On the edge of an inverse cascade

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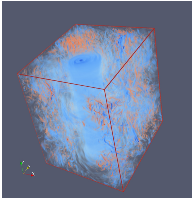
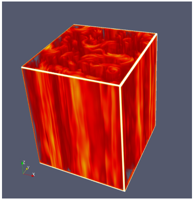
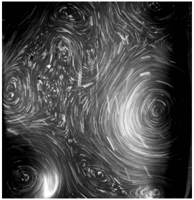
Flowing Matter Across the Scales
ROME
26-03-2015

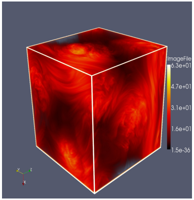
Forward and inverse cascades

There are some systems ...









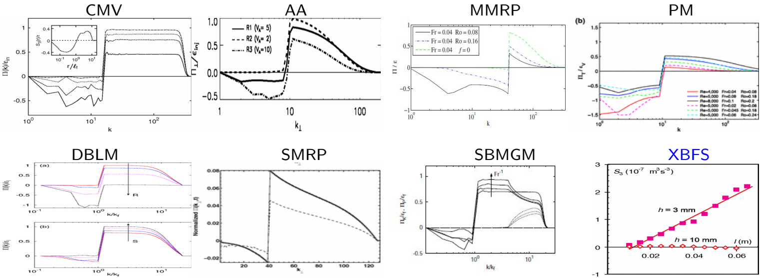
- Fast rotating flows ($Ro \equiv U/\Omega\ell \ll 1$)
- Flows in the presence of a magnetic field ($M \equiv U/B_0 \ll 1$)
- Confined flows (thin geometries) ($\Gamma \equiv h/\ell_f \ll 1$)
- Helical MHD flows ($h_M \equiv$ helicity injection/energy injection $\cdot k_f$)
- ...

for which the inverse cascade depends on a parameter

$$\mu = Ro, \Gamma, M, h_M, \ldots$$

Motivation

Fluxes in: Thin layers/Rotating/Stratified/Magnetic fields ...



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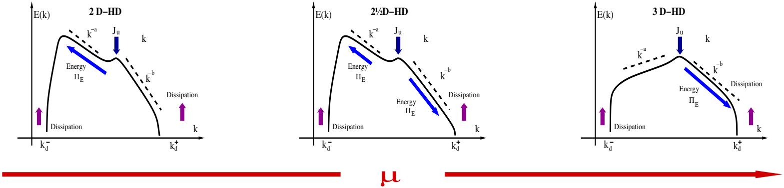
A turbulence to turbulence transition ...



- the system transitions from one turbulent state (inverse cascading) to an other (forward cascading) varying a parameter μ. (μ is not Re)
- the transition occurs in the presence of turbulent noise
- these transitions are not only observed as dimensional (ie 2D to 3D), but weak to strong, HD to MHD, ...
- these transitions are not only observed for the energy cascade but also for other invariants (magnetic helicity, square vector potential, wave action, ...)

Breaking the enstrophy conservation

Most models have a varying inverse cascade of energy due to a dimensional transition from a 3D to a 2D

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Are there computationally more tractable models that show a transition from forward to inverse cascade?

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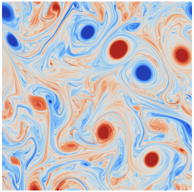
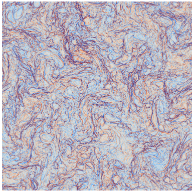
Are there computationally more tractable models that show a transition from forward to inverse cascade?

Transitioning from 2D-HD to 2D-MHD





2D-HD vs 2D-MHD

Equations:

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega \quad = \qquad\qquad + [\nu_n^+ \nabla^{2n} \omega + \nu_n^- \nabla^{-2n} \omega] + F_\omega$$

where

$$\omega = \hat{\mathbf{e}}_n \cdot \nabla \times \mathbf{u}$$

Nonlinearity conserves

$$E = \frac{1}{2} \int \mathbf{u}^2 dv, \qquad \Omega = \frac{1}{2} \int \omega^2 dv$$

2D-HD vs 2D-MHD

Equations:

$$\begin{aligned}\partial_t \omega + \mathbf{u}\cdot\nabla\omega &= \mathbf{b}\cdot\nabla j + [\nu_n^+ \nabla^{2n}\omega + \nu_n^+ \nabla^{-2n}\omega] + F_\omega \\ \partial_t a + \mathbf{u}\cdot\nabla a &= \qquad\qquad + [\eta_n^+ \nabla^{2n} a + \eta_n^- \nabla^{-2n} a] + F_a\end{aligned}$$

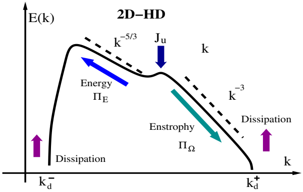
where

$$\omega = \hat{\mathbf{e}}_n \cdot \nabla \times \mathbf{u}, \quad \mathbf{b} = \nabla \times (\hat{\mathbf{e}}_n a), \quad j = \hat{\mathbf{e}}_n \cdot \nabla \times \mathbf{b}$$

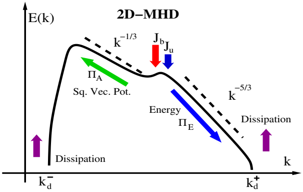
Nonlinearity conserves

$$E = \frac{1}{2}\int \mathbf{u}^2 + \mathbf{b}^2 dv, \qquad A = \frac{1}{2}\int a^2 dv$$

2D-HD



2D MHD

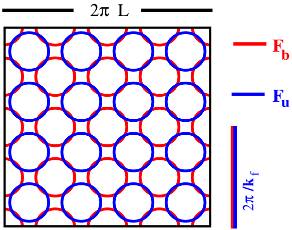


$\mathbf{F}_b = 0 \Rightarrow \mathbf{b} =$ No dynamo theorem for 2D flows!

$\lvert\mathbf{F}_u\rvert > 0,\ \mathbf{F}_b = 0$	$\lvert\mathbf{F}_u\rvert > 0,\ \lvert\mathbf{F}_b\rvert > 0$	$\mathbf{F}_u = 0,\ \lvert\mathbf{F}_b\rvert > 0$
Inverse cascade of E	**?**	Forward cascade of E
Forward cascade of Ω	**?**	not conserved
Forward cascade of A	**?**	Inverse cascade of A

What is the fate of the forward/inverse cascade as we vary $\mathbf{F}_u, \mathbf{F}_b$?

Set-up of Numerical Experiments



2D square periodic box of side $2\pi L$
No mean magnetic field $\langle \mathbf{b} \rangle = \mathbf{0}$

$$F_\omega(x,y) = f_u k_f^{+1} \sin(k_f x)\sin(k_f x)$$
$$F_a(x,y) = f_b k_f^{-1} \cos(k_f x)\cos(k_f x)$$

hypodissipation $\nu_n^- \nabla^{-2n}$ & hyperdissipation $\nu_n^+ \nabla^{+2n}$ with n=2.

Control Parameters / Non-dimensional Numbers

$$\mu_f \equiv \frac{f_b}{f_u} \qquad k_f L \qquad Re_f^- = \frac{f_u^{1/2} k_f^{1/2+2n}}{\nu_n^-} \qquad Re_f^+ = \frac{f_u^{1/2} k_f^{1/2-2n}}{\nu_n^+}$$

$$P_M^- \equiv \nu_n^- / \eta_n^- = 1, \qquad P_M^+ \equiv \nu_n^+ / \eta_n^+ = 1$$

Quantifying the cascades

Inverse and Forward cascades of energy:

$$\epsilon_E^- \equiv \nu_n^- \langle (\nabla^{-n}\mathbf{u})^{\mathbf{2}} + (\nabla^{-\mathbf{n}}\mathbf{b})^{\mathbf{2}} \rangle, \qquad \epsilon_{\mathbf{E}}^+ \equiv \nu_{\mathbf{n}}^+ \langle (\nabla^{+\mathbf{n}}\mathbf{u})^{\mathbf{2}} + (\nabla^{+\mathbf{n}}\mathbf{b})^{\mathbf{2}} \rangle$$

$$\epsilon_E \equiv \epsilon_E^- + \epsilon_E^+ \qquad 0 \leq \frac{\epsilon_E^-}{\epsilon_E} \leq 1,$$

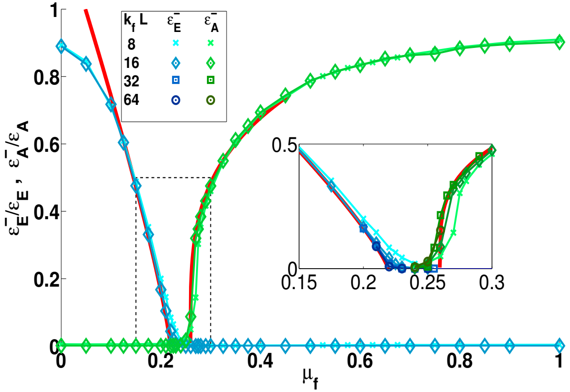
Inverse and Forward cascades of square vector potential:

$$\epsilon_A^- \equiv \nu_n^- \langle (\nabla^{-n} a)^2 \rangle, \qquad \epsilon_A^+ \equiv \nu_n^+ \langle (\nabla^{+n} a)^2 \rangle$$

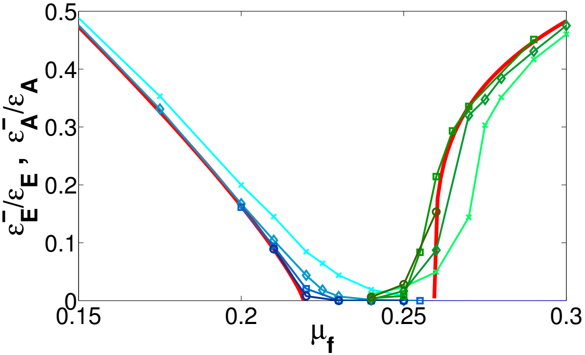
$$\epsilon_A \equiv \epsilon_E^- + \epsilon_E^+ \qquad 0 \leq \frac{\epsilon_A^-}{\epsilon_A} \leq 1,$$

A Critical transition

Varying μ_f for different box-size and fixed Re_n^+.



A Critical transition



critical behavior:

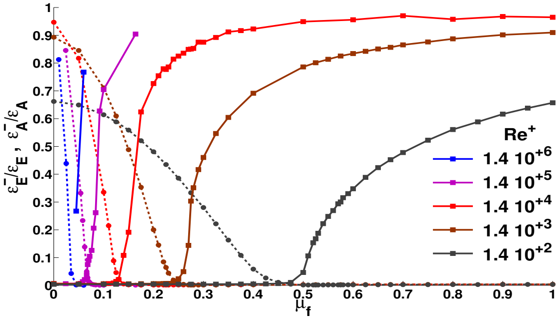
$$\epsilon_E^- \propto (\mu_{c_E} - \mu)^{\gamma_E} \quad and \quad \epsilon_A^- \propto (\mu - \mu_{c_A})^{\gamma_A}$$

a best fit leads to:

$$\mu_{c_E} \simeq 0.22\ldots, \ \gamma_E \simeq 0.82 \quad and \quad \mu_{c_A} \simeq 0.25\ldots, \ \gamma_A \simeq 0.27$$

Critical point dependence on Re_n^+

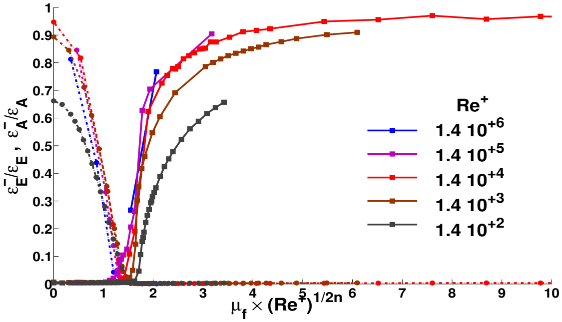
Varying μ_f for different values of Re_n^+ and fixed box-size.



- $\mu_c = \mu_c(Re_n^+)$

Critical point dependence on Re_n^+ (rescaling)

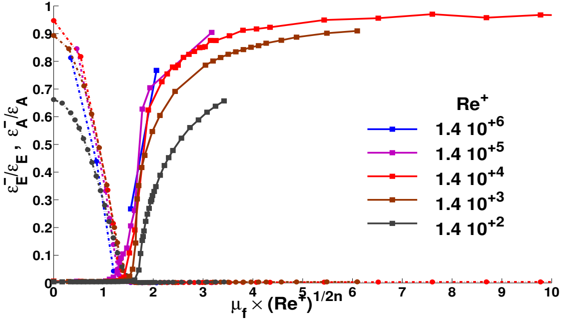
Varying μ_f for different values of Re_n^+ and fixed Re_n^-.



- $\mu_c \propto (Re_n^+)^{-1/2n}$

Critical point dependence on Re_n^+ (rescaling)

Varying μ_f for different values of Re_n^+ and fixed Re_n^-.



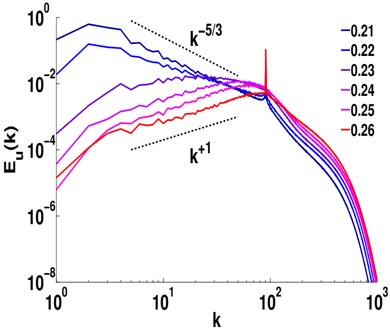
- $\mu_c \propto (Re_n^+)^{-1/2n}$

Magnetic tension determines the transition:

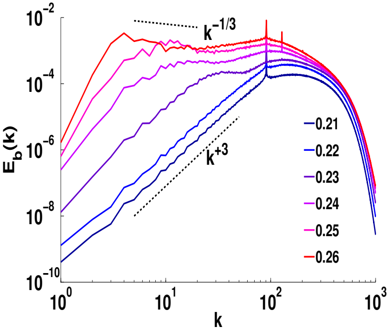
$$\mu_b \equiv \frac{b^2 k_f}{f_u} \propto \mu_f^2 \left(\frac{k_d^+}{k_f}\right)^2 \propto \mu_f^2 [Re^+]^{1/n}$$

Varying μ_f for large box-sizes $k_f L \gg 1$.
$\mu_{c_E} \simeq 0.22\ldots,$ & $\mu_{c_A} \simeq 0.25\ldots$

Kinetic energy spectra

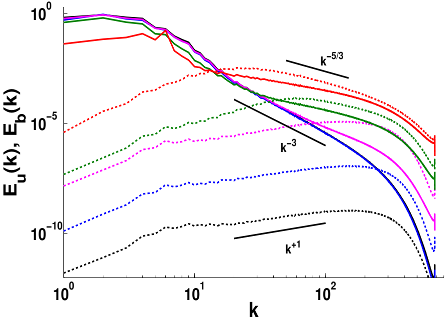


Magnetic energy spectra



Small scale spectra

Varying μ_f for $Re_n^+ \gg 1$.



For small μ

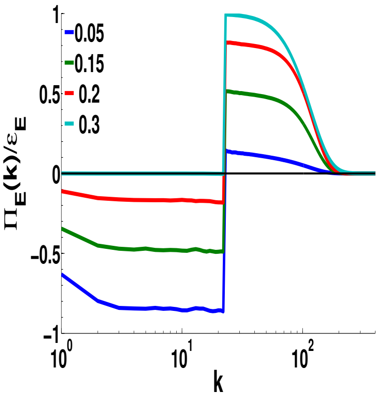
- magnetic energy at the smallest scales is $b_\ell^2 \propto \mu^2 \ell_d^{-2}$ (passive advectio)
- kinetic energy at the smallest scales is $u_\ell^2 \propto \epsilon_\Omega^{2/3} \ell_d^2$ (enstrophy cascade)

Nonlinearity starts when

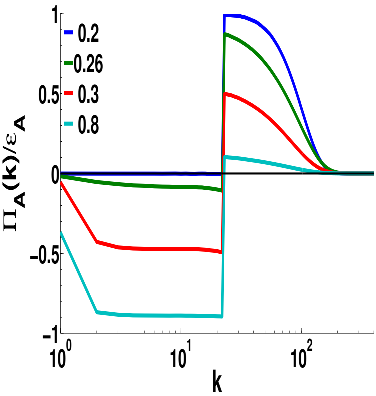
$$\mu \geq \mu_{NL} \propto \ell_d^2 \propto Re^{-1/n}$$

Variable forward and backward fluxes

Energy flux

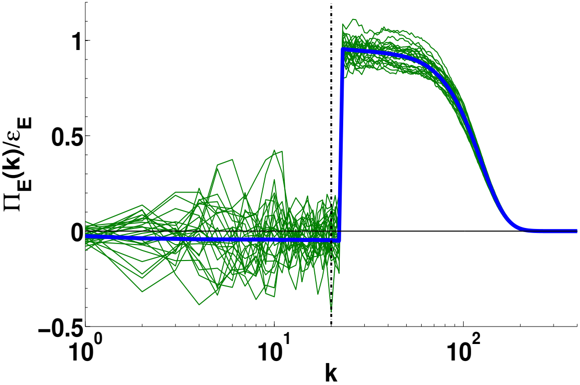


Sq.Vec.Pot. flux



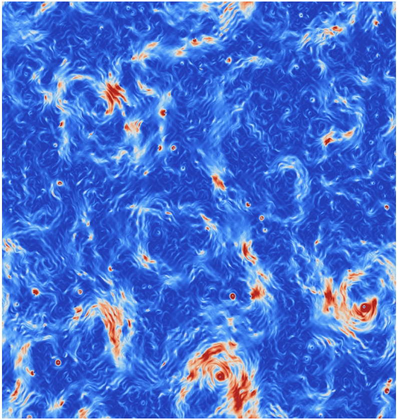
Instantaneous and time averaged fluxes

Strong fluctuations of the energy fluxes .....

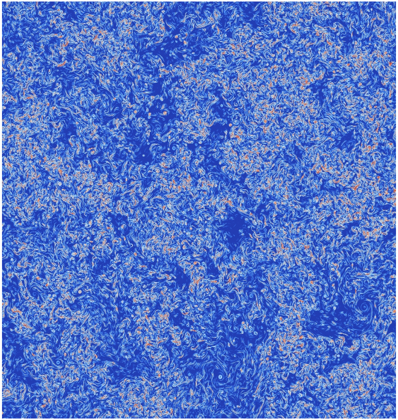


Large scale Structures — 2D-HD to 2D-MHD

$$\mu = 0.21 \cdots \lesssim \mu_c$$



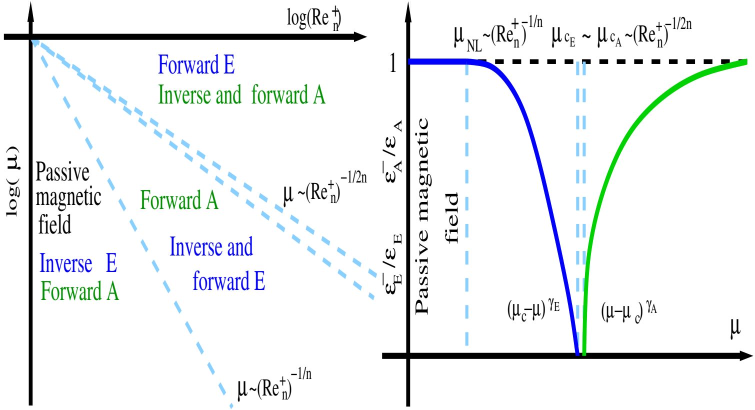
Kinetic energy

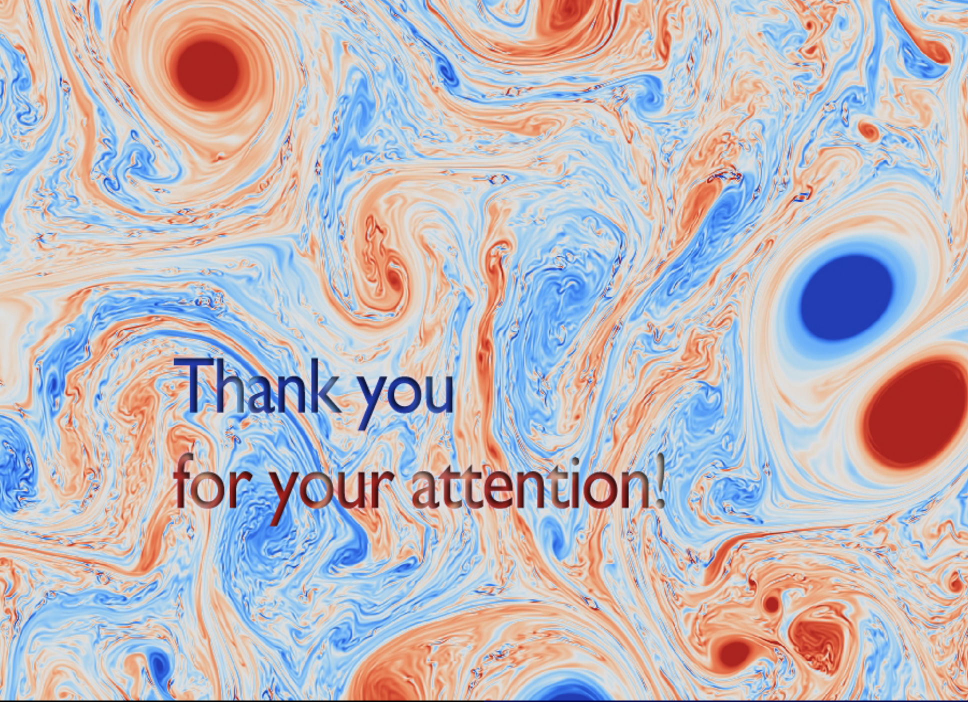


Magnetic energy

Phase diagram

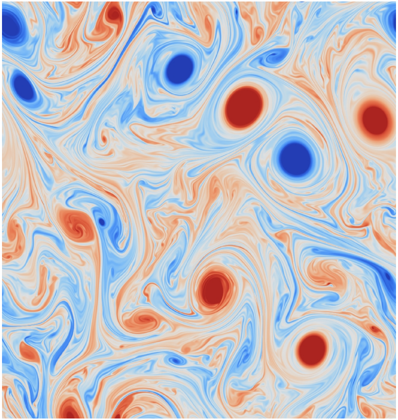
In the limit $k_f L \to \infty$



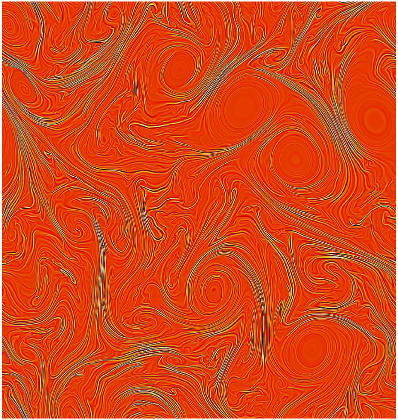

Thank you
for your attention!

Small scale Structures — 2D-HD to 2D-MHD

$$\mu \ll \mu_{NL}$$



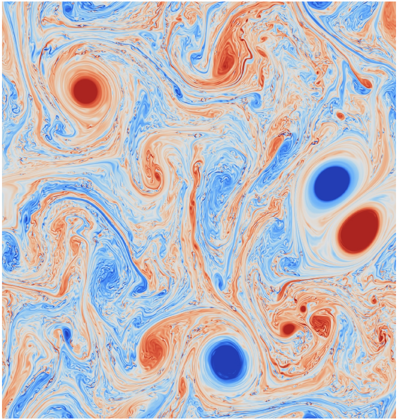
Vorticity



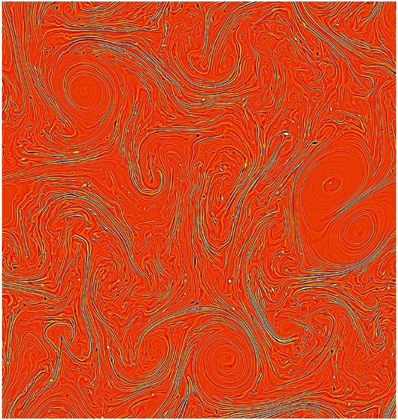
Current density

Small scale Structures — 2D-HD to 2D-MHD

$$\mu_{NL} \lesssim \mu \ll \mu_c$$



Vorticity



Current density

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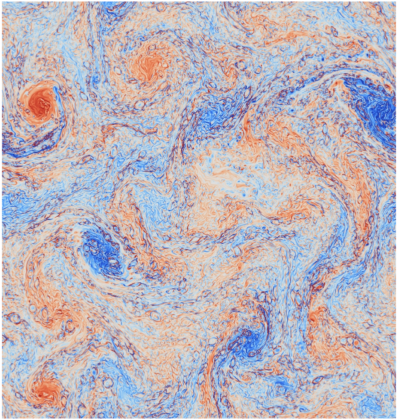

Vorticity

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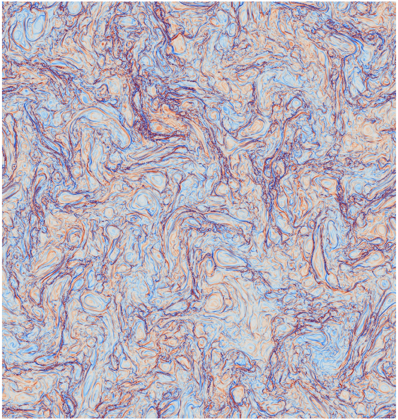
Small scale Structures — 2D-HD to 2D-MHD

$$\mu_{NL} \ll \mu \lesssim \mu_c$$



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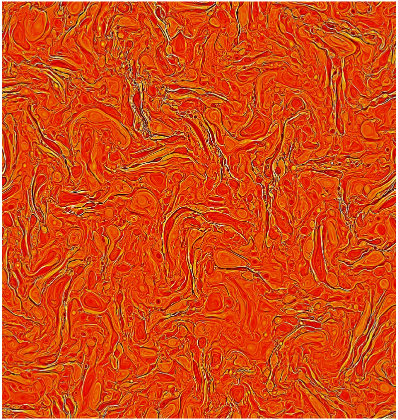

Current density

Large scale Structures — 2D-HD to 2D-MHD

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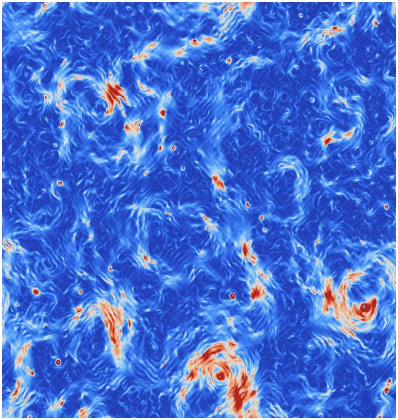
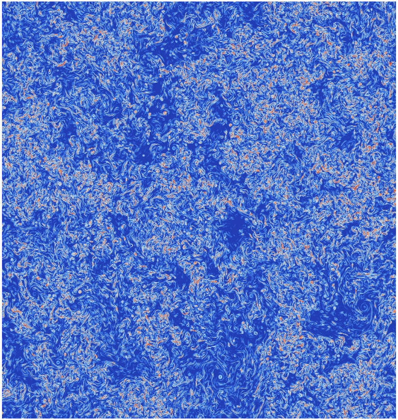


Kinetic energy



Magnetic energy

Large scale Structures — 2D-HD to 2D-MHD

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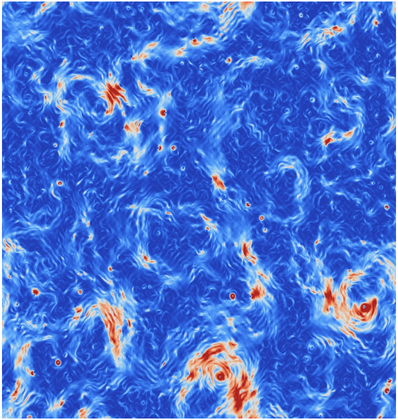
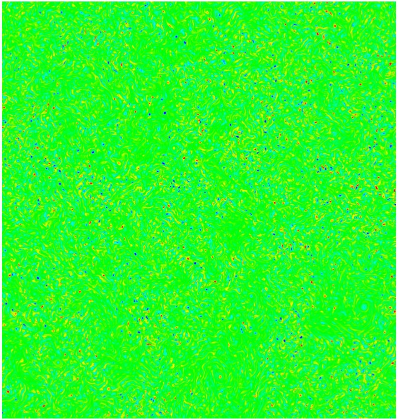


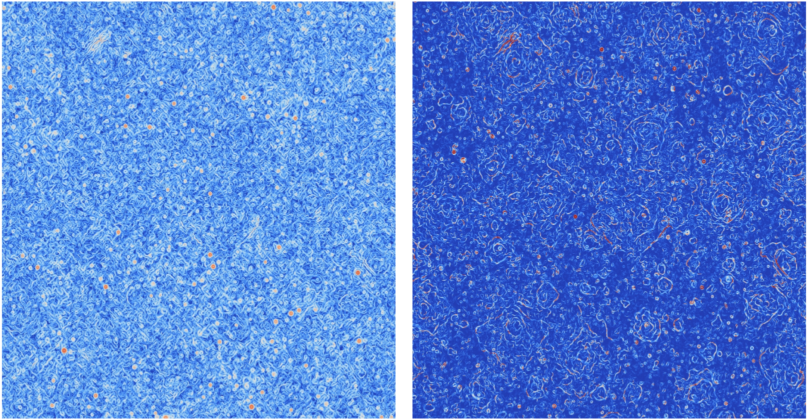
Kinetic energy



Vector Potential

Large scale Structures — 2D-HD to 2D-MHD

$$\mu = 0.26 \cdots \gtrsim \mu_c$$



Kinetic energy   Magnetic energy

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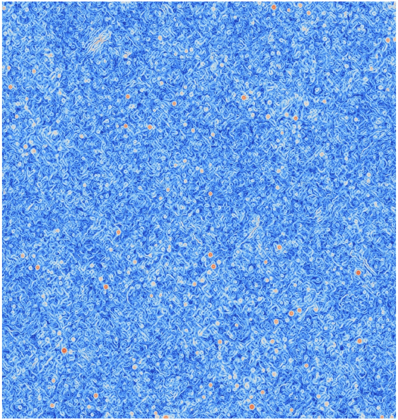
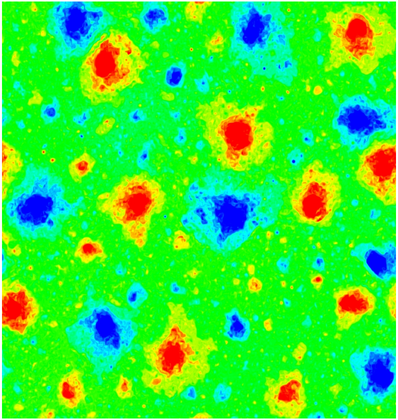
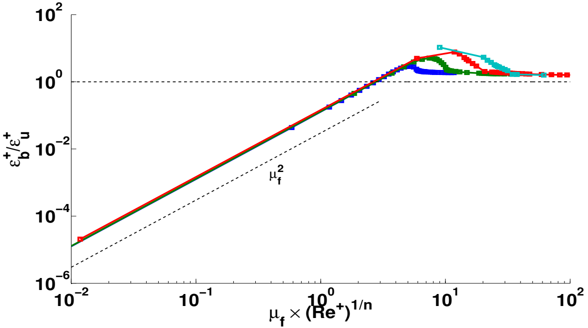


Kinetic energy



Vector Potential

Small scale dissipations



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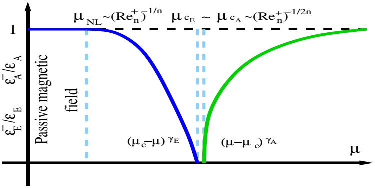
Nonlinearity starts when

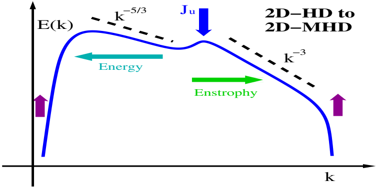
$$\mu \geq \mu_{NL} \propto \ell_d^2 \propto Re^{-1/n}$$

A cartoon summary

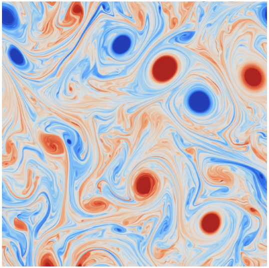
Transition Diagram



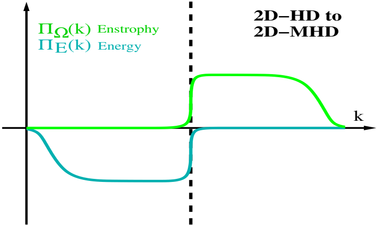
Energy Spectra



Stream Function



Fluxes



A cartoon summary I

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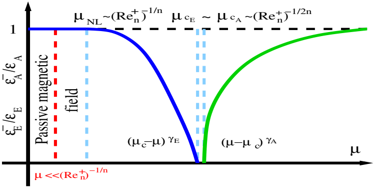
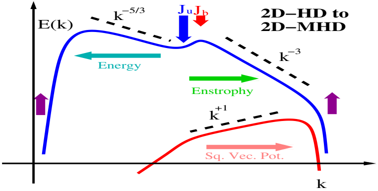
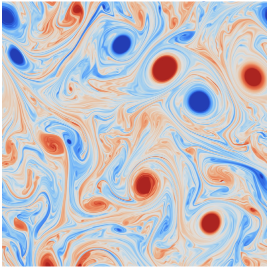


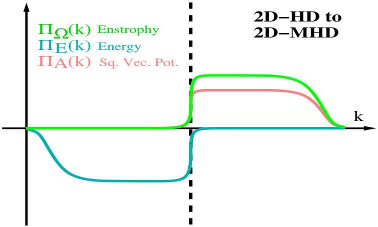
Energy Spectra



Stream Function

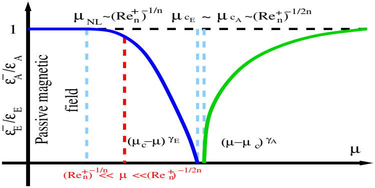


Fluxes

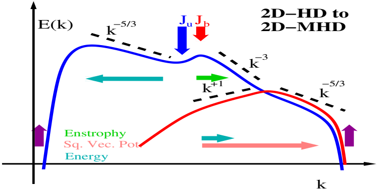


A cartoon summary II

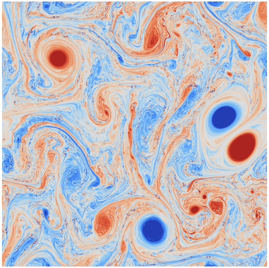
Transition Diagram



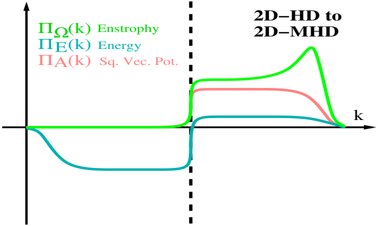
Energy Spectra



Stream Function

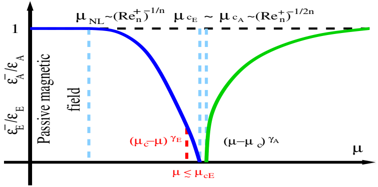


Fluxes

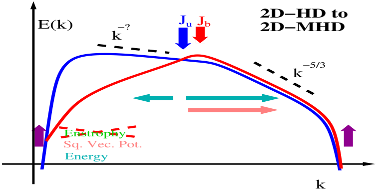


A cartoon summary III

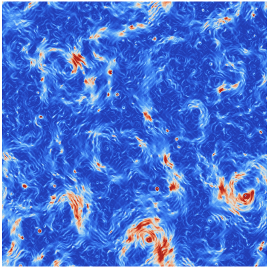
Transition Diagram



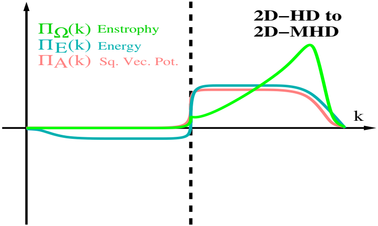
Energy Spectra



Kinetic Energy

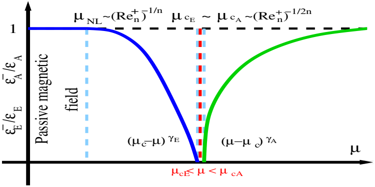


Fluxes

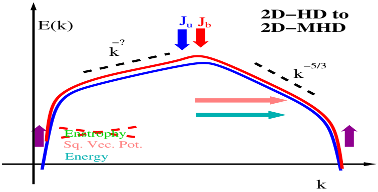


A cartoon summary IV

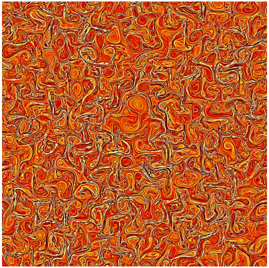
Transition Diagram



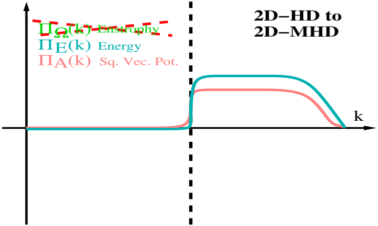
Energy Spectra



Current density

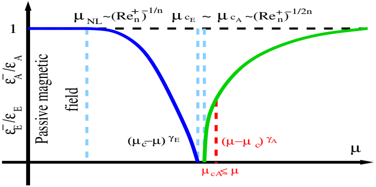


Fluxes

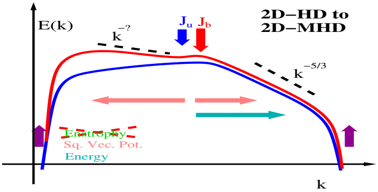


A cartoon summary V

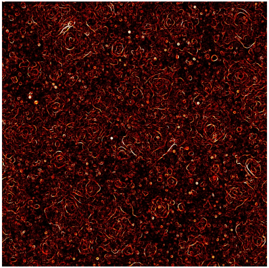
Transition Diagram



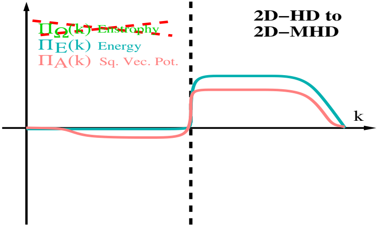
Energy Spectra



Magnetic Energy

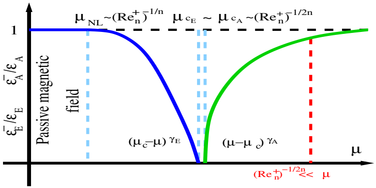


Fluxes

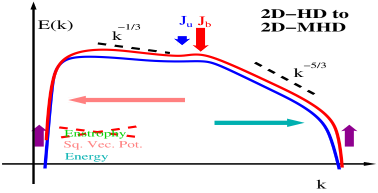


A cartoon summary VI

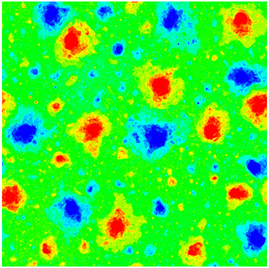
Transition Diagram



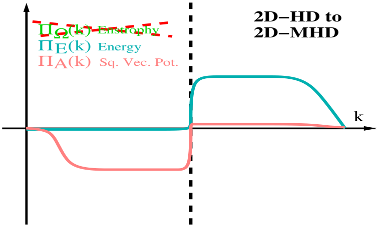
Energy Spectra



Vector Potential

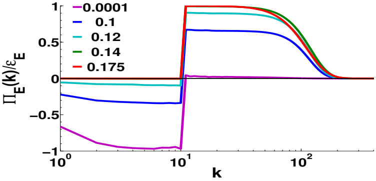


Fluxes

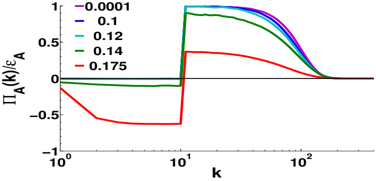


Break down of the enstrophy conservation

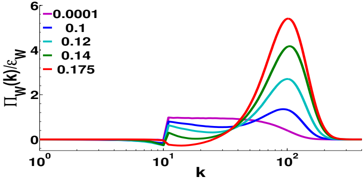
Energy flux



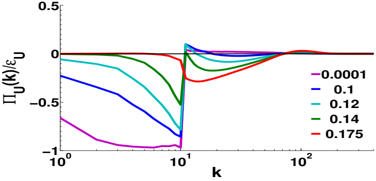
Sq.Vec.Pot. flux



Enstrophy flux

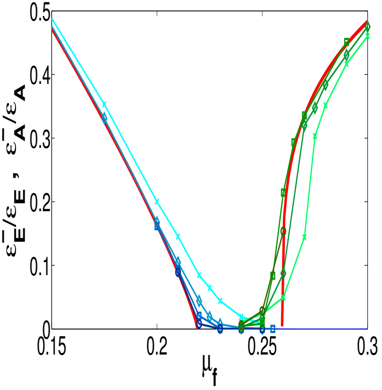


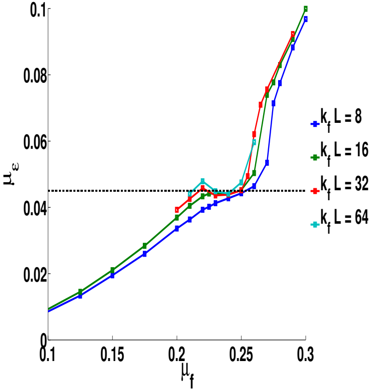
Kinetic energy flux



Digging in in the transition

xxx





Digging in in the transition

xxx