Clustering of gyrotactic algae in a turbulent ocean

Filippo De Lillo
Dipartimento di Fisica and INFN, Università di Torino

Turbulence drives microscale patches of motile phytoplankton
Durham, *et al*. Nature Comm. **4**, 2148 (2013).
Turbulent Fluid Acceleration Generates Clusters of Gyrotactic Microorganisms
De Lillo, *et al*. Phys. Rev. Lett. **112**, 044502 (2014).
Gyrotactic trapping in laminar and turbulent Kolmogorov flow
Santamaria, *et al*., Phys. Fluids **26**, 111901 (2014).

Flowing Matter – COST Action MP1305

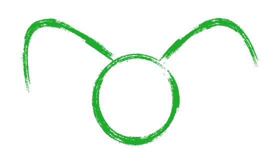
Flowing Matter across the scales
Rome
XXIV-XXVII March
A.D. MMXV

PEOPLE INVOLVED

FD, Guido Boffetta
Dipartimento di Fisica and INFN, Università di Torino
Massimo Cencini
ISC-CNR Roma

for small scale clustering (1st part)
Roman Stocker, Michael Barry
Ralph M. Parsons Laboratory, Department of Civil and Environmental Engineering, MIT
William Durham
Department of Zoology, University of Oxford
Eric Climent
Institut de Mécanique des Fluides, Université de Toulouse INPT-UPS-CNRS

for thin gyrotactic trapping (2nd part)
Francesco Santamaria
Dipartimento di Fisica and INFN, Università di Torino



…and Chlamy

Motivations

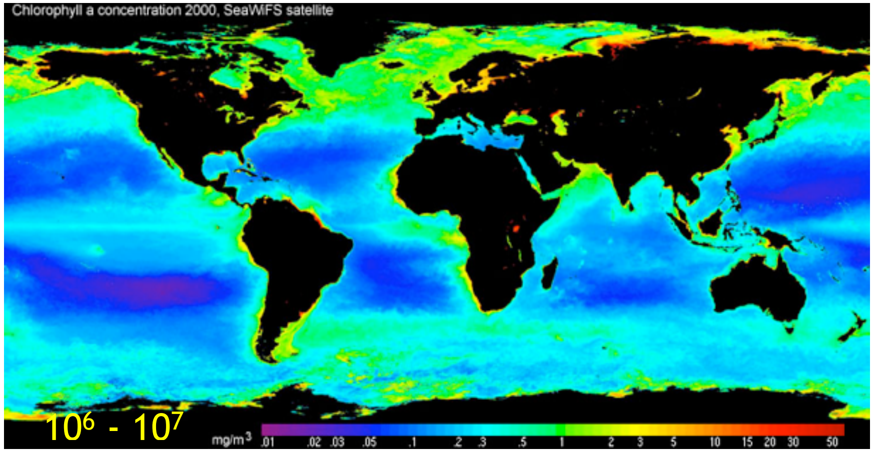
Phytoplankton is composed by one-celled organisms able to perform phototaxis
It is at the basis of the oceanic food web
It is the source of about 50% of the oxygen "produced" on the earth
It is fundamental for carbon cycle

Many phytoplankters are able to swim as a way to control their position in the water column (some others control their buoyancy)
Doing so allows them to stay in the "photic" layer…
Phytoplankton is "patchy" at several scales which affects:
- exploitation of nutrients
- predation
- mating (when reproducing sexually)
- access to light (mutual shading…)

Large scale patchiness

Small scale patchiness

Plankton is "patchy" over many scales



Phytoplankton (chlorophyll) distribution on a global scale

Large scale

"ecological" causes influenced by flow (e.g., presence and distribution of nutrients)

Algal blooms (*N. scintillans* in NZ)

Still, some large formations may have other origins (more about this later…)

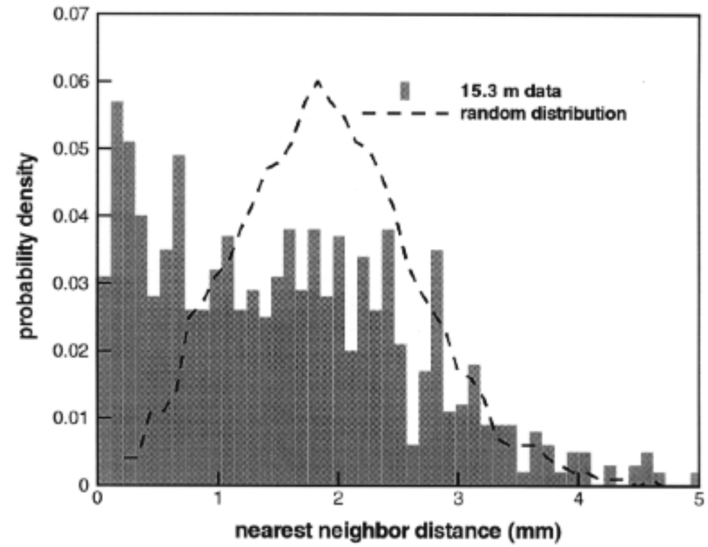


Figure 10. Measured nearest-neighbour distances and expected values from a random distribution.

E.Malkiel, O.Alquaddoomi, J.Katz
Meas. Sci. Technol. **10** (1999)

Patchiness at mm/cm-scale

Motile phytoplankton **more patchy** than non-motile one

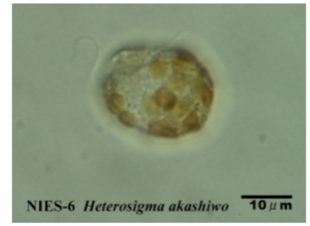
A fluid-dynamic explanation?

See also F Toschi's and I Pagonabarraga's works (and talks in this conference) for related phenomena

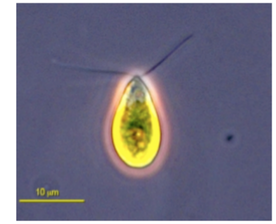
MODEL GYROTACTIC ALGAE

- "bottom heavy" swimmers, e.g. Chlamydomonas
- dilute -> no interaction, no feed-back on the fluid
- Translational and rotational diffusion negligible
- Modulus of the swimming velocity constant

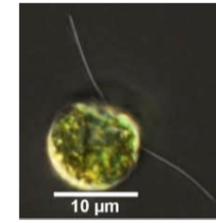
Heterosigma akashiwo



Dunaliella tertiolecta



Chlamydomonas reinhardtii

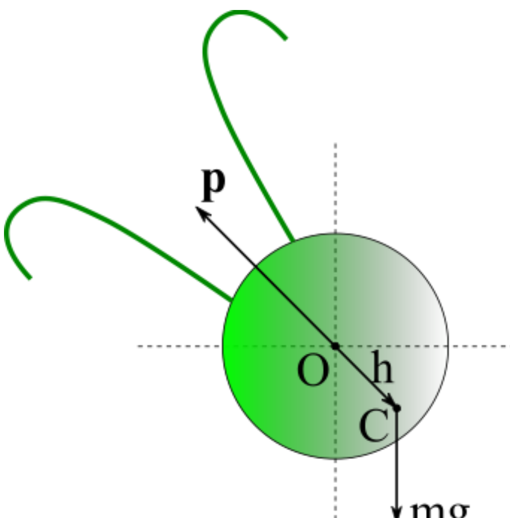


$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{u}(\boldsymbol{X}, t) + v_s \mathbf{p}$$

$$\frac{d\mathbf{p}}{dt} = \frac{1}{2B}\left[\hat{\mathbf{k}} - (\hat{\mathbf{k}} \cdot \mathbf{p})\mathbf{p}\right] + \frac{1}{2}\boldsymbol{\omega} \times \mathbf{p} \qquad |\mathbf{p}| = 1$$

GRAVITAXIS

fluid vorticity



$$v_s \simeq 100\mu\mathrm{m/s}$$

$$B = \frac{3\nu}{hg} \simeq 1 \div 6\,\mathrm{s}$$

ν kinematic viscosity

g gravitational acceleration

JO Kessler (Nature 1985)

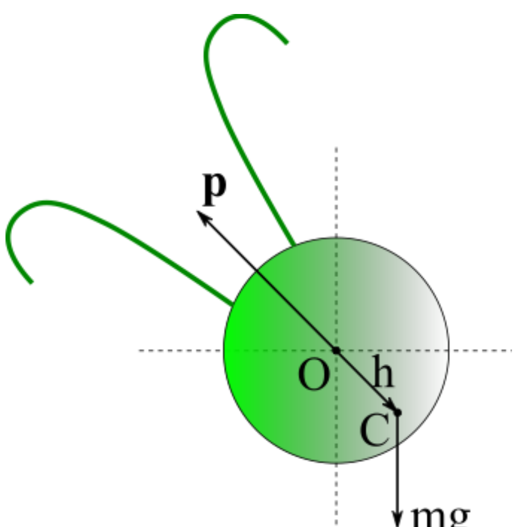
T Pedley and JO Kessler (Annu. Rev. Fluid. Mech. 1992)

Model gyrotactic algae

- "bottom heavy" swimmers, e.g. Chlamydomonas
- dilute -> no interaction, no feed-back on the fluid
- Translational and rotational diffusion negligible
- Modulus of the swimming velocity constant

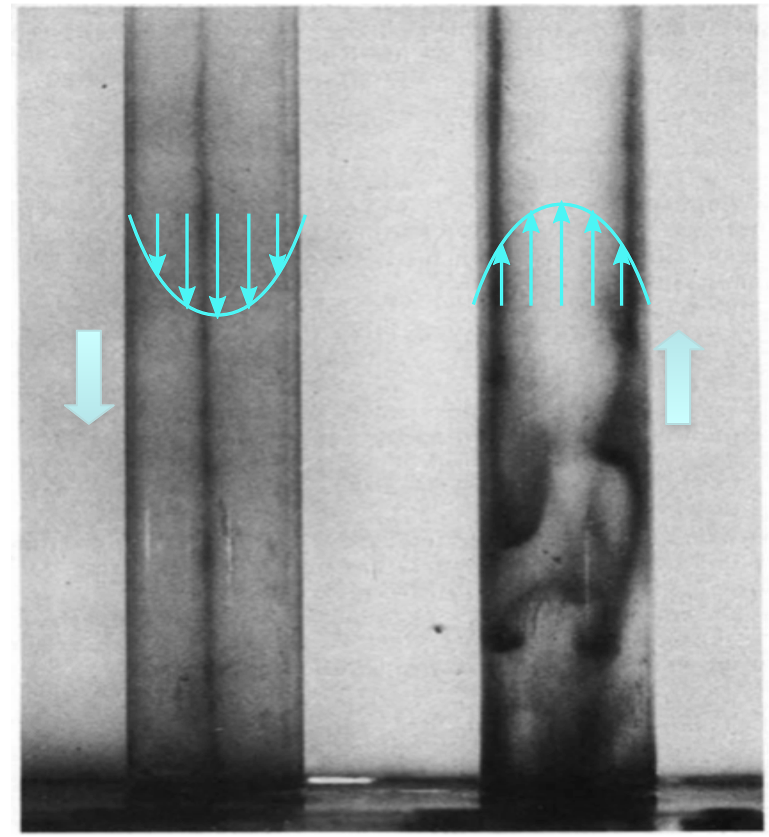
$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{u}(\boldsymbol{X}, t) + v_s \mathbf{p}$$

$$\frac{d\mathbf{p}}{dt} = \frac{1}{2B}\left[\hat{\mathbf{k}} - (\hat{\mathbf{k}} \cdot \mathbf{p})\mathbf{p}\right] + \frac{1}{2}\boldsymbol{\omega} \times \mathbf{p} \qquad |\mathbf{p}| = 1$$



$$v_s \simeq 100 \mu\mathrm{m/s}$$

$$B = \frac{3\nu}{hg} \simeq 1 \div 6\,\mathrm{s}$$



Gyrotactic focusing in pipe flows

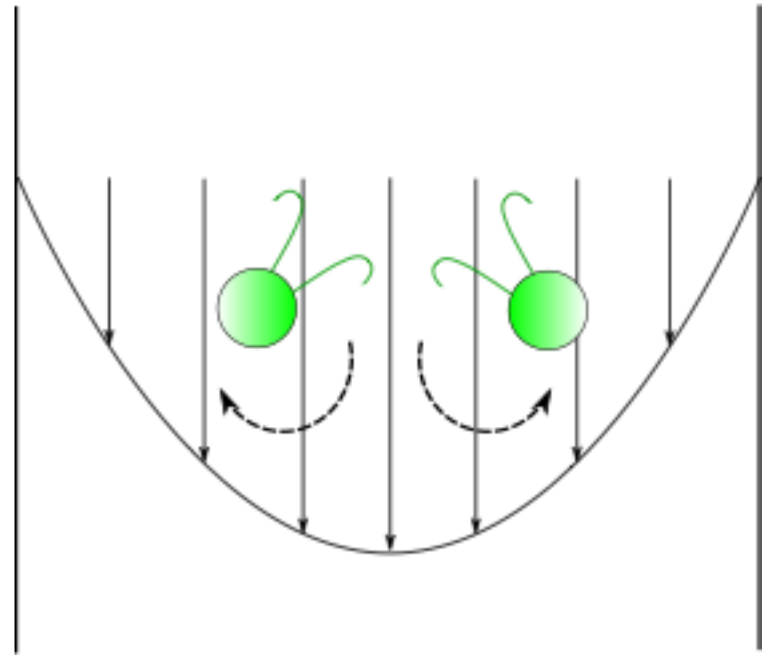
*JO Kessler, Nature **313**, 218 (1985)*

JO Kessler (Nature 1985)

T Pedley and JO Kessler (Annu. Rev. Fluid. Mech. 1992)

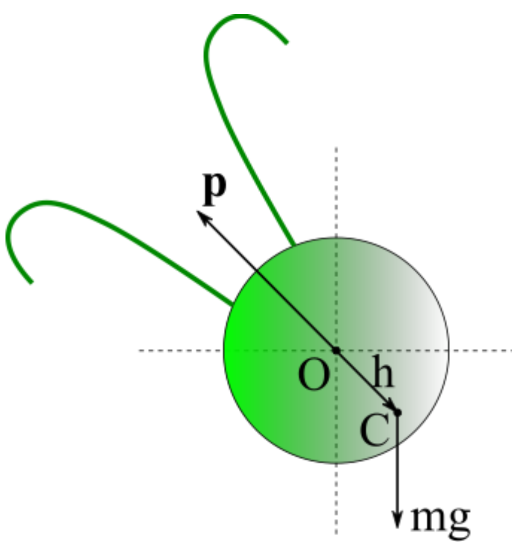
Model gyrotactic algae

- "bottom heavy" swimmers, e.g. Chlamydomonas
- dilute -> no interaction, no feed-back on the fluid
- Translational and rotational diffusion negligible
- Modulus of the swimming velocity constant



$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{u}(\boldsymbol{X}, t) + v_s \mathbf{p}$$

$$\frac{d\mathbf{p}}{dt} = \frac{1}{2B}\left[\hat{\mathbf{k}} - (\hat{\mathbf{k}}\cdot\mathbf{p})\mathbf{p}\right] + \frac{1}{2}\boldsymbol{\omega}\times\mathbf{p} \qquad |\mathbf{p}| = 1$$



$$v_s \simeq 100\mu\mathrm{m/s}$$

$$B = \frac{3\nu}{hg} \simeq 1 \div 6\,\mathrm{s}$$

Gyrotactic focusing in pipe flows

*JO Kessler, Nature **313**, 218 (1985)*

JO Kessler (Nature 1985)

T Pedley and JO Kessler (Annu. Rev. Fluid. Mech. 1992)

IN THIS TALK...

Small scale patchiness

scales: mm to cm

Role of
turbulence in:

Thin Phytoplankton Layers
(formation and dissolution)

scales: some 10 cm to some m
vertically

up to some km
horizontally

IN THIS TALK...

Small scale patchiness

scales: mm to cm

Role of turbulence in:

Thin Phytoplankton Layers (formation and dissolution)

scales: some 10 cm to some m vertically

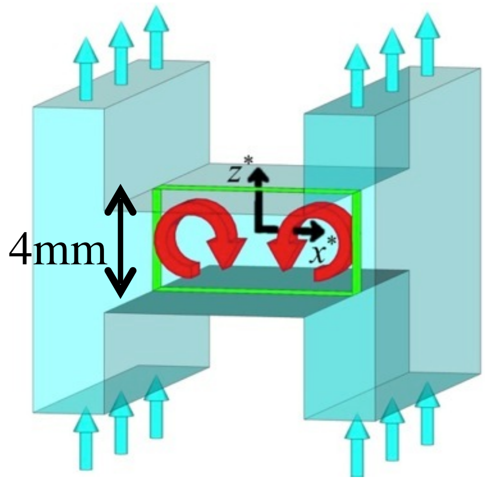
up to some km horizontally

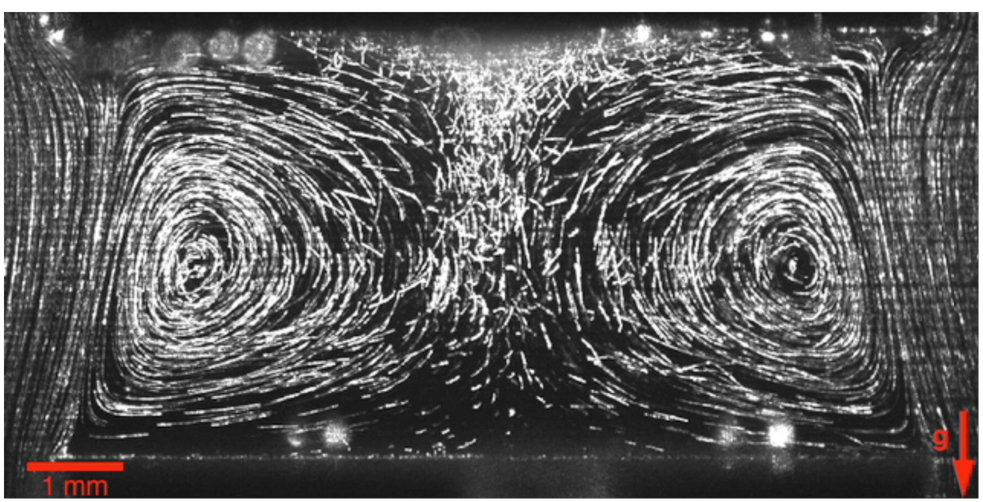
Experiment in vortical flow

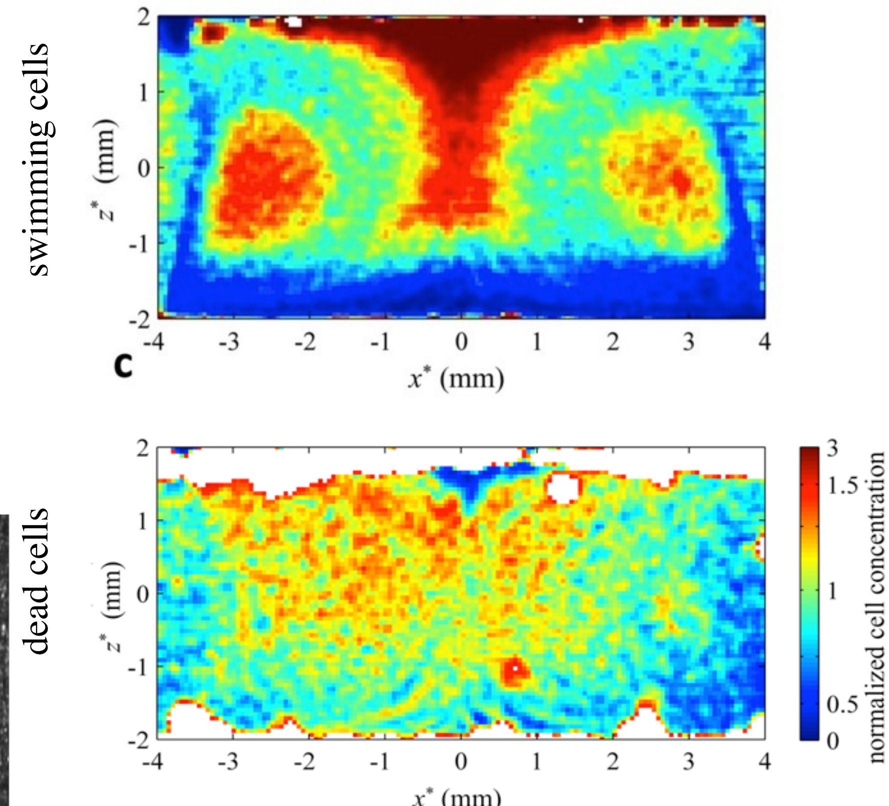
Durham *et al*. Nature Comm. **4**, 2148 (2013)

Steady vortical flow in microfluidic apparatus (at MIT)

Density of *Heterosigma akashiwo*







Cells accumulate in the downwelling region and slightly in the vortex cores

What happens in real turbulent flows ?

numerical simulation

DIRECT NUMERICAL SIMULATIONS

Durham *et al*. Nature Comm. **4**, 2148 (2013)

Simulation of the complete set of equations

Resolutions up to 256^3

Three dimensionless numbers

$$\mathrm{Re} = \frac{UL}{\nu} \qquad \Phi = \frac{v_s}{u_\eta} \qquad \Psi = B\omega_{\mathrm{rms}}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{f}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t) + v_s\mathbf{p}$$

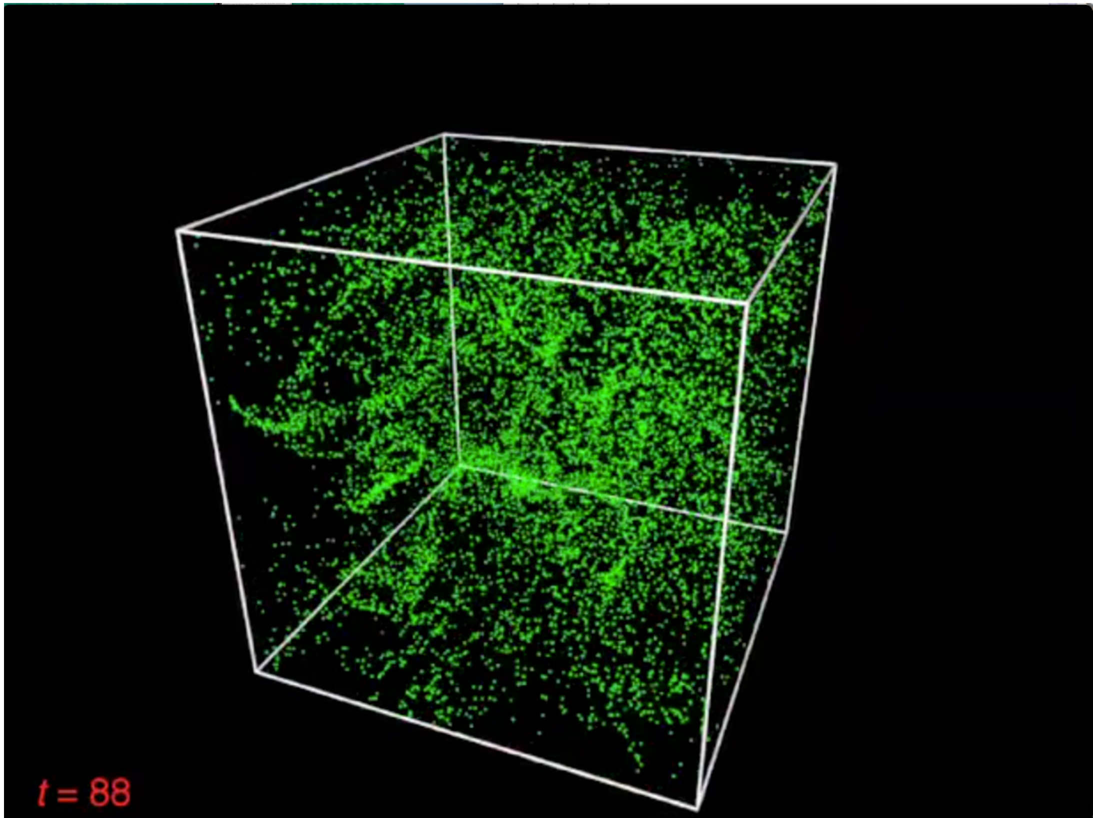
$$\frac{d\mathbf{p}}{dt} = \frac{1}{2B}\left[\hat{\mathbf{k}} - (\hat{\mathbf{k}}\cdot\mathbf{p})\mathbf{p}\right] + \frac{1}{2}\boldsymbol{\omega}\times\mathbf{p}$$

A dissipative dynamical system

Phase space contraction rate

$$\sum_{i=1}^{d} \frac{\partial \dot{X}_i}{\partial X_i} + \frac{\partial\, \dot{\mathrm{p}}_i}{\partial\, \mathrm{p}_i} = -\frac{d-1}{2v_o} g p_z$$

Clustering on a **fractal set**



Direct Numerical Simulations

Typical conditions in the ocean mixing layer $\epsilon = 10^{-7}\, m^2\, s^{-3}$

$$\eta = (\nu^3/\epsilon)^{1/4} \simeq 2\, mm$$

$$\tau_k = (\nu/\epsilon)^{1/2} \simeq 3\, s$$

$$u_k = (\nu\epsilon)^{1/4} \simeq 0.5\, mm/s$$

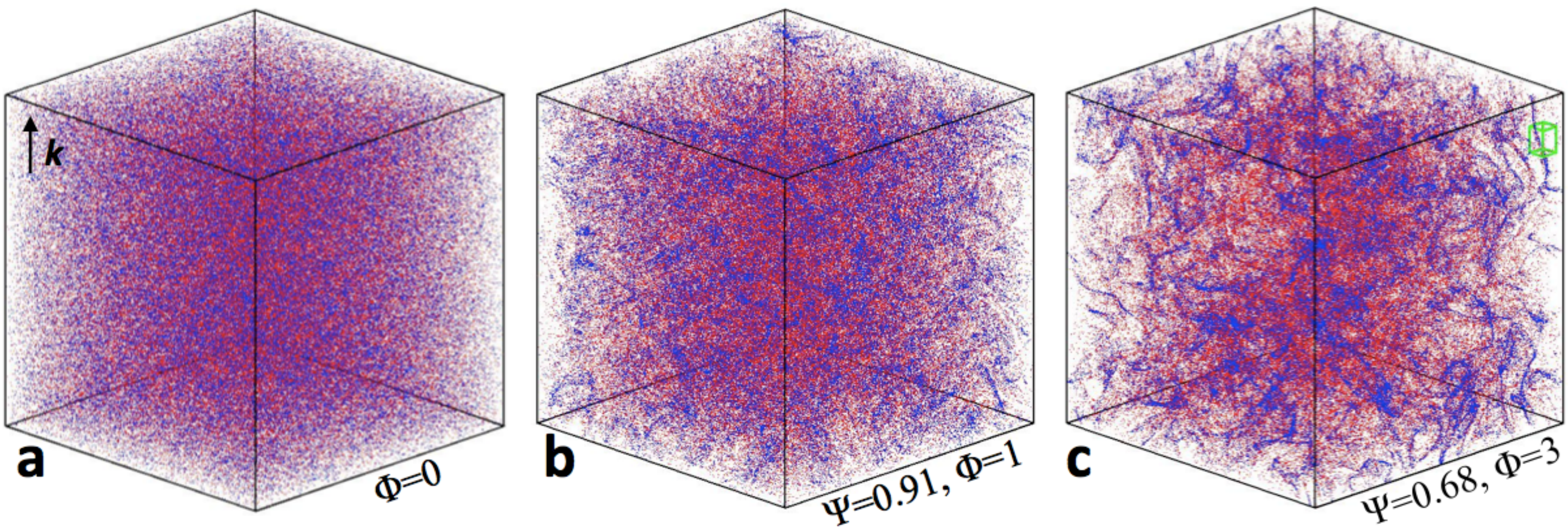
$$\Phi = \frac{v_s}{u_\eta} \simeq 0.4$$

$$\Psi = \omega_{\rm rms} B \simeq 0.3$$

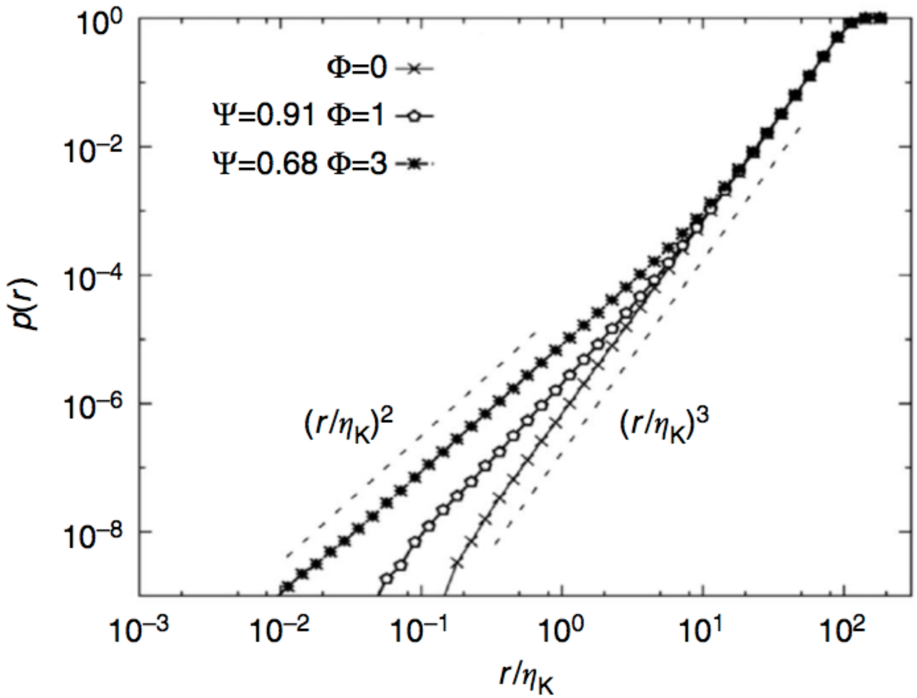
$Re_\lambda = 65$

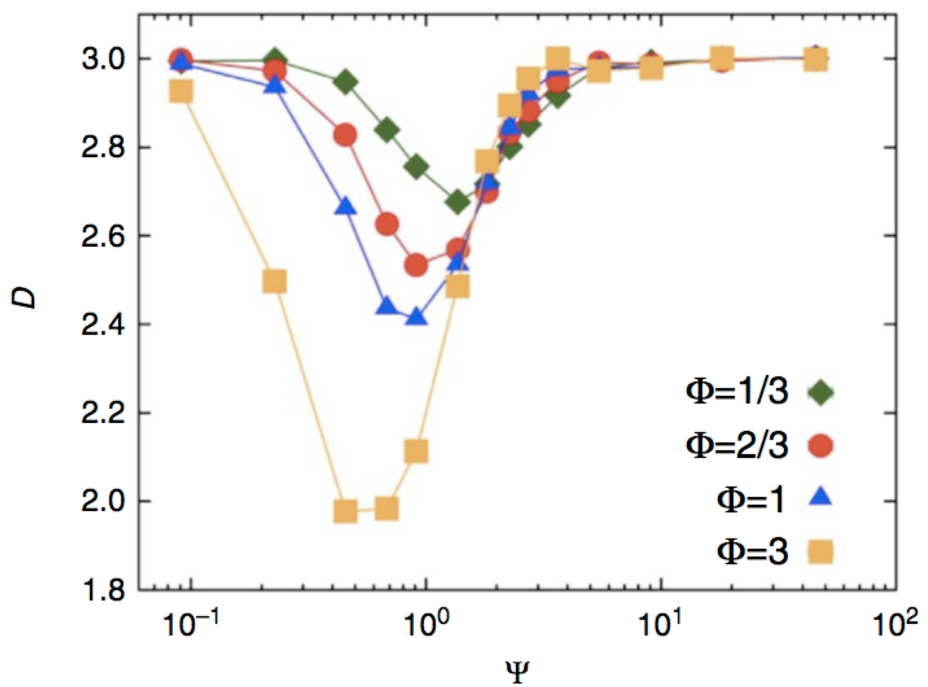
10^6 cells

$n > 2\langle n \rangle$



Fractal clustering





p(r): probability to have two cells closer than *r*

p(r) ~ r^D , *D*: correlation dimension

Clustering is maximum where D is minimum

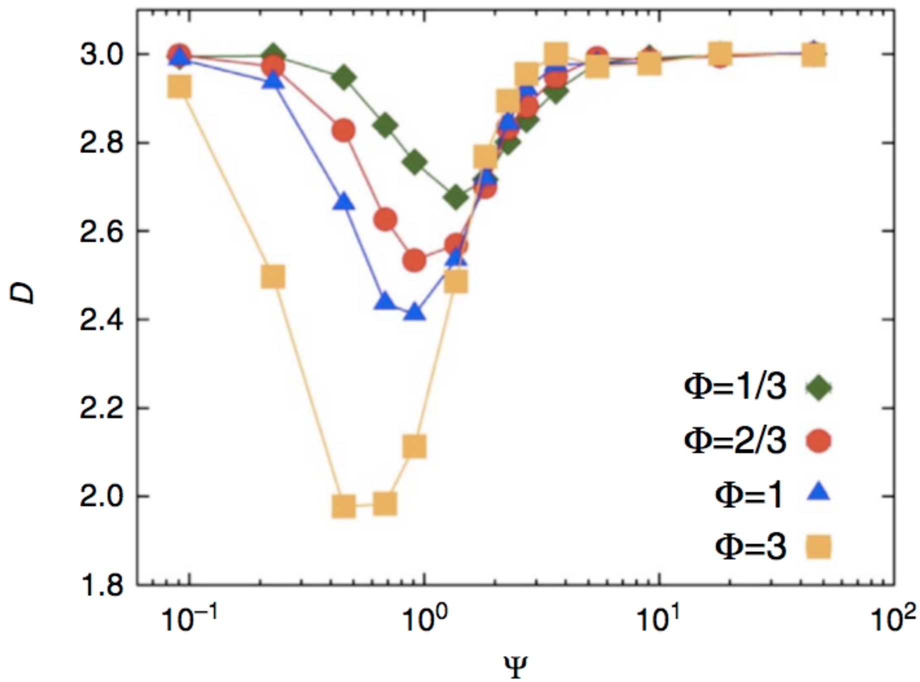
$$\Phi = \frac{v_s}{u_\eta}$$

$$\Psi = B\omega_{\rm rms}$$

Fractal clustering

D: correlation dimension

Clustering is maximum (D minimum) for $\Psi \simeq 1$ and increases with Φ



Homogeneous distribution in both limits

$\Psi \to 0$ (vertical swimming)

and

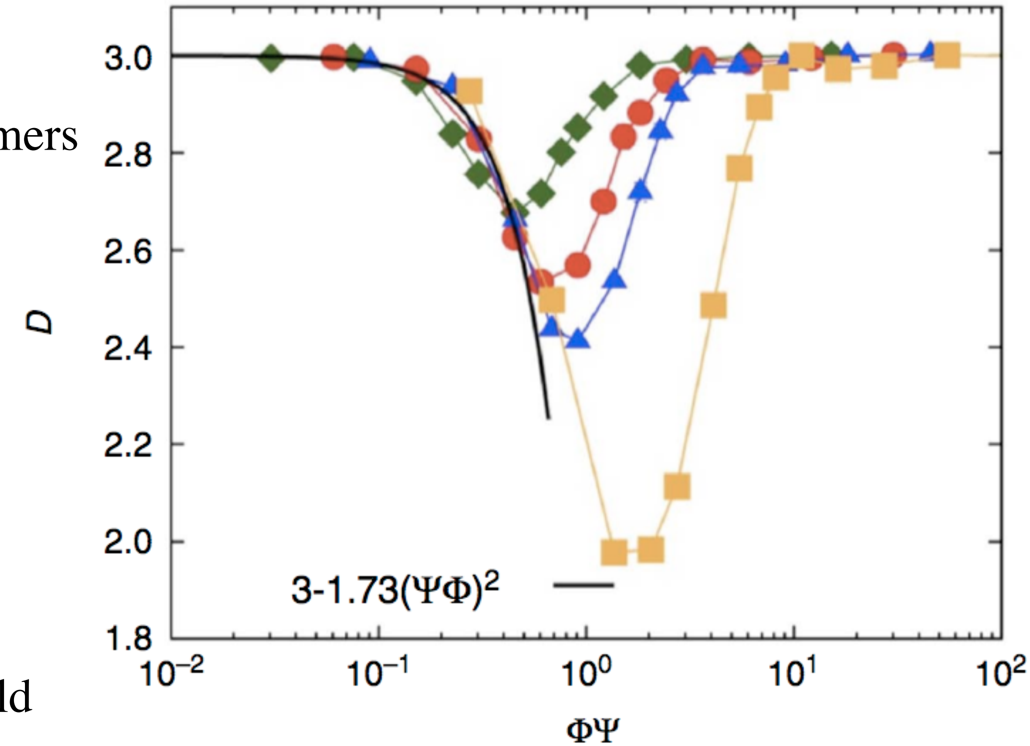
$\Psi \to \infty$ (random directions)

$$\Phi = \frac{v_s}{u_\eta}$$

$$\Psi = B\omega_{\rm rms}$$

See also Bernhard Mehlig’s talk and his paper
K. Gustavsson, *et al*. arXiv:1501.02386 [physics.flu-dyn] (2015)

Prediction for small Ψ (fast orientation)



Dimensionless form of equations for swimmers

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x},t) + \Phi\mathbf{p}$$

$$\frac{d\mathbf{p}}{dt} = \frac{1}{2\Psi}\left[\mathbf{k} - (\mathbf{k}\cdot\mathbf{p})\mathbf{p}\right] + \frac{1}{2}\omega\times\mathbf{p}$$

For small Ψ, at first order we have

$$\mathbf{p} = (\Psi\omega_y, -\Psi\omega_x, 1)$$

passive tracers in an effective velocity field

$$\mathbf{v} = \mathbf{u} + \Phi\mathbf{p}$$

with divergence

$$\nabla\cdot\mathbf{v} = \Phi\nabla\cdot\mathbf{p} = -\Psi\Phi\nabla^2 u_z$$

Fractal codimension and therefore $\quad 3 - D \propto (\Phi\Psi)^2$

$$\boxed{D = 3 - a(\Phi\Psi)^2}$$

For a weakly compressible flow

$$\boldsymbol{v} = \boldsymbol{u} + \alpha\boldsymbol{w} \qquad \nabla\cdot\boldsymbol{w} \neq 0 \qquad \nabla\cdot\boldsymbol{u} = 0$$

$$d - D \propto \alpha^2$$

G Falkovich, A Fouxon, MG Stepanov, *Nature* **419**, 151-154 (2002).
I Fouxon, *Phys. Rev. Lett.* **108**, 134502 (2012).

WHERE DO CELLS CLUSTER ?

Swimmers as tracers transported by a weakly compressible flow **v**

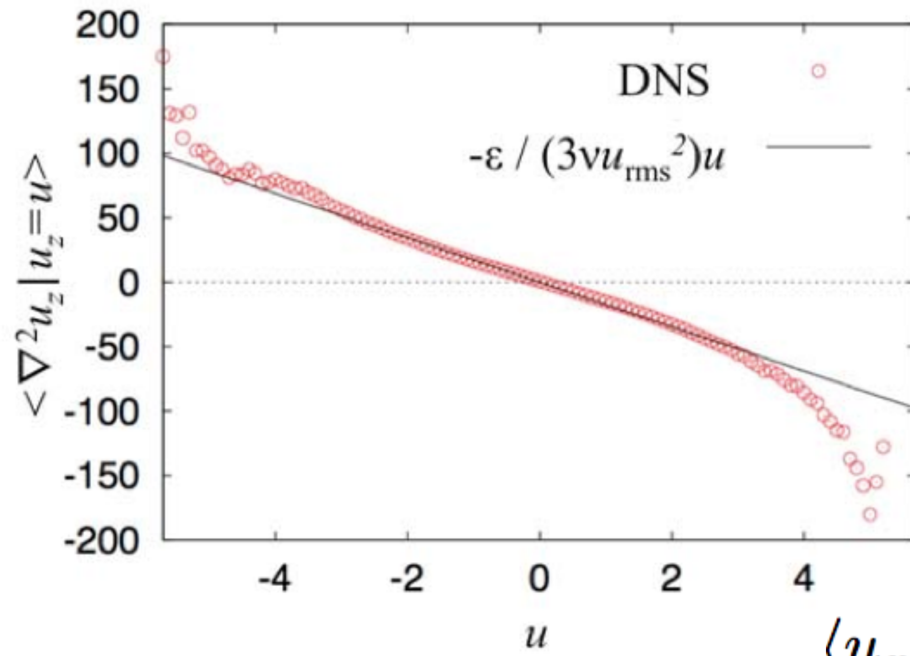
$$\nabla \cdot \mathbf{v} = \Phi \nabla \cdot \mathbf{p} = -\Psi \Phi \nabla^2 u_z$$

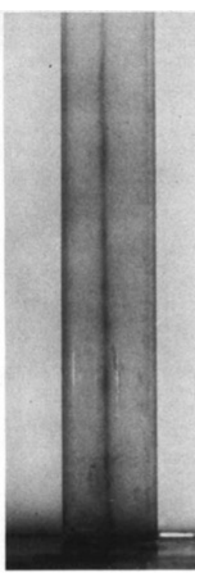
and concentrate on regions where $\nabla^2 u_z > 0$

In homogeneous, isotropic turbulence

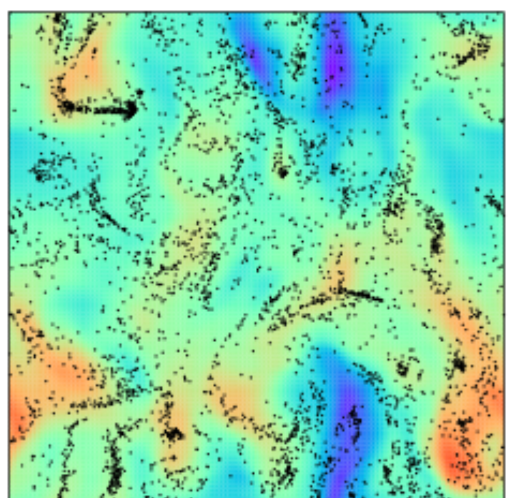
$$\epsilon = \nu \langle (\nabla \mathbf{u})^2 \rangle = -3\nu \langle u_z \nabla^2 u_z \rangle$$

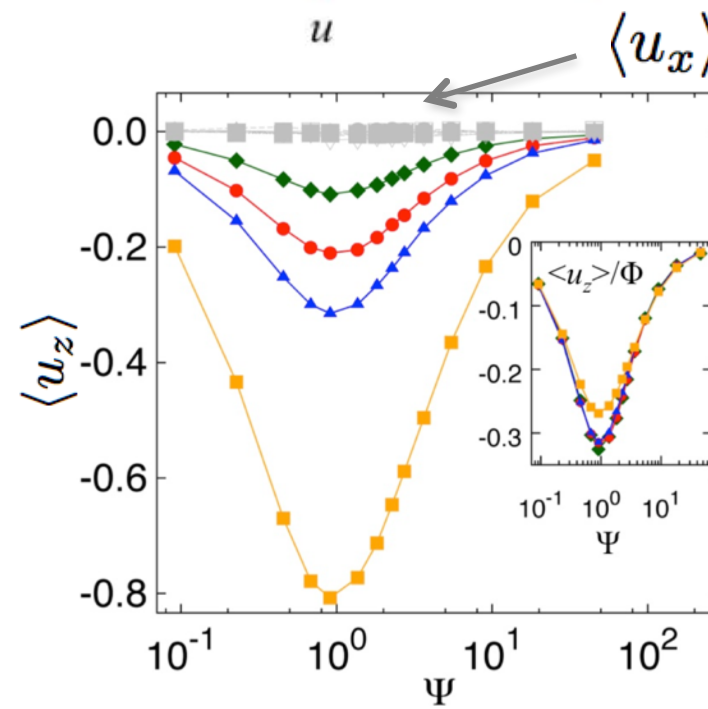
and therefore $\nabla^2 u_z > 0$ means $u_z < 0$





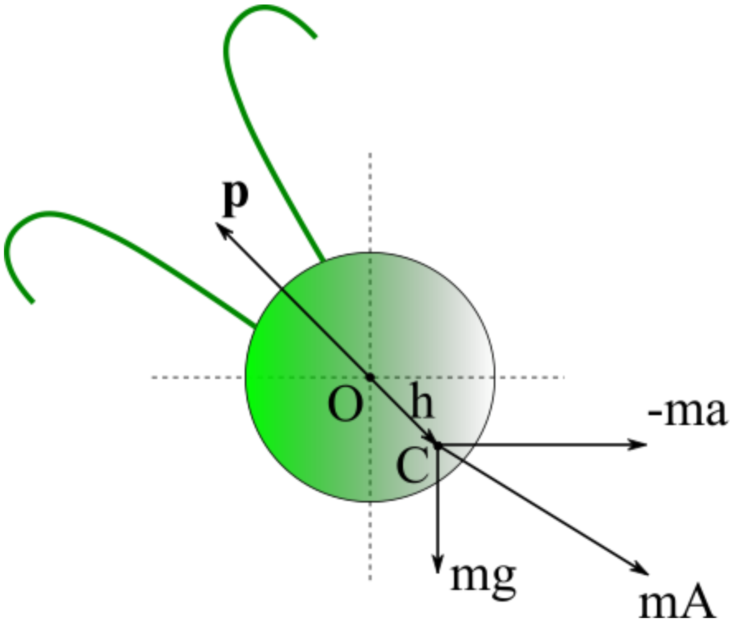
Swimming cells accumulate in downwelling regions, where $u_z < 0$





With large accelerations

De Lillo *et al*. Phys. Rev. Lett. **112**, 044502 (2014)



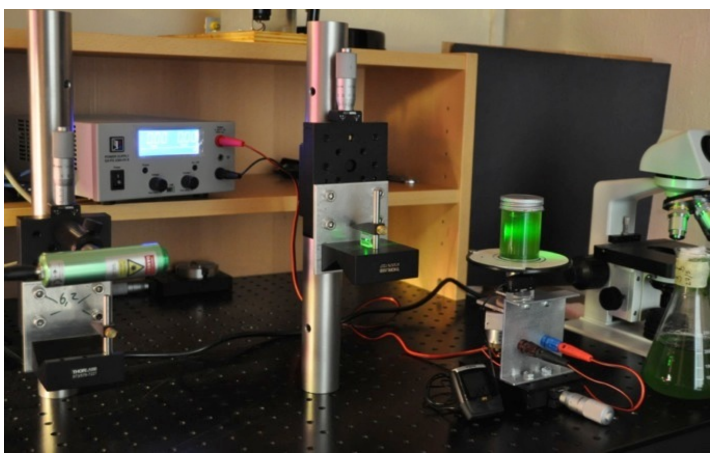
Gyrotactic algae should feel **inertial forces** too!

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{u}(\boldsymbol{X}, t) + v_s \mathbf{p}$$

$$\frac{d\mathbf{p}}{dt} = -\frac{1}{2v_0}\left[\boldsymbol{A} - (\boldsymbol{A}\cdot\mathbf{p})\mathbf{p}\right] + \frac{1}{2}\boldsymbol{\omega}\times\mathbf{p}$$

$$\boldsymbol{A} = \boldsymbol{g} - \boldsymbol{a}$$

$$v_0 = 3\nu/h \qquad B = \frac{v_0}{g} \simeq 1 \div 6\,\mathrm{s}$$



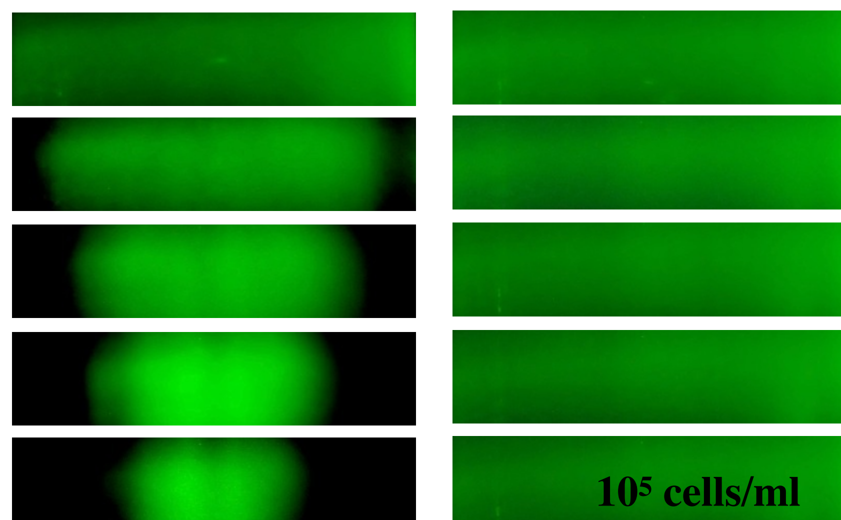
Experiment in a rotating tank

r=2 cm
f=5 Hz

C. augustae

alive | dead

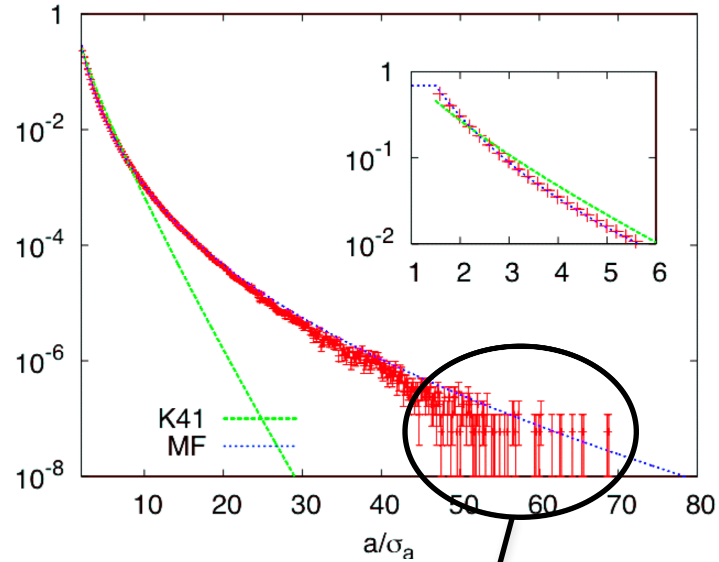
t=30 s
t=120 s
t=180 s
t=240 s
t=270 s

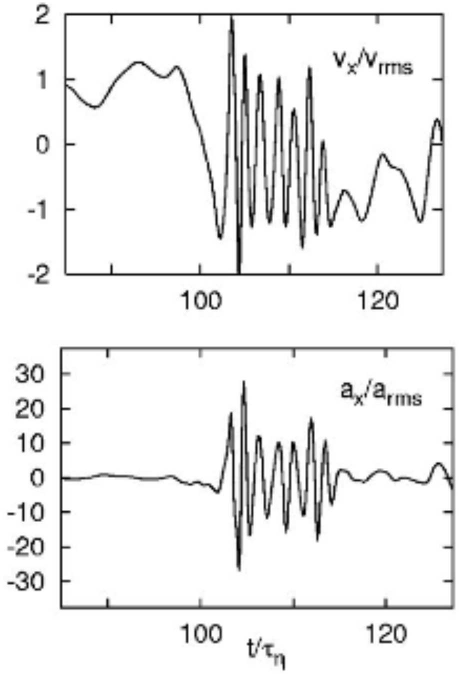


Accelerations in turbulence

Trapping of particles in small scale vortices

Frequency in vortex $\simeq \tau_\eta^{-1}$
Trapping time $10 - 20\tau_\eta$





Biferale, Boffetta, Celani, Devenish, Lanotte, Toschi
Phys. Rev. Lett. **93**, 064502 (2004)

WITH LARGE ACCELERATIONS

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{u}(\boldsymbol{X}, t) + v_s \mathbf{p}$$

$$\frac{d\mathbf{p}}{dt} = -\frac{1}{2v_0}\left[\boldsymbol{A} - (\boldsymbol{A}\cdot\mathbf{p})\mathbf{p}\right] + \frac{1}{2}\boldsymbol{\omega}\times\mathbf{p}$$

$$\boldsymbol{A} = \boldsymbol{g} - \boldsymbol{a}$$

If $\quad a_{\rm rms} \gg g$

orientation is controlled by $\quad \Psi_a = \dfrac{v_0\omega_{\rm rms}}{a_{\rm rms}}$

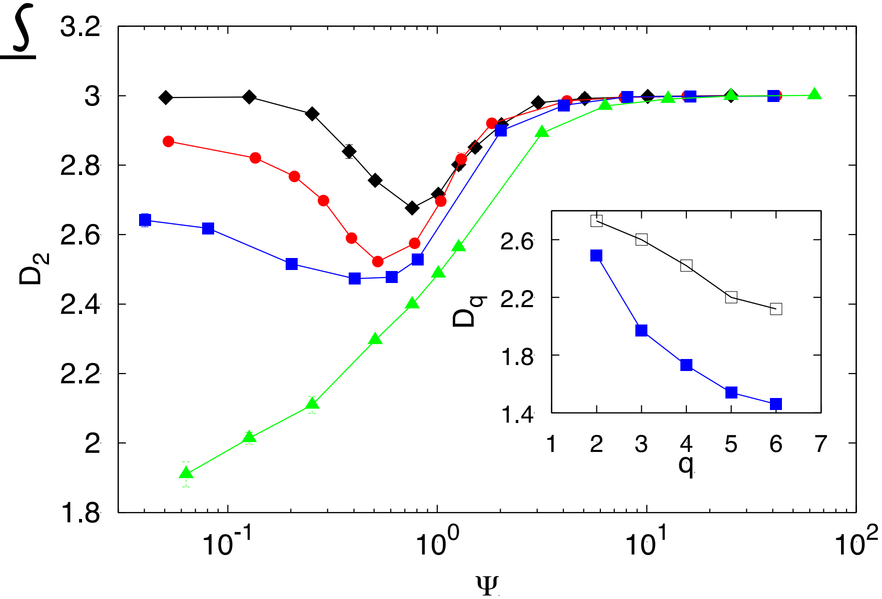
Phase-space contraction:

$$\sum_{i=1}^{d} \frac{\partial \dot{X}_i}{\partial X_i} + \frac{\partial\, \dot{\mathrm{p}}_i}{\partial\, \mathrm{p}_i} = -\frac{d-1}{2v_o}\left[\cancel{g p_z} + \boldsymbol{a}\cdot\mathbf{p}\right]$$

If $\quad \Psi \ll 1 \quad \Rightarrow \quad \mathbf{p} \to \hat{\boldsymbol{a}} = \dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}$

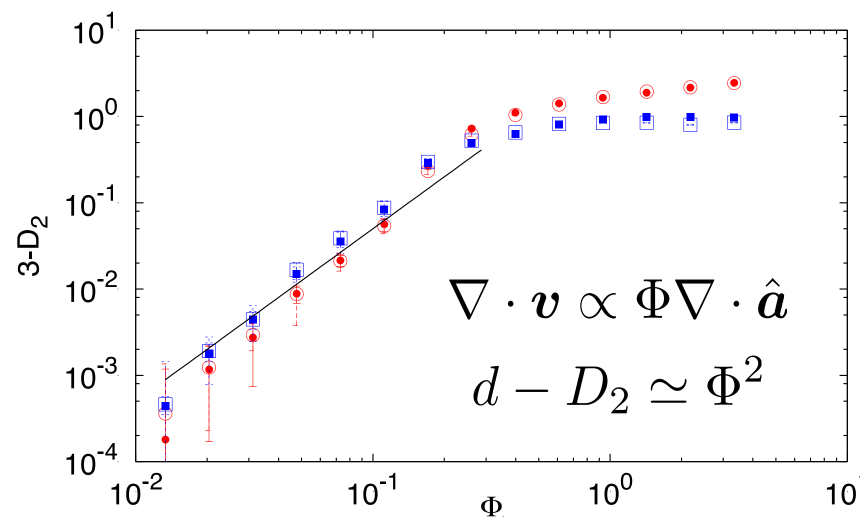
particle velocity: $\boldsymbol{v} \approx \boldsymbol{u} + v_s\hat{\boldsymbol{a}}$

Typical accelerations in the ocean are not large enough to observe this effect!



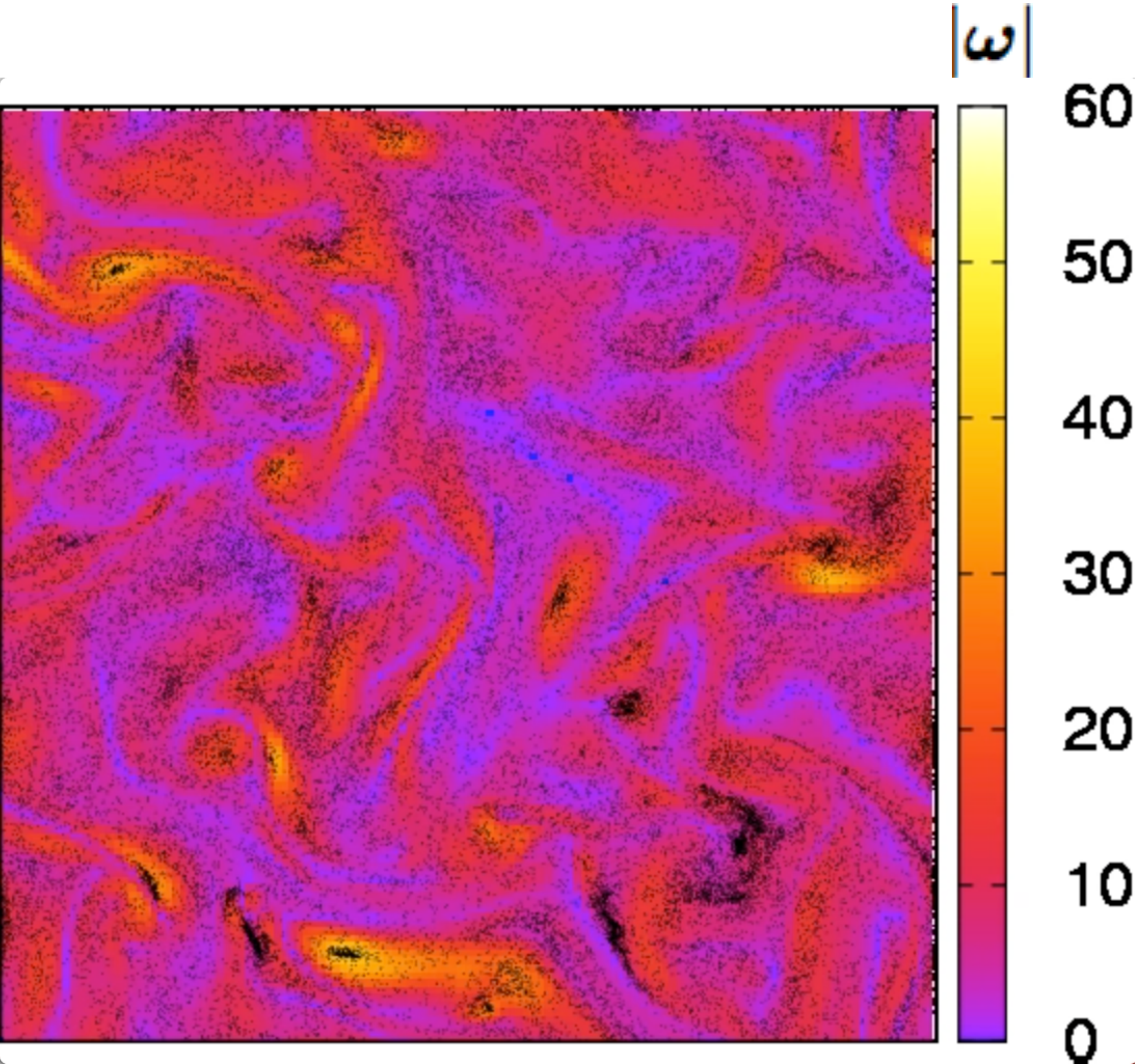
Black line ➔ $\mathrm{Re}_\lambda = 62$, $\boldsymbol{A} = \boldsymbol{g}$

$Re_\lambda = 20 \qquad Re_\lambda = 36 \qquad Re_\lambda = 62$



De Lillo, *et al.*
Phys. Rev. Lett. **112**, 044502 (2014).

With large accelerations



effective velocity:

$$\boldsymbol{v} \approx \boldsymbol{u} + v_s \hat{\boldsymbol{a}}$$

So algae should accumulate inside vortices (like bubbles….)

$\mathrm{Re}_\lambda = 62$
$\Phi = 1$
$\Psi = 1.5$

In this talk...

Small scale patchiness

scales: mm to cm

Role of turbulence in:

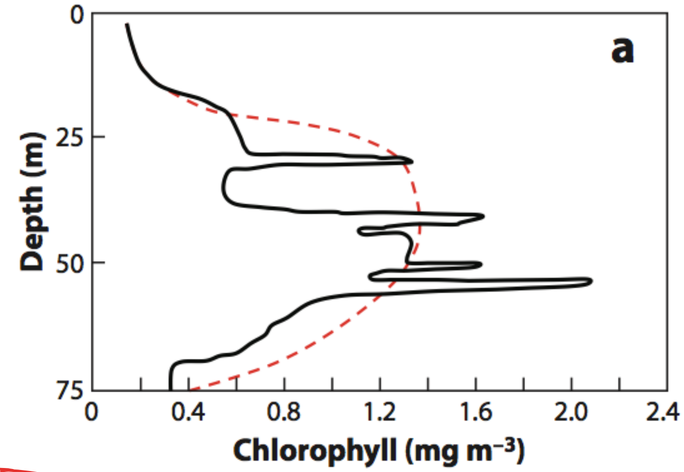
Thin Phytoplankton Layers (formation and dissolution)

scales: some 10 cm to some m vertically

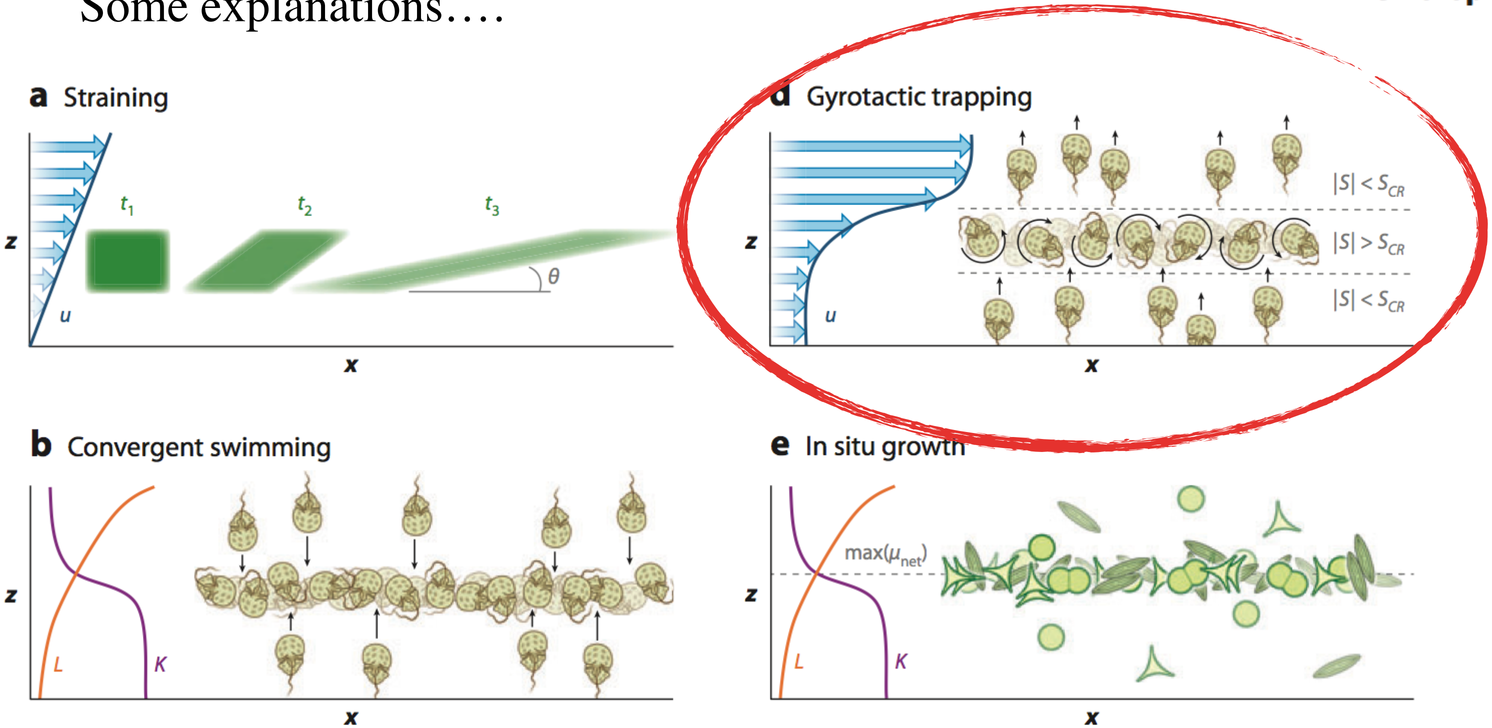
up to some km horizontally

Thin phytoplankton layers

- Thin layers of high phytoplankton concentration.
- Vertical thickness cm to m
- Horizontal size up to km
- Persistence up to days



Some explanations….



Durham and Stoker, Annu, Rev. Marine Sci. **4**, 177 (2012)

Gyrotactic trapping in shear flows

W.M. Durham, J.O. Kessler and R. Stocker, Science **323**, 1067 (2009)

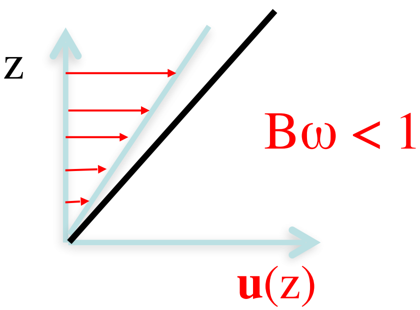
One possible explanation for the formation of layers at the bottom of the mixed layer, where vertical shear is present.

Thin layers of *Heterosigma akashiwo* near Shannon Point (WA)

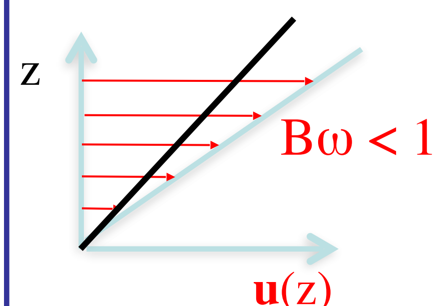
Trapping in a shear

$$\boldsymbol{u} = (u(z), 0, 0)$$

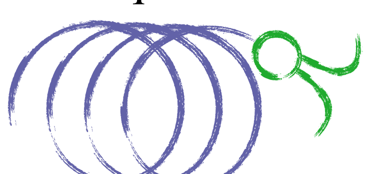
Equation for the angle
$$\dot{\theta} = \frac{1}{2}(\omega - B^{-1}\sin\theta) \qquad \omega = \partial_z u(z)$$

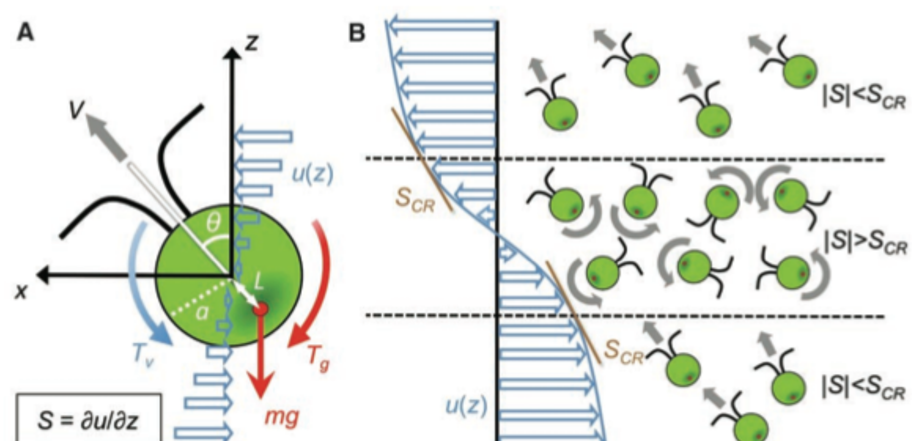


equilibrium direction
$$\sin\theta_{eq} = B\omega$$



no equilibrium…





Gyrotactic trapping in shear flows

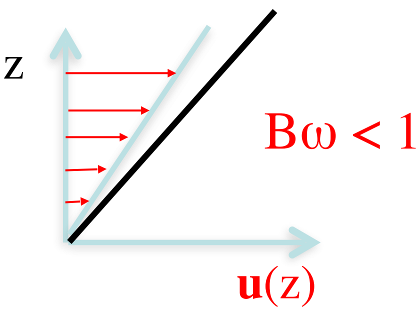
W.M. Durham, J.O. Kessler and R. Stocker, Science **323**, 1067 (2009)

One possible explanation for the formation of layers at the bottom of the mixed layer, where vertical shear is present.

Thin layers of *Heterosigma akashiwo* near Shannon Point (WA)

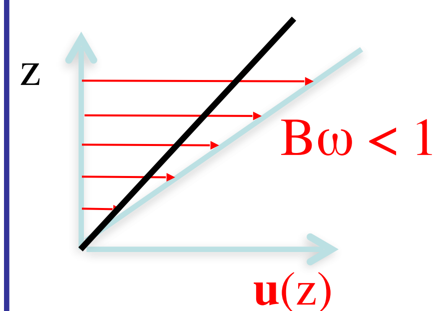
Trapping in a shear $\boldsymbol{u} = (u(z), 0, 0)$

Equation for the angle $$\dot{\theta} = \frac{1}{2}(\omega - B^{-1}\sin\theta) \qquad \omega = \partial_z u(z)$$

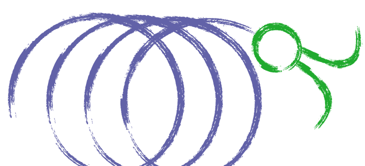


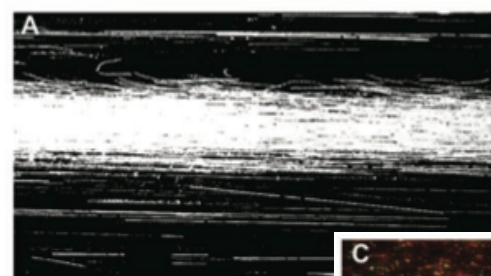
equilibrium direction

$$\sin\theta_{eq} = B\omega$$

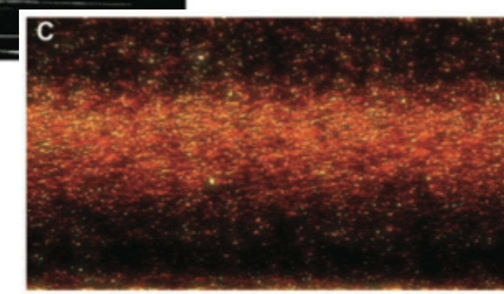


no equilibrium…





C. rehinartii



H. akashiwo

Lab experiments by Durham *et al.*(2009)

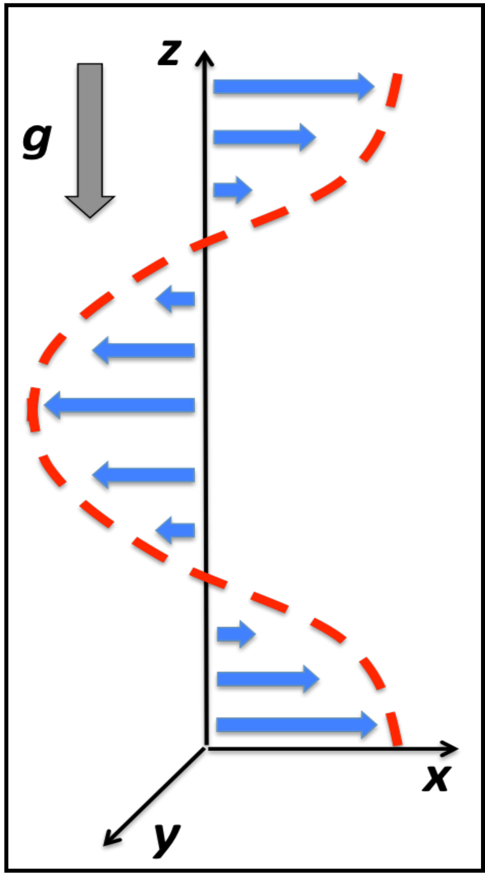
What about turbulence? Kolmogorov flow

Navier-Stokes equations for incompressible velocity field $\boldsymbol{u}$

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i P + \nu \partial^2 u_i + g_i$$

Kolmogorov body force: $g_i = \delta_{i,1} F \cos z$

Stationary solution: $\boxed{U_i = \delta_{i,1}\, U_0 \cos\left(Z/L\right)}$



For $Re \equiv UL/\nu > \sqrt{2}$ the laminar solution is linearly unstable

For $Re \gg \sqrt{2}$ the flow becomes turbulent: DNS are necessary.

Why this flow?

because it is the simplest periodic shear flow

because the mean profile of the **turbulent** flow **is still a cosine**

G. I. Sivashinsky, *Physica D* (1985)

Musacchio and Boffetta, Physical Review E (2014)

SWIMMERS IN LAMINAR KOLMOGOROV FLOW

Santamaria *et al.*, Phys Fluids **26**, 111901 (2014).

$$\dot{Z} = \Phi \mathrm{p}_z$$

$$\dot{\mathrm{p}}_x = -\frac{1}{2\Psi}\mathrm{p}_x\mathrm{p}_z - \frac{1}{2}\sin Z\,\mathrm{p}_z$$

$$\dot{\mathrm{p}}_y = -\frac{1}{2\Psi}\mathrm{p}_y\mathrm{p}_z$$

$$\dot{\mathrm{p}}_z = \frac{1}{2\Psi}(1-\mathrm{p}_z^2) + \frac{1}{2}\sin Z\,\mathrm{p}_x$$

In analogy with the constant shear case we could expect

$\Psi < 1$ quasi-equilibrium solutions with $\dot{\mathbf{p}} = 0$ exist for all Z, **swimmers can escape**

$\Psi > 1$ for some Z rotation due to shear dominates, **swimmers are trapped**

$$\mathcal{C}(\mathbf{p}, Z) = \mathrm{p}_y e^{Z/(2\Phi\Psi)}$$

$$\mathcal{H}(\mathbf{p}, Z) = \Phi e^{\frac{Z}{2\Phi\Psi}}\left[\mathrm{p}_x - \frac{\Psi(2\Phi\Psi\cos Z - \sin Z)}{1+4\Phi^2\Psi^2}\right]$$

Two constants of motion

the system is integrable

If a swimmer is not trapped, Z is not limited, $\mathrm{p_y} \to 0$

The system becomes 2D.

$$\dot{\theta} = \tfrac{1}{2\Psi}\cos\theta + \tfrac{1}{2}\sin Z$$

$$\dot{Z} = \Phi\sin\theta$$

$$\dot{\theta} = G(Z)\partial_Z\mathcal{H}$$

$$\dot{Z} = -G(Z)\partial_\theta\mathcal{H}$$

$$G(Z) = e^{-\frac{Z}{2\Phi\Psi}}$$

G(Z):inverse integrating factor

A Zöttl, H Stark, *PRL* (2012)
for a similar approach for prolate cells in Poiseuille flow

Swimmers in laminar Kolmogorov flow

Conservation of $\mathcal{H}$ implies that for large Z (**untrapped swimmers**)

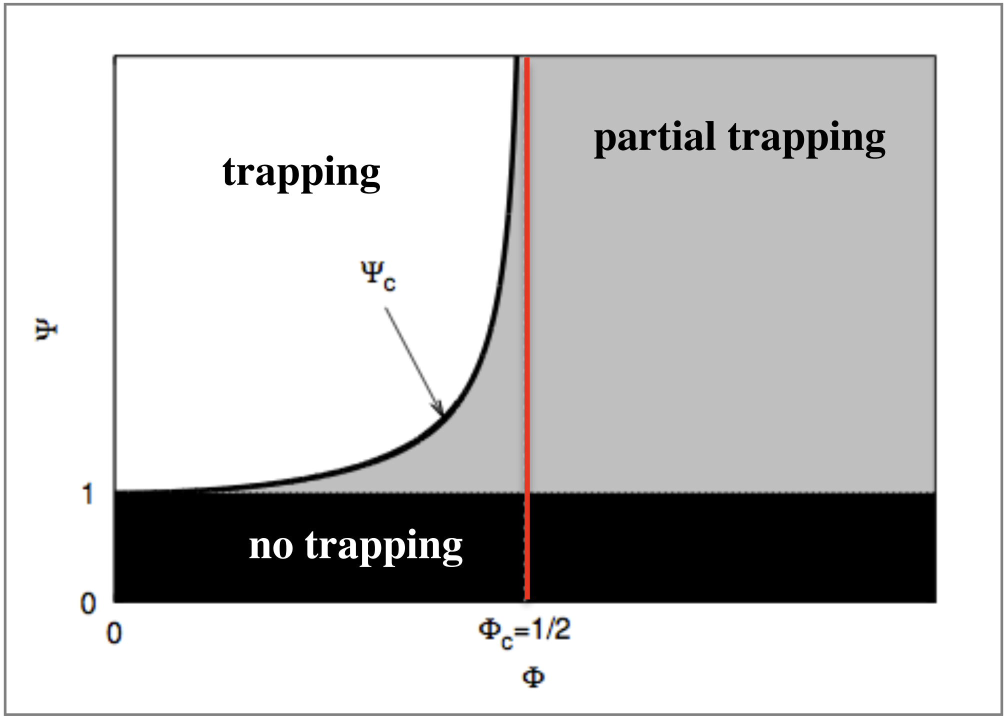
$$\mathcal{H}(\mathbf{p}, Z) = \Phi e^{\frac{Z}{2\Phi\Psi}} \left[\mathrm{p}_x - \frac{\Psi(2\Phi\Psi\cos Z - \sin Z)}{1 + 4\Phi^2\Psi^2} \right]$$

$$p_x = \frac{\Psi(2\Phi\Psi\cos Z - \sin Z)}{1 + 4\Phi^2\Psi^2}$$

This defines a $\Phi_c = \dfrac{1}{2}$

$\Phi < \Phi_c$ solutions for $\Psi \leq \Psi_c = \left(1 - 4\Phi^2\right)^{-1/2}$

$\Phi > \Phi_c$ solutions for any Ψ

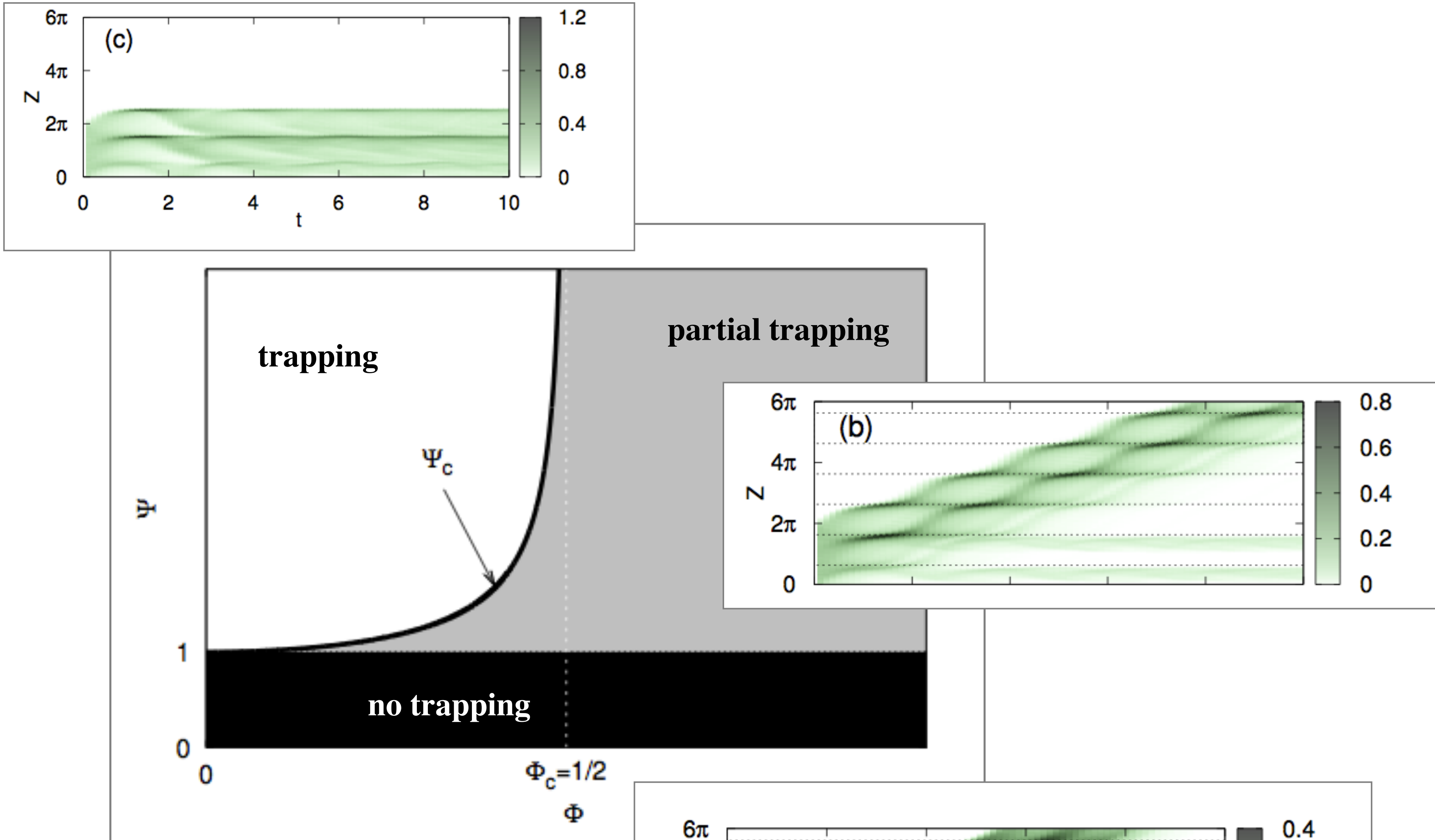


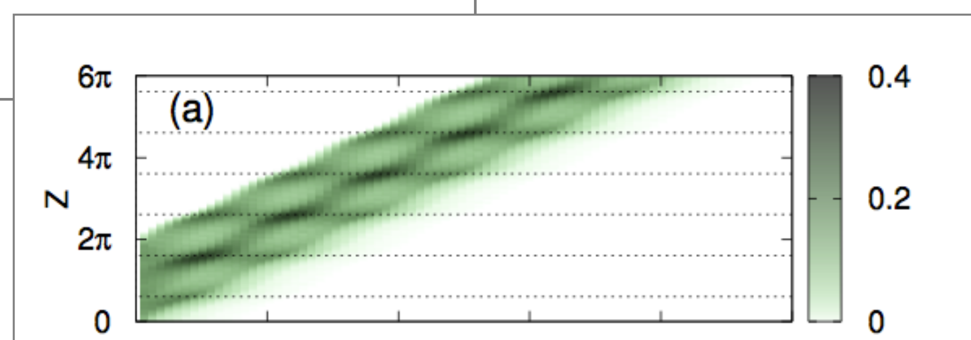
swimming number

$$\Phi = v_s/U_0$$

$$\Psi = BU_0/L$$

stability number

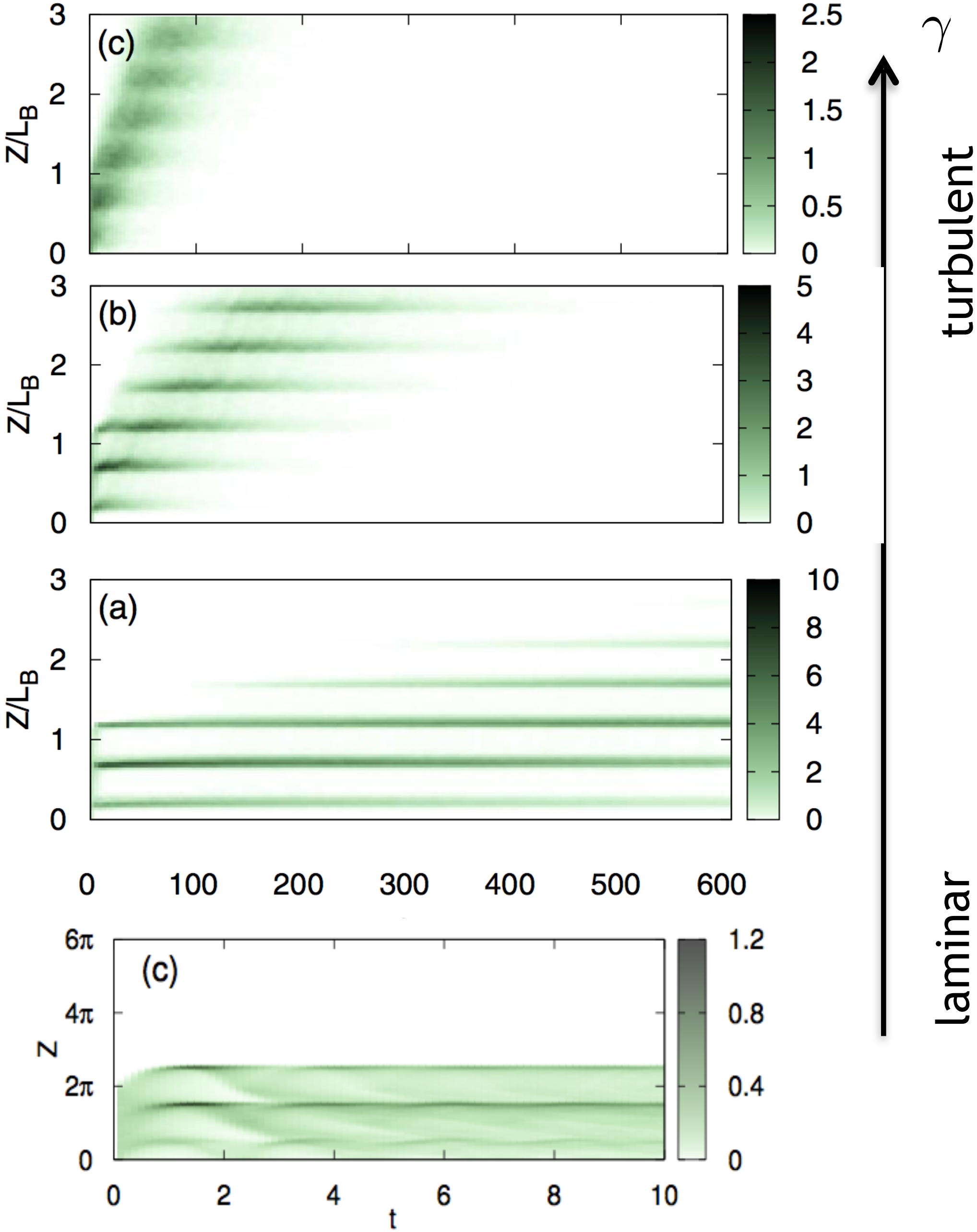




$$\Psi_c = (1 - 4\Phi^2)^{-1/2}$$

SWIMMERS IN TURBULENT KOLMOGOROV FLOW

Santamaria *et al*., Phys Fluids **26**, 111901 (2014).



Effective diffusion due to turbulence **makes trapping transient**

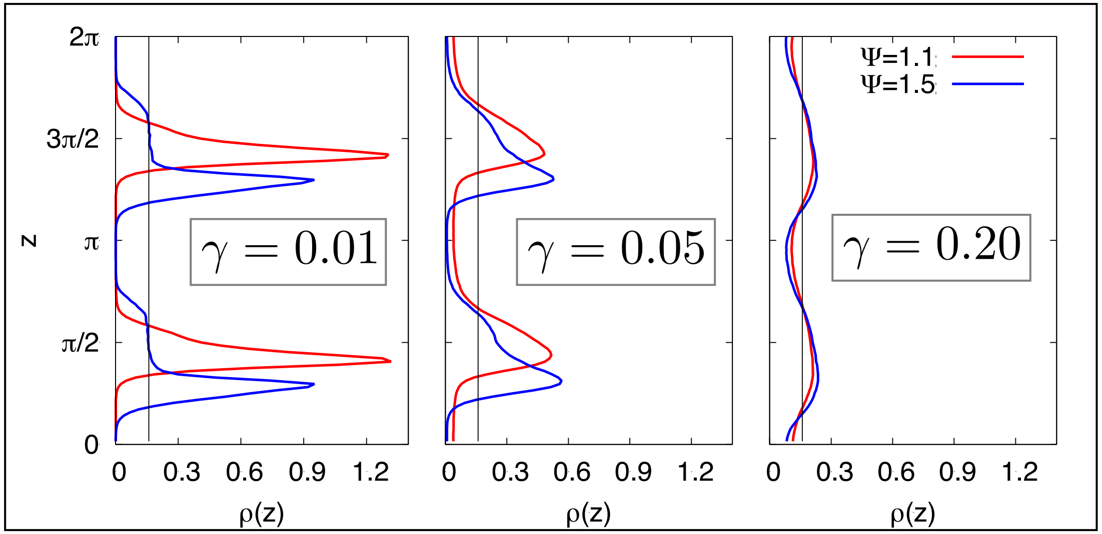
Technical note: in a turbulent Kolmogorov flow the relative intensity of fluctuations is constant. We change it by decomposing the velocity field

$$\mathbf{u} = \langle \mathbf{u} \rangle + \gamma \mathbf{u}'$$

From now on we consider only conditions that would give trapping in a laminar flow

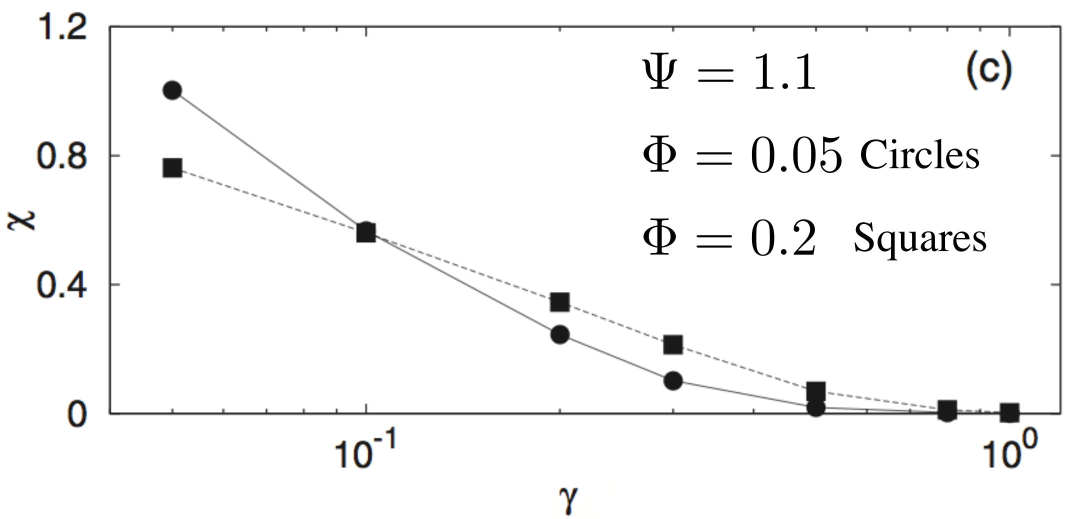
$$\Psi > \Psi_c$$

DENSITY PROFILES



Turbulence intensity

$$\Phi = 0.05$$

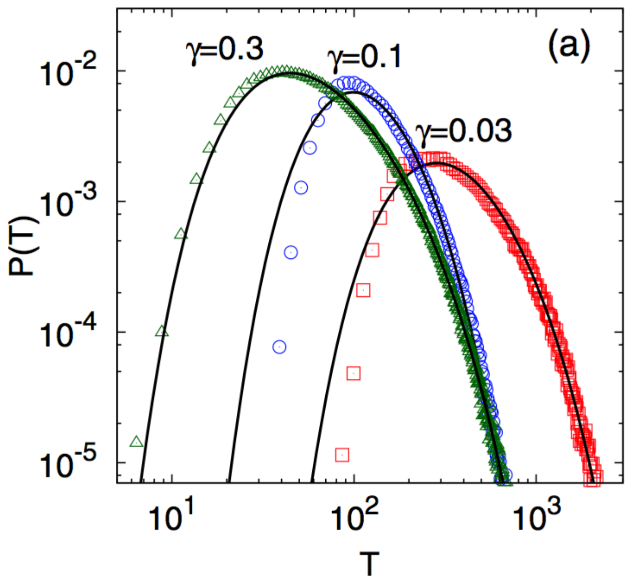


$$\chi = \frac{1}{\rho_0}\sqrt{\langle(\rho - \rho_0)^2\rangle}$$

$$\gamma > 0.5 \;\Rightarrow\; \chi \simeq 0$$

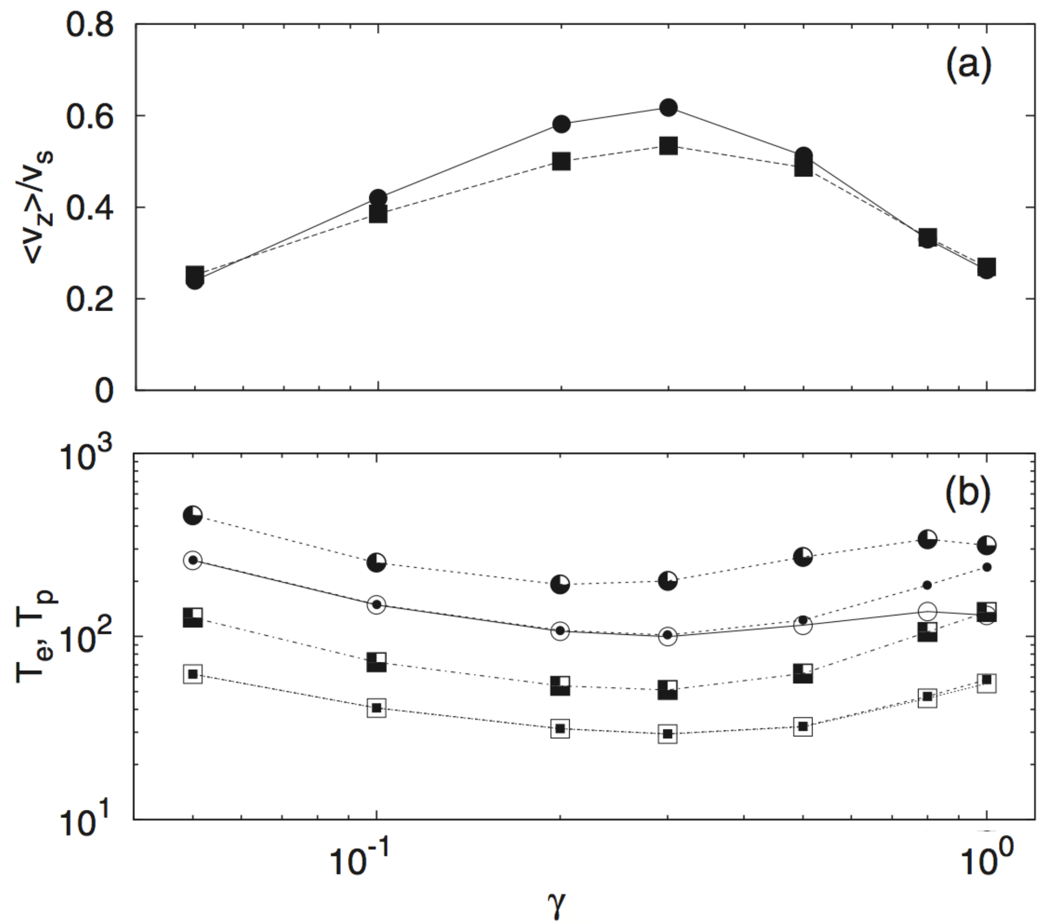
HOW LONG DO LAYERS LAST?

Exit time: time to swim half a period **upwards**



Exit time distribution compared with an **inverse gaussian** (i.e. the shape for a pure diffusion with drift)

If typical parameters for the ocean are used
hours < T_p < days



$\Phi = 0.05$ Circles

$\Phi = 0.2$ Squares

○ □ $\langle T \rangle$

● ■ $\pi / \langle V_z \rangle$

$\int_0^{T_p} p(T)dT \approx 0.9$ ◔ ◪ T_p

CONCLUSIONS

- We consider a simple model for gyrotaxis
- We studied the effects of turbulence on small scale patchiness and on the formation of thin phytoplankton layers

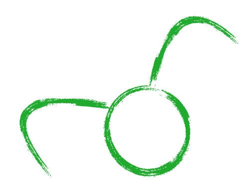
- Indications that **turbulence can induce small scale clustering** of swimming algae
- Gyrotactic algae tend to **cluster on downwelling regions**
- Clustering **controlled by the orientation time**
- Effects of fluid acceleration can be dramatic…but not in the ocean

Durham, *et al*. Nature Comm. **4**, 2148 (2013).
De Lillo, *et al*. Phys. Rev. Lett. **112**, 044502 (2014).

- **Analytical conditions** for the **formation of TLs** in laminar Kolmogorov flow can be derived
- Turbulence causes layers to **dissolve in a finite time**. We discussed some estimates of the **lifetime of layers**, with the correct orders of magnitude.

Santamaria *et al*., Phys Fluids **26**, 111901 (2014).

Grazie!

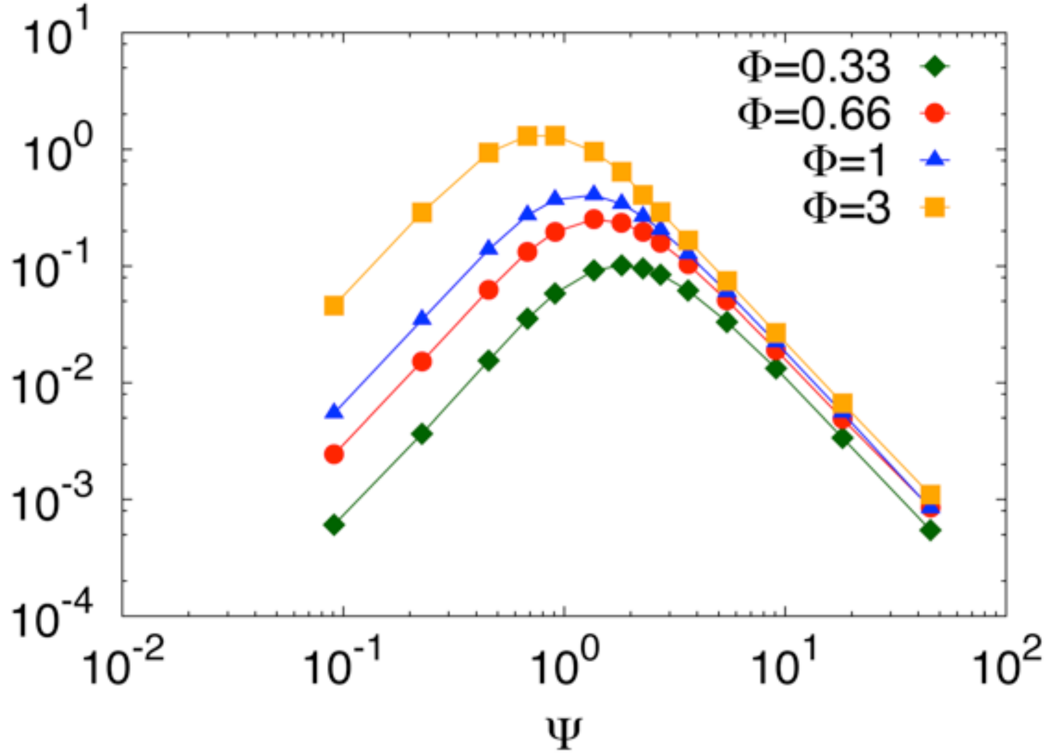


…and thanks to Chlamy!

Accumulation index

Another measure of clustering: the deviation from a Poisson distribution

$$N = \frac{\sigma - \sigma_p}{\langle n \rangle}$$



$$\sigma = \left(\langle n^2 \rangle - \langle n \rangle^2 \right)^{1/2}$$

N is related to the fractal dimension indeed if $3\text{-}D_2 \ll 1$ one can show that

$$N \approx \frac{(3 - D_2) \langle n \rangle^{1/2}}{2} \ln \left(\frac{L_B}{\Lambda} \right)$$

Dubrulle and Lachiéze-Rey A&A (1994)

A turbulence primer....

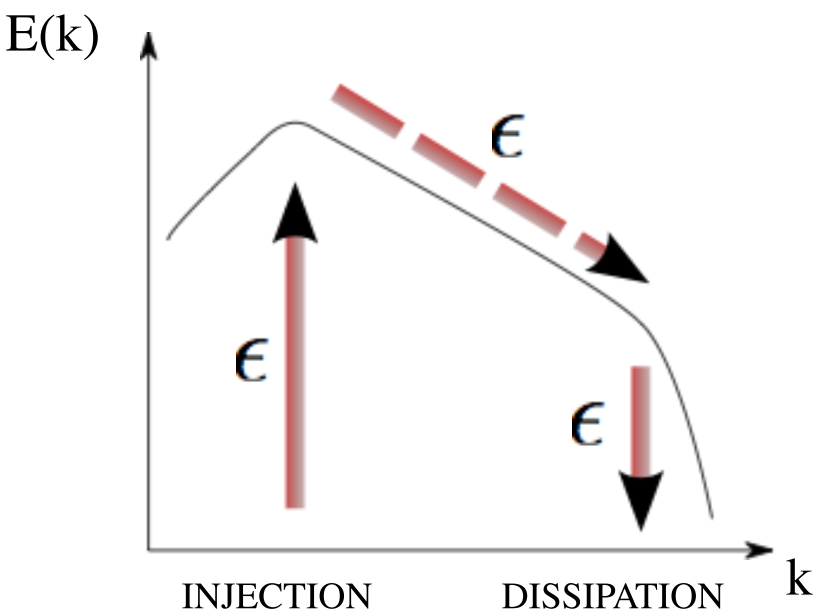
What we mean by turbulence (your neighbour might give a different definition):

$$\mathrm{Re} = \frac{UL}{\nu}$$

-a solution of the Navier-Stokes equation at large Re

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f}$$

-"more than chaotic": many active scales



$$\epsilon = \frac{U^3}{L} = \frac{u_\eta^3}{\eta}$$ Energy flux across scales constant in the inertial range

η dissipative scale (Kolmogorov's scale)

u_η typical velocity at scale η

τ_η typical eddy-turn-over time at scale η

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}} \quad \tau_\eta = \left(\frac{\nu}{\epsilon}\right)^{\frac{1}{2}}$$

$$u_\eta = (\nu\epsilon)^{1/4}$$

Important for what follows: small scales are determined by ν and ϵ only!