32 exercises of varying difficulties, from simple estimates to elaborate calculations, with detailed solutions to help readers understand fluid mechanics.

Gregory Falkovich is a Professor in the Department of Physics of Complex Systems, Weizmann Institute of Science. He has researched in plasma, condensed matter, fluid mechanics, statistical and mathematical physics and cloud physics and meteorology, and has won several awards for his work.

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FALKOVICH Fluid Mechanics

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Fluid Mechanics

A Short Course for Physicists

GREGORY FALKOVICH

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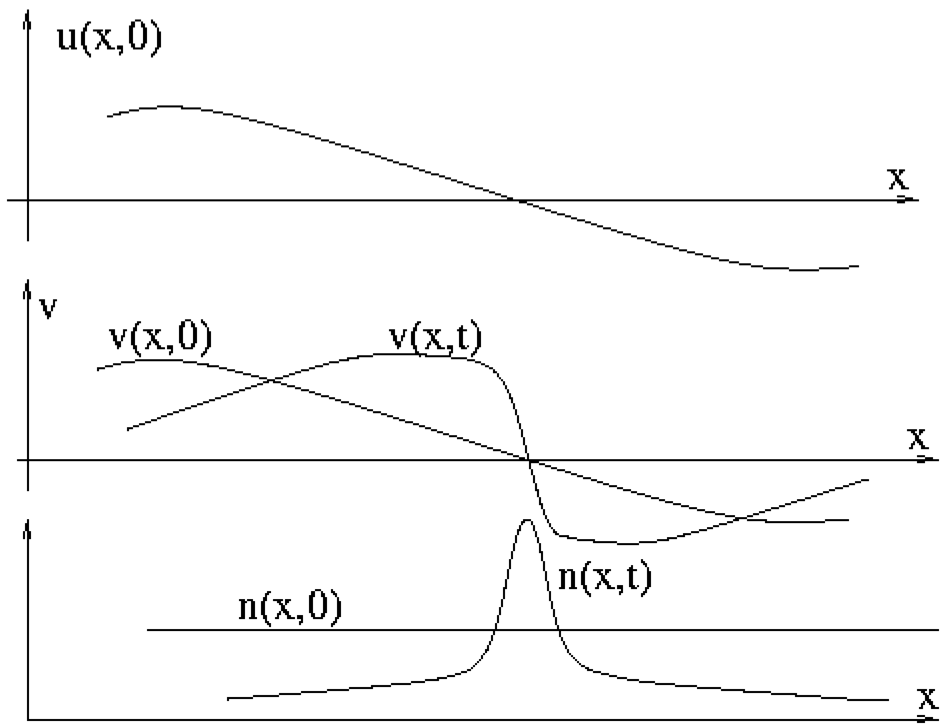
$$\sigma_{ii} = \partial v_i/\partial x_i < -\tau^{-1} \Rightarrow \sigma_{ii} = (t_0 - t)^{-1} \propto n(q,t)$$



$$\tilde{\gamma}_k = t^{-1} \ln\langle |R|^k \rangle$$