Clustering and coalescence from multiplicative noise: the Kraichnan ensemble

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Abstract
We study the dynamics of the two-point statistics of the Kraichnan ensemble
which describes the transport of a passive pollutant by a stochastic turbulent
flow characterized by scale invariant structure functions. The fundamental
equation of this problem consists of the Fokker–Planck equation for the two-
point correlation function of the density of particles performing spatially
correlated Brownian motions with scale invariant correlations. This problem is
equivalent to the stochastic motion of an effective particle driven by a generic
multiplicative noise. In this paper, we propose an alternative and more intuitive
approach to the problem than the original one (Gawedzki and Vergassola 2000
Physica D **138** 63) leading to the same conclusions. The general features of
this new approach make possible to fit it to other more complex contexts.

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1. Introduction

Correlated random walks have always received interest from different scientific disciplines
thanks to their ability to generate complex patterns and collective behaviors [2–8]. A case of
particular relevance amounts to considering a particle distribution in which single particles
perform an ordinary Brownian motion, but the trajectories of different particles are spatially
correlated [9, 10]. A rich phenomenology occurs in this case ranging from simple diffusion,
to clustering and coalescence of particles [11, 12]. The so-called Kraichnan ensemble
[13, 14], introduced to describe the transport of passive scalars by turbulent flows, belongs to
this class of models. In this approach, the turbulent flow is modeled as a stochastic velocity
field with no time correlations (i.e., white in time) and scale invariant spatial correlations.
Passive particles (e.g. pollutant) are considered to have vanishing mass (no inertia), and are
simply transported by the flow without affecting it. As we discuss below, from the point
of view of non-equilibrium statistical physics, this ensemble of models is equivalent to the

general problem of a d-dimensional Langevin equation driven by multiplicative noise only. The whole ensemble has been solved in [1, 15] using the mathematical theory of boundary conditions of elliptic differential operators developed in rigorous quantum mechanics [16].

In this paper, besides showing the strict relationship between Kraichnan turbulence and the physics of multiplicative noise, we present an alternative approach to these models leading to the same quantitative classification of the phases occurring in the Kraichnan ensemble. Our approach deals directly with the equation for the two-point correlation function of the density of pollutant instead of the correlation function of the passive scalar. The two equations are known to be equivalent being one the adjoint of the other. The reason to propose this alternative solution is twofold: (i) it represents a more intuitive technique based on a simple regularization of the equation at small scales. This regularization amounts to studying the integrated mass of pollutant surrounding a generic pollutant particle. (ii) The new method is so general that can be potentially applied to more complex cases which, up to now, have been solved only under particular conditions [17, 18].

The key point of the present method is an 'exponent hunter' technique allowing an exact classification of particle–particle correlations at vanishing interparticle distance where the dynamics presents a singularity.

The equation we study describes three different regimes for the pollutant concentration which depend on: space dimensionality, anisotropy and the scaling properties of the velocity field correlations. These regimes correspond to three different boundary like situations for the singularity of the equation at small interparticle separation:

 (i) *Phase of strong flow compressibility*. There is only one possible solution of the problem, as different particles collide with vanishing relative velocity. Thus particles coalesce in finite time and no stationary state is approached by the system.
 (ii) *Phase of weak compressibility*. Particles never collide and the system converges toward a stationary state characterized by power-law or 'fractal-like' density correlations.
(iii) *Phase of intermediate compressibility* (called also *sticky* phase [15]). Particles collide in finite time with nonzero relative velocity. Therefore both previous cases are possible depending on the boundary condition, absorbing or reflecting, assigned 'by hand' at vanishing interparticle separation.

Note that for the first two cases there is no chance to impose an arbitrary boundary condition at vanishing interparticle separation because the solution of the fundamental equation is unique.

The paper is structured as follows. In section 2, we briefly review the basic properties of the Kraichnan ensemble. In section 3, we discuss the method of solution of the Fokker–Planck-like equation describing the evolution of the two-point correlation function of the pollutant concentration. Finally, section 4 is devoted to discussion and conclusions.

2. The Kraichnan ensemble and basic equations

We start the discussion by introducing briefly, in the context of stochastic processes, the definition of the Kraichnan ensemble. The microscopic concentration (or particle density) of the N passive pollutant particles in the d-dimensional volume V is

$$\rho(\boldsymbol{x}, t) = \sum_{i=1}^{N} \delta[\boldsymbol{x} - \boldsymbol{x}_i(t)],$$

in which $\boldsymbol{x}_i(t)$ represents the position at time t of the ith pollutant particle transported by the turbulent flow. The particle distribution is supposed to be spatially homogeneous so that

$\langle \rho(\boldsymbol{x}, t) \rangle = \rho_0 = \lim_{V \to \infty} N/V > 0$. Each particle is considered to have a negligible mass, and consequently it will follow the trajectory defined by the Lagrangian flow,

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{v}[\boldsymbol{x}(t), t], \tag{1}$$

where $\boldsymbol{v}(\boldsymbol{x}, t)$ is the Gaussian stochastic velocity field representing the 'synthetic' turbulent flow advecting the pollutant. The spatio-temporal correlation properties of this field are taken to be

$$\begin{cases} \overline{\boldsymbol{v}(\boldsymbol{x}, t)} = 0 \\ \overline{v_\mu(\boldsymbol{x}, t) v_\nu(\boldsymbol{x}', t')} = \delta(t - t') c_{\mu\nu}(\boldsymbol{x} - \boldsymbol{x}'), \end{cases} \tag{2}$$

with $\mu, \nu = 1, \ldots, d$. In other words, the velocity field is considered white in time and colored in space.

We limit our study to the description of the two-particle dynamics and correlations. The higher order statistics has been characterized in [1]. The relative vector distance $\boldsymbol{r}(t) = \boldsymbol{x}_i(r) - \boldsymbol{x}_j(t)$ between any pair of particles satisfies the closed Langevin equation in Îto representation[3]

$$\frac{\mathrm{d}\boldsymbol{r}(t)}{\mathrm{d}t} = \boldsymbol{w}(t). \tag{3}$$

The quantity $\boldsymbol{w}(t) = \boldsymbol{v}[\boldsymbol{x}_i(t), t] - \boldsymbol{v}[\boldsymbol{x}_j(t), t]$ is a Gaussian noise whose correlation properties are

$$\begin{cases} \overline{\boldsymbol{w}(t)} = 0 \\ \overline{w_\mu(t) w_\nu(t')} = 2\delta(t - t') d_{\mu\nu}[\boldsymbol{r}(t)], \end{cases} \tag{4}$$

where $d_{\mu\nu}(\boldsymbol{r}) = [c_{\mu\nu}(0) - c_{\mu\nu}(\boldsymbol{r})]$ is the so-called *structure tensor* of the velocity field. Equations (3) and (4) indicate that the quantity $\boldsymbol{r}(t)$ satisfies the most general Langevin equation for multiplicative noise [19, 20].

The Kraichnan ensemble is defined by the scale invariant choice

$$d_{\mu\nu}(\boldsymbol{r}) = a r^\xi \delta_{\mu\nu} + b r^{\xi-2} r_\mu r_\nu \tag{5}$$

with a and b constant such that $c_{\mu\nu}(\boldsymbol{r})$ is a non-negative definite rank 2 tensor. This means that its Fourier transform (power spectrum tensor)

$$\tilde{c}_{\mu\nu}(\boldsymbol{k}) = \int \mathrm{d}^d r \, c_{\mu\nu}(\boldsymbol{r}) \, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$$

has non-negative eigenvalues for all $\boldsymbol{k}$. Such a condition implies also that $0 < \xi \leqslant 2$ [10]. The one-dimensional case is recovered when setting $d = 1$ and $b = 0$. In the paper [1] and all related literature, the tensor $d_{\mu\nu}(\boldsymbol{r})$ is written in terms of two other constants, A and B, such that

$$\begin{cases} a = A + (d + \xi - 1)B \\ b = \xi(A - B). \end{cases} \tag{6}$$

The positive definiteness of the covariance tensor $c_{\mu\nu}(\boldsymbol{r})$ is guaranteed by taking $A, B \geqslant 0$, which is equivalent to the condition $-(\xi a)/(d + \xi - 1) \leqslant b \leqslant \xi a$. These two constants are introduced as $A = 0$ corresponds to the incompressible case where $\nabla \cdot \boldsymbol{v} = 0$ and $B = 0$ to the purely potential one with $\boldsymbol{v} = \nabla \phi$. The quantities $\mathcal{S}^2 = A + (d - 1)B$ and $\mathcal{C}^2 = A$

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[3] A simple way to understand why equation (1) has to be interpreted in Îto rather than Stratonovich representation is that the corresponding Fokker–Planck equation must not contain the drift term, as the stochastic advecting flow is considered isotropic.

are proportional to $\langle \|\nabla \boldsymbol{v}\|^2 \rangle$ and $\langle (\nabla \cdot \boldsymbol{v})^2 \rangle$ respectively, and they satisfy the inequalities $\mathcal{S}^2 \geqslant \mathcal{C}^2 \geqslant 0$. The degree of compressibility is defined as the ratio $\mathcal{P} = \mathcal{C}^2/\mathcal{S}^2 \in [0, 1]$. The bounds $\mathcal{P} = 0$ and $\mathcal{P} = 1$ define a completely incompressible and a completely compressible flow respectively. In one dimension, $\mathcal{S}^2 = \mathcal{C}^2 \geqslant 0$ and therefore $\mathcal{P} = 1$. Clearly, in real flows equation (5) can hold only in a limited range of scales, as in general $c_{\mu\nu}(\boldsymbol{r})$ must vanish for $r \to \infty$, and accordingly $d_{\mu\nu}(\boldsymbol{r})$ must converge to the constant $c_{\mu\nu}(0)$.

As a consequence of equation (5), the point $\boldsymbol{r} = 0$ is a stationary point of equation (3). As shown below, it may determine, for particular values of the parameters $\mathcal{P}, d$ and ξ, an absorbing phase for the dynamics. We will discuss this problem in the Fokker–Planck formalism.

The Langevin equation (3) is equivalent [21] to the d-dimensional Fokker–Planck equation (FPE)

$$\partial_t P(\boldsymbol{r}, t | \boldsymbol{r}_0, t_0) = \sum_{\mu,\nu}^{1,d} \partial_{\mu\nu}^2 [d_{\mu\nu}(\boldsymbol{r}) P(\boldsymbol{r}, t | \boldsymbol{r}_0, t_0)], \tag{7}$$

describing the evolution of the probability density function (PDF) $P(\boldsymbol{r}, t | \boldsymbol{r}_0, t_0)$ of the interparticle distance $\boldsymbol{r}$ at time t given the initial value $\boldsymbol{r}_0$ at time t_0. In equation (7) $\partial_t = \frac{\partial}{\partial t}$ and $\partial_{\mu\nu}^2 = \frac{\partial^2}{\partial r_\mu \partial r_\nu}$ with r_μ being the μth component of vector $\boldsymbol{r}$. In turbulence literature (e.g., see [1, 14]) instead of equation (7), one usually studies the adjoint equation which is called, in the stochastic processes context, a 'backward' FPE, while equation (7) is called the 'forward' FPE. They are practically equivalent, even though the differential operators in the two equations are defined on different functional spaces [22]. The two equations coincide for incompressible flows.

Equation (7) is a d-dimensional FPE without drift, typical of generalized multiplicative noise. If the tensor $d_{\mu\nu}(\boldsymbol{r})$ never vanishes in the interior of the domain of definition of equation (7), a unique solution exists when the initial [e.g. $P(\boldsymbol{r}, t_0 | \boldsymbol{r}_0, t_0) = \delta(\boldsymbol{r} - \boldsymbol{r}_0)$] and the boundary conditions are given [21, 23]. However, if $d_{\mu\nu}(\boldsymbol{r})$ vanishes at some point, the solution could be no more unique and an additional boundary condition might be necessary at such a singularity.

We develop a novel method to provide, not only the qualitative classification of the singularity $\boldsymbol{r} = 0$ for the Kraichnan ensemble (see appendix A), but also the exact solution of equation (7) in the form of a particular series expansion in the neighborhood of the singularity. This represents an alternative and more intuitive approach than the original one [1]. This method can be potentially generalized and applied to different kind of FPE with singular points in the domain of definition.

It can happen that the singularity $\boldsymbol{r} = 0$ behaves as an additional regular boundary[4]. Then an absorbing, reflecting or mixed boundary condition needs to be assigned. In all the other cases, the singularity can be seen as a boundary at which the condition is unique and automatically fixed by the form of equation (7) itself. In particular when the singularity corresponds to an adhesive boundary (see appendix A), the diffusion of the effective particle described by equation (7) is such that the boundary is reached in a finite time and it behaves automatically as an absorbing boundary [23]. Naively one can say that the effective particle, whose position is $\boldsymbol{r}(t)$, arrives at this boundary in a finite time but with vanishing velocity obliging it to stay at that point indefinitely. Instead when the singularity behaves as a natural

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[4] We use here the definitions given by Van Kampen in chapter XII of its book [23]. This classification differs from that given originally by Feller [22] and reproposed by Gardiner [21]. The two are related directly by the following simple positions: 'adhesive boundary' becomes 'exit boundary' in the latter, 'natural repulsive' becomes 'entrance' or 'natural' depending if the average time to be expelled at finite distance from the singularity is respectively finite or infinite, and finally 'natural attractive' also becomes simply 'natural'.

attractive boundary the same effective particle approaches it, but only in an infinite time. Therefore there is no need of external boundary conditions at the singularity. Finally, for a singularity which is a natural repulsive boundary for equation (7), the effective particle, when placed arbitrarily close to this point, is expelled far from it. We will see that this last situation, in turn, splits into two sub-cases in which the time to be expelled at a finite distance is respectively finite or infinite. The details about the Van Kampen classification will be discussed in appendix A.

3. Method and solution

In this paper, instead of considering the PDF $P(\boldsymbol{r}, t|\boldsymbol{r}_0, t_0)$, we study the equivalent problem of the evolution of the two-point correlation function of the pollutant particle density at time t,

$$\Gamma(\boldsymbol{r}, t) = \frac{\langle \rho(\boldsymbol{x}, t)\rho(\boldsymbol{x} + \boldsymbol{r}, t)\rangle}{\rho_0} - \delta(\boldsymbol{r}), \tag{8}$$

where we have subtracted the Dirac delta contribution in $\boldsymbol{r} = 0$, to exclude the self-correlation of each particle with itself [24]. In this way, $\Gamma(\boldsymbol{r}, t)$ gives exactly the average conditional density of other particles seen by a generic particle at separation $\boldsymbol{r}$ from itself. The symbol $\langle \cdots \rangle$ indicates the ensemble average over the realizations of the process (the hypothesis of statistical spatial homogeneity is done implicitly). It is noteworthy that this average in equation (8) implies both the average $\overline{(\cdots)}$ over the realizations of the velocity field and the ensemble average over the initial particle configurations. From the knowledge of $P(\boldsymbol{r}, t|\boldsymbol{r}_0, t_0)$, we can write

$$\Gamma(\boldsymbol{r}, t) = \int \mathrm{d}^d r_0 \, \Gamma(\boldsymbol{r}_0, t_0) P(\boldsymbol{r}, t|\boldsymbol{r}_0, t_0).$$

Consequently, $\Gamma(\boldsymbol{r}, t)$ satisfies the same forward FPE as $P(\boldsymbol{r}, t|\boldsymbol{r}_0, t_0)$, which can be cast to the tensorial form

$$\partial_t \Gamma(\boldsymbol{r}, t) = \vec{\nabla} \cdot \{\vec{\nabla} \cdot [\hat{d}(\boldsymbol{r})\Gamma(\boldsymbol{r}, t)]\}, \tag{9}$$

where $\hat{d}(\boldsymbol{r})$ is the flow structure tensor which, in intrinsic form, reads

$$\hat{d}(\boldsymbol{r}) = ar^\xi \hat{I} + br^{\xi-2}(\boldsymbol{r} \otimes \boldsymbol{r}), \tag{10}$$

with $\hat{I}$ being the identity tensor and $\boldsymbol{r} \otimes \boldsymbol{r}$ the tensor product of $\boldsymbol{r}$ for itself. Let us now integrate explicitly equation (9) over the sphere $S(r)$ of radius r around the origin. The quantity

$$N(r, t) = \int_{S(r)} \mathrm{d}^d r' \, \Gamma(\boldsymbol{r}', t) = \int_0^r \mathrm{d}r' r'^{d-1} \int_{\Omega_T} \mathrm{d}\Omega \, \Gamma(\boldsymbol{r}', t) \tag{11}$$

is the number of other particles seen in average by a particle in the sphere of radius r centered on it, and Ω_d is the total solid angle in d dimensions. Note that from equation (11) we have

$$\int_{\Omega_d} \mathrm{d}\Omega \, \Gamma(\boldsymbol{r}, t) = \frac{1}{r^{d-1}} \partial_r N(r, t). \tag{12}$$

Using equation (11) and the divergence theorem of integral calculus, we can write equation (9) as

$$\partial_t N(r, t) = \int_{\partial S(r)} \mathbf{dS} \cdot \{\vec{\nabla} \cdot [\hat{d}(\boldsymbol{r})\Gamma(\boldsymbol{r}, t)]\}, \tag{13}$$

where $\partial S(r)$ indicates the surface of the sphere $S(r)$, and $\mathbf{dS} = r^{d-1}\mathrm{d}\Omega \hat{\imath}_r$ is the vector surface element of the sphere of radius r along the radial unitary vector $\hat{\imath}_r$.

In order to transform the integral in equation (13) with condition (10) into a more treatable expression, we apply the tensor calculus in $d = 3$, and then we see how to extend the

description to arbitrary dimensions d. The three-dimensional vector gradient operator in spherical coordinates in the local orthogonal spherical reference frame is

$$\vec{\nabla} = \hat{\imath}_r \partial_r + \frac{\hat{\imath}_\theta}{r} \partial_\theta + \frac{\hat{\imath}_\phi}{r \sin \theta} \partial_\phi.$$

In the same frame, the tensor product $r \otimes r$ acquires the simple form

$$r \otimes r = r^2 (\hat{\imath}_r \otimes \hat{\imath}_r),$$

while the identity operator is obviously

$$I = \hat{\imath}_r \otimes \hat{\imath}_r + \hat{\imath}_\theta \otimes \hat{\imath}_\theta + \hat{\imath}_\phi \otimes \hat{\imath}_\phi.$$

Note that $\hat{\imath}_{r(\theta,\phi)} \otimes \hat{\imath}_{r(\theta,\phi)}$ is the local projector operator along the direction $\hat{\imath}_{r(\theta,\phi)}$.

Using the expression (10) and the rules of tensor calculus in the spherical coordinates frame[5], one can write

$$\vec{\nabla} \cdot [\hat{d}(r)\Gamma(r,t)] = [(a+b)\partial_r(r^\xi \Gamma) + 2br^{\xi-1}\Gamma]\hat{\imath}_r$$

$$+ ar^{\xi-1}\left[(\partial_\theta \Gamma)\hat{\imath}_\theta + \frac{\partial_\phi \Gamma}{\sin \theta}\hat{\imath}_\phi \right], \tag{14}$$

and therefore simply

$$\hat{\imath}_r \cdot \{\vec{\nabla} \cdot [\hat{d}(r)\Gamma(r,t)]\} = (a+b)\partial_r[r^\xi \Gamma(r,t)] + 2br^{\xi-1}\Gamma(r,t). \tag{15}$$

Note that the radial direction is the only important spherical component of equation (14). This is exactly the reason why we have used the local spherical orthogonal frame.

In d dimensions the gradient operator in local hyper-spherical coordinates is

$$\vec{\nabla} = \hat{\imath}_r \partial_r + \sum_{j=1}^{d-1} \hat{\imath}_{\psi_j} \nabla_{\psi_j}, \tag{16}$$

where ∇_{ψ_j} is the component of the gradient along the orthogonal angular direction $\hat{\imath}_{\psi_j}$ on the d-dimensional sphere of radius r. Since the curvature of the sphere is constant and the hyper-spherical frame is orthogonal, it is simple to show that for any $j = 1, \ldots, d-1$ we have[6]

$$\nabla_{\psi_j}\hat{\imath}_r = \frac{\hat{\imath}_{\psi_j}}{r}$$

and therefore

$$\hat{\imath}_{\psi_j} \cdot \nabla_{\psi_j}(\hat{\imath}_r \otimes \hat{\imath}_r) = \frac{\hat{\imath}_r}{r}.$$

This observation permits to generalize the result (15) to any dimension d in the following way:

$$\hat{\imath}_r \cdot \{\vec{\nabla} \cdot [\hat{d}(r)\Gamma(r,t)]\} = (a+b)\partial_r[r^\xi \Gamma(r,t)] + (d-1)br^{\xi-1}\Gamma(r,t). \tag{17}$$

In this relation, the only dependence on the angular variables lies on $\Gamma(r,t)$. Plugging equations (17) and (12) into equation (13), we finally arrive at the following closed equation for $N(r,t)$:

$$\partial_t N(r,t) = (a+b)r^{d-1}\partial_r[r^{\xi-d+1}\partial_r N(r,t)] + (d-1)br^{\xi-1}\partial_r N(r,t). \tag{18}$$

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[5] In particular we use that any derivatives of the identity operator I trivially vanishes that $\partial_r \hat{\imath}_r = 0$, $\partial_\theta \hat{\imath}_r = \hat{\imath}_\theta$ and $\partial_\phi \hat{\imath}_r = \sin \theta \hat{\imath}_\phi$, and that $\hat{\imath}_{r,\theta,\phi} \otimes \hat{\imath}_{r,\theta,\phi}$ are projection operators.

[6] Due to the constant curvature of the sphere and the orthogonality of the coordinate frame, in order to find $\nabla_{\psi_j}\hat{\imath}_r$, it is sufficient to consider the equatorial circumference in the direction $\hat{\imath}_{\psi_j}$ and therefore the variation of $\hat{\imath}_r$ along this. Clearly, as all equatorial circumferences passing through a point of the spherical surface are a rotation one of each other, the modulus of $\nabla_{\psi_j}\hat{\imath}_r$ is the same for any direction $\hat{\imath}_{\psi_j}$ and depends only on r. It is finally simple to find that such a dependence is $1/r$.

The singularity $r = \mathbf{0}$ of equation (9) now constitutes the boundary $r = 0$ of equation (18) (the other one being $r \to \infty$). Equation (18) can be rewritten in a form which is useful for the singularity classification following Feller [22] or Van Kampen [23] (appendix A), and to face explicitly the limit case $\xi = 2$ (appendix B). Let us introduce $n(r, t) = \partial_r N(r, t)$ such that $n(r, t)\mathrm{d}r$ is the average number of particles seen by the particle at the origin in the spherical shell of radius r and infinitesimal thickness $\mathrm{d}r$ around it. Equation (18) in terms of this function becomes a one-dimensional Fokker–Planck equation

$$\partial_t n(r, t) = (a + b)\partial_r^2[r^\xi n(r, t)] - (d - 1)a\partial_r[r^{\xi-1}n(r, t)], \tag{19}$$

which depends explicitly on the dimension d only through the coefficient $(d - 1)$ in the right-hand side.

3.1. The exponent hunter method for the 'rough' case $\xi < 2$

We now introduce a new technique for the rough flow case $\xi < 2$ through which we not only recover the above Van Kampen's classification, but also find the exact small r behavior of all the possible solutions to equation (18), and therefore to equation (9). The smooth case $\xi = 2$ is presented in appendix B.

Since $N(r, t)$ is by definition smooth and finite at small r, this method basically amounts to looking for the solution of equation (18) in the form of the appropriate power expansion for $N(r, t)$. Let us assume $\xi < 2$ and that $N(r, t)$ at any time can be expanded at small r in some power series,

$$N(r, t) = \sum_{l=0}^{\infty} c_l(t)r^{\beta_l} \qquad \text{with} \qquad \beta_i < \beta_j \quad \text{if} \quad i < j, \tag{20}$$

where all β_l are independent of t. Then we construct a recursive solution, by substituting this expression into equation (18),

$$\sum_{l=0}^{\infty} \dot{c}_l(t)r^{\beta_l} = \sum_{l=0}^{\infty} \beta_l[(a + b)(\xi + \beta_l - d) + (d - 1)b]c_l(t)r^{\xi+\beta_l-2}. \tag{21}$$

The only possibility to have a solution of equation (21), being $\xi < 2$, is that either (i) $\beta_0 = 0$ or (ii) $[(a + b)(\xi + \beta_0 - d) + (d - 1)b] = 0$.[7] In order to accept or not these conditions, we have to take into account the statistical and physical meaning of $N(r, t)$. This requires $N(r, t)$ has to be finite and non-negative for all finite r and t. Moreover $\partial_r N(r, t)$ has to be the same non-negative, as the number of particles seen cannot decrease by increasing the radius of the sphere. Let us analyze separately the two cases:

(i) Assume $\beta_0 = 0$. The series (20) satisfying equation (21) needs to have $\forall n \geqslant 0$ coefficients such that:

$$\begin{cases} \beta_n = n(2 - \xi) \\ \dot{c}_n(t) = \gamma_n c_{n+1}(t) \\ \gamma_n = (n + 1)(2 - \xi)\{(a + b)[\xi - d + (n + 1)(2 - \xi)] + (d - 1)b\}. \end{cases} \tag{22}$$

This solution corresponds to an expansion into integer powers of $r^{2-\xi}$. Moreover, we see immediately that, as $\beta_i < \beta_j$ for $i < j$, $\beta_0 = 0$ implies $N(r, t)$ to be always finite at finite r and t. Note also that, as $(a + b) > 0$ and $\xi < 2$, the inequality $\gamma_i < \gamma_j$ holds for $i < j$ too. Therefore the necessary and sufficient condition preventing both $N(r, t)$ and

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[7] Otherwise the lowest order power at the right-hand side of the equation cannot be matched by any term of the left-hand side.

$\partial_r N(r, t)$ from becoming negative at sufficiently small r is $\gamma_0 \geqslant 0$, implying that $\gamma_i \geqslant 0$ for all integer i. Hence, the inequality

$$\gamma_0 \equiv (2 - \xi)[(a + b)(-d + 2) + (d - 1)b] \geqslant 0$$

provides the only condition of acceptability of this solution. It can be rewritten in terms of the positive ratio $a/(a + b)$ as

$$\frac{a}{a + b} \leqslant \frac{1}{d - 1}, \tag{23}$$

i.e., $\mathcal{P} \geqslant (d - 2 + \xi)/(2\xi)$ for the compressibility. The most important feature of the solution with $\beta_0 = 0$ is that, when $\mathcal{P} > (d - 2 + \xi)/(2\xi)$ strictly, the function $N(r, t)$ converges for $r \to 0$ to a positive value $c_0(t)$ growing with time. This means that particles coalesce more and more in time forming massive point clusters. Indeed the condition $c_0(t) > 0$ means that the average conditional particle density $\Gamma(\boldsymbol{r}, t)$ develops a contribution proportional to the Dirac delta function $\delta(\boldsymbol{r})$ whose increasing weight is exactly $c_0(t)$. From the point of view of diffusion theory this corresponds either to an *adhesive* or to a *regular absorbing* boundary at $r = 0$ for equation (19). The second important feature is given by the first correction $c_1(t) r^{\beta_1}$, with $\beta_1 = (2 - \xi)$, to the leading constant term of $N(r, t)$. This implies that at sufficiently small $r > 0$ the function[8] $\Gamma(\boldsymbol{r}, t)$ is proportional to $r^{2 - \xi - d}$ with the amplitude proportional to $c_1(t)$.

Instead for $\mathcal{P} = (d - 2 + \xi)/(2\xi)$ we have $\gamma_0 = 0$ identically. As a consequence, if $c_0(0) = 0$ it will be $c_0(t) = 0$ and $N(0, t) = 0$ at all t. That is $\Gamma(\boldsymbol{r}, t)$ develops no Dirac delta function contribution and no coalescence occurs. In other words the probability to find any two particles at vanishing distance is zero at all time. However, at small r the function $\Gamma(\boldsymbol{r}, t)$ is proportional to $r^{2 - \xi - d}$ (i.e. diverging and integrable) which denotes a simple clustering of the particle distribution. This behavior is typical of the singularity $\boldsymbol{r} = \boldsymbol{0}$ acting as either a repulsive entrance boundary, or a reflecting regular boundary.

(ii) Assume now that $(a + b)(\xi + \beta_0 - d) + (d - 1)b = 0$. The related solution of equation (21) is given by the equalities

$$\begin{cases} \beta_0 = d - \xi - \dfrac{(d - 1)b}{a + b} = 1 - \xi + \dfrac{(d - 1)a}{a + b} \\ \beta_n = \beta_0 + n(2 - \xi) \\ \dot{c}_n(t) = \lambda_n c_{n+1}(t) \\ \lambda_n = [\beta_0 + (n + 1)(2 - \xi)] [(n + 1)(2 - \xi)(a + b)]. \end{cases} \tag{24}$$

Again the solution can be found as an expansion in integer powers of $r^{2-\xi}$, but there is also a 'singular' multiplicative contribution r^{β_0}. In order to have finite $N(r, t)$ at finite r the following restriction has to be imposed:

$$\beta_0 = d - \xi - \frac{(d - 1)b}{a + b} \geqslant 0,$$

which can be recast to

$$\frac{a}{a + b} \geqslant \frac{1 - \xi}{d - 1}, \tag{25}$$

i.e., $\mathcal{P} \leqslant d/\xi^2$ in terms of compressibility. This is the only necessary and sufficient condition for the acceptability of this solution. In fact, since $\xi < 2$, if $\beta_0 \geqslant 0$ we have from the last of equation (24) $\lambda_n > 0$ for all n, and this guarantees both $N(r, t)$ and $\partial_r N(r, t)$ to be finite and non-negative.

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[8] Or more precisely to its angular average $(1/\Omega_d) \int d\Omega \Gamma(\boldsymbol{r}, t)$.

When $a/(a + b) > (1 - \xi)/(d - 1)$ (i.e. $\mathcal{P} < d/\xi^2$) strictly, one has $N(0, t) = 0$ for all t and no coalescence occurs. In other words different particles have zero probability at any time to be found at the same spatial point. In terms of diffusion theory this is possible only if the point $r = 0$ behaves either as a repulsive entrance boundary or as a regular reflective one. Moreover at sufficiently small scale $N(r, t) \simeq c_0(t)r^{\beta_0}$ with $\beta_0 = [d - \xi - b(d - 1)/(a + b)] > 0$. This implies that $\Gamma(\boldsymbol{r}, t)$ is proportional to $c_0(t)r^{\beta_0-d}$. Note that the case $a/(a + b) = 1/(d - 1)$ above is in the present class and in fact the exponents coincide for this value of $a/(a + b)$. The quantity $D = \beta_0$ plays the role of a local (density) fractal dimension of the particle distribution.

Finally, for $a/(a + b) = (1 - \xi)/(d - 1)$ (i.e. $\mathcal{P} = d/\xi^2$) we have again $\beta_0 = 0$ and this solution belongs to the class of solutions with the adhesive or absorbing behavior at $r = 0$ seen above for $a/(a + b) > 1/(d - 1)$ and developing particle coalescence.

At this point we can draw the following conclusions for the rough stochastic flows with $\xi < 2$. The physics described by equation (7) can be summarized in three different regimes (phases) depending on the values of the compressibility parameter $\mathcal{P}$:

(i) For $\mathcal{P} \leqslant (d - 2 + \xi)/(2\xi)$ (weak compressibility) the solution is unique and coincides with the 'repulsive' solution characterized by $\beta_0 = [d - \xi - b(d - 1)/(a + b)] > 0$ showing no coalescence but simple clustering. This confirms that for these values of the compressibility $\mathcal{P}$, the singularity at $r = 0$ acts as a repulsive entrance boundary: particles never collide.

(ii) For $(d - 2 + \xi)/(2\xi) < \mathcal{P} < d/\xi^2$ (intermediate compressibility) both attractive and repulsive solutions are possible, and clearly also all their compositions. This confirms that the singularity $r = 0$ works as a regular boundary because particles hit one each other in finite time with nonzero relative velocity. Accordingly, it is necessary to select an absorbing reflecting or mixed boundary condition to determine a single solution.

In a realistic situation, the choice of appropriate boundary conditions is, of course, suggested by physical considerations on the specificity of the interactions among pollutant particles in the fluid environment. Moreover, in real flows, an important role is played by the existence, at small separation, of two characteristic scales ℓ_ν, ℓ_κ related to purely viscous and diffusive motion respectively. Below them, the Kraichnan picture does not hold anymore. The attractive solution has to be selected when $\{\ell_\nu, \ell_D\} \to 0$ in such a way the Schmidt number $\mathrm{Sc} = \nu/\kappa$ diverges fast enough [15, 25, 26]

(iii) Finally, for $\mathcal{P} \geqslant d/\xi^2$ (strong compressibility) the solution is again unique showing particle coalescence signed by the development of an increasing delta function at $r = 0$ in $\Gamma(\boldsymbol{r}, t)$. It confirms that for such strong compressibility, $r = 0$ works as an adhesive (or exit) boundary for which particles collide in finite time but with vanishing relative velocity.

It is important to stress that, due to the structure of the equation system for the coefficients $c_n(t)$, the only possible stationary solution is given by $c_0 > 0$ and $c_n = 0$ for $n \geqslant 1$. Clearly, this is a real and physically meaningful stationary solution only for the natural repulsive case at low compressibility $\mathcal{P} \leqslant (d - 2 + \xi)/(2\xi)$ or for the regular boundary case at intermediate compressibility $(d - 2 + \xi)/(2\xi) < \mathcal{P} < d^2/2\xi$ with reflecting boundary condition. For the other cases this stationary solution is unphysical amounting to placing initially all the system particles at the same spatial point. It is in fact known from the theory of stochastic processes [23] that when either an adhesive boundary or a regular absorbing boundary are present no stationary solution is possible.

4. Discussion and conclusions

In this paper, we have presented a different approach to analyze and classify the solutions of the Kraichnan ensemble. Different from the previous methods [1, 15], our approach focuses on the equation for the two-point correlation function of the pollutant density instead of the correlation function of the passive scalar which is the customary quantity studied in turbulence. The two equations are known to be equivalent one being the adjoint of the other. The reason to suggest this alternative method is two-fold. First, it represents a more intuitive approach to the problem based on a natural regularization of the basic diffusion equation at small scales. Second, the new method is so general that it can be potentially applied to more complex flows.

The key points of the present approach are: (i) transforming the fundamental diffusion equation for two-point correlations $\Gamma(r, t)$ into an equation for the integrated mass of pollutant $N(r, t)$ surrounding a generic pollutant particle for which the singularity is smoother; (ii) developing an 'exponent hunter' technique, consisting in finding the appropriate power-series expansion allowing an exact and quantitative classification of the particle–particle correlations at finite inter-particle distance.

In this way, all the possible behaviors of the small separation singularity is obtained directly from the explicit solution of the equation for $N(r, t)$, and the classification becomes straightforward.

Finally, it is noteworthy to observe that the crossover from intermediate to strong compressibility corresponds in multiplicative noise field theories to the non-equilibrium second-order transition from an active to an absorbing phase. In particular, the adhesive behavior of the singularity for strong compressibility Kraichnan models stands for the absorbing phase in field theories [19].

Appendix A. The Van Kampen's classification

In this appendix, we apply to Kraichnan ensemble the boundary (or singularity) classification introduced by Van Kampen in [23] for the general one-dimensional Fokker–Planck equation

$$\partial_t f(r, t) = \tfrac{1}{2} \partial_r^2 [D(r) f(r, t)] - \partial_r [K(r) f(r, t)]. \tag{A.1}$$

describing the evolution of the PDF $f(r, t)$ of the position r of a particle at time t. Indeed equation (19) is exactly of this type with $D(r) = 2(a + b)r^\xi$ and $K(r) = (d - 1)ar^{\xi-1}$, and with the singularity to be classified $r = 0$. Van Kampen's classification for a singularity $r = 0$ is based on the analysis of the behavior for $\epsilon \to 0$ of the integrals

$$\begin{cases} L_1 = \displaystyle\int_\epsilon^{r_0} dr\, e^{\phi(r)} \\[2mm] L_2 = \displaystyle\int_\epsilon^{r_0} dr\, e^{\phi(r)} \int_{r_0}^r dr'\, \frac{e^{-\phi(r')}}{D(r')} \\[2mm] L_3 = \displaystyle\int_\epsilon^{r_0} dr\, \frac{e^{-\phi(r)}}{D(r)}, \end{cases} \tag{A.2}$$

where

$$\phi(r) = -2 \int_{r_0}^r dr'\, \frac{K(r')}{D(r')}$$

and with $x, r_0 > 0$. In general one can show that:

(i) If for $\epsilon \to 0$ we have $L_1 \to +\infty$, the singularity $r = 0$ behaves as a *natural repulsive boundary* as the particle starting from $r_0 > 0$ has zero probability to reach $r = 0$ [23].

This means that starting the motion close to $r = 0$, the particle runs away from the singularity never touching it. Consequently, equation (A.1) does not need an additional boundary condition at $r = 0$ and the solution, once the initial and the other possible boundary conditions are given, is unique.

In general, one distinguishes two sub-cases depending whether the conditional mean escape time from the singularity to a finite distance is finite or infinite. In the first sub-case, one can show that the solution converges in time to a stationary state. The conditional mean escape time can be evaluated as follows. Let us put at $r = \epsilon > 0$ (which is a regular point) a reflective boundary and start the dynamics from a generic $r_0 \in (\epsilon, r_1)$ with $r_1 > \epsilon$. It is possible to show [23] that the mean escape time through r_1 conditioned to starting the dynamics from r_0 and having a reflecting boundary at ϵ is

$$\tau(r_1, \epsilon | r_0) = \frac{\int_{r_0}^{r_1} \mathrm{d}r\, \mathrm{e}^{\phi(r)}}{\int_{\epsilon}^{r} \mathrm{d}r'\, \mathrm{e}^{\phi(r')}}. \tag{A.3}$$

We have an entrance boundary at $r = 0$ if τ remains finite for $r \to \epsilon \to 0$. Instead, if τ diverges, $r = 0$ is a proper natural repulsive boundary. We show below how to distinguish these two possibilities in our Kraichnan case.

(ii) If for $\epsilon \to 0$ we have $L_1 < +\infty$ and $L_2 \to +\infty$, the point $r = 0$ behaves as a *natural attractive boundary*. The particle starting at finite r approaches $r = 0$ but in an infinite mean time. As above the solution is unique and no boundary is needed to be fixed. Different from above, no stationary state is reached.

(iii) If for $\epsilon \to 0$ we have $L_1, L_2 < +\infty$ and $L_3 \to +\infty$, the point $r = 0$ behaves as an *adhesive boundary*. The particle starting at finite r approaches $r = 0$ in a finite time but reaches it with vanishing velocity. Therefore once the particle has reached $r = 0$ it stays there forever. In other words $r = 0$ works 'naturally' as an absorbing boundary. Again the solution is unique and no boundary condition has to be fixed by hand. Since the probability to find at $t > 0$ the particle at $r = 0$ is finite and increases in time, no stationary state is reached and $f(r, t)$ develops a Dirac delta function at $r = 0$ with a time increasing coefficient.

(iv) Finally, if in the same limit $L_1, L_2, L_3 < +\infty$, the point $r = 0$ behaves as a *regular boundary*. One can now show that the particle reaches $r = 0$ in a finite time but with a nonzero velocity. Consequently, a solution to the equation is determined once a boundary condition at $r = 0$ is explicitly fixed. This condition can be either absorbing, or reflecting, or mixed. Only if a purely reflecting condition is fixed the solution runs toward a stationary state. Otherwise, as in the previous case, a time increasing Dirac delta contribution appears at $r = 0$.

Before applying this classification to the Kraichnan ensemble, it is useful to mention again that in this case $D(r) = 2(a + b)r^\xi$, $K(r) = (d - 1)ar^{\xi-1}$ and r is not the position of a single particle but the relative distance between two system particles, so that $r = 0$ means a collision between these two particles. The above classification can be translated as follows:

Case 1 is obtained for $a/(a + b) \geqslant 1/(d - 1)$, i.e., for $\mathcal{P} \leqslant (d - 2 + \xi)/(2\xi)$ (weak compressibility). Moreover it is simple to show that for $\xi < 2$ (rough flow) the conditional mean escape time from $r = 0$, equation (A.3), for $r_0 \to \epsilon \to 0$ is finite and it behaves as an entrance boundary, while for $\xi = 2$ (smooth flow) this time diverges and $r = 0$ behaves as a proper natural repulsive boundary. In other words for $\xi < 2$ nearby particles almost surely

never collide and get far away from one each other in finite time, while for $\xi = 2$ nearby particles again never collide but increase their relative distance logarithmically.

Case 2 is obtained only when $\xi = 2$ and $a/(a + b) < 1/(d - 1)$, i.e., $\mathcal{P} > d/4$ (strong compressibility for smooth flow). Now any pair of particles at finite relative distance almost surely do not collide approaching one another very slowly. The difference between the two sub-cases with $\xi = 2$ is very subtle. This is the reason why in the original Feller's classification they were included in a unique class of 'natural boundaries'. We see in appendix B how to distinguish these solutions.

Case 3 is obtained when $\xi < 2$ and $a/(a + b) \leqslant (\xi - 1)/(d - 1)$, i.e. $\mathcal{P} \geqslant d/\xi^2$ (strong compressibility). In this case any two particles at finite initial relative distance almost surely collide in a finite time, but with vanishing relative velocity and consequently coalesce.

Finally, case 4 is obtained for $\xi < 2$ and $(\xi - 1)/(d - 1) < a/(a + b) < 1/(d - 1)$, i.e., for $(d - 2 + \xi/(2\xi) < \mathcal{P} < d/\xi^2$ (intermediate compressibility). Now any pair of particles almost surely collide in a finite time with non-vanishing relative velocity. Therefore to fix the solution of the equation, one has to decide if collisions are either completely elastic (reflecting boundary at $r = 0$) or completely inelastic (absorbing boundary at $r = 0$) or intermediate (mixed boundary). This is the only case in which it is necessary to fix by hand a boundary condition at $r = 0$ to determine the solution of equation (9).

All this qualitative analysis coincides with the one given by [1] through the boundary condition theory of elliptic operators.

Appendix B. The 'smooth' case $\xi = 2$

We have seen that in Van Kampen's classification the solution of the 'smooth' case $\xi = 2$ to our problem is unique and corresponds to a singularity at $r = 0$ behaving as a natural (attractive or repulsive) boundary. In $d = 1$ the problem $\xi = 2$ has been solved by [11] which found the solution to equation (7) for $r_0 \to 0$. The d-dimensional case has been extensively studied in [27] by analyzing the time evolution of the moments of the separation between two system particles. Here we give the exact solution for the average conditional radial density of particles whose evolution is described by equation (19). This direct solution clarifies the meaning of $r = 0$ as a natural boundary and the distinction between an attractive and a repulsive case.

First of all, we see that any power-law function $\eta(t)r^\alpha$ is a solution of the equation (19) with $\xi = 2$ if $\eta(t)$ satisfies

$$\eta(t) = \eta(0)\, \mathrm{e}^{\gamma(\alpha)t} \tag{B.1}$$

with $\gamma(\alpha) = (\alpha + 1)[(a + b)(\alpha + 2) - (d - 1)a]$. By the physical definition of $n(r, t)$ the exponent α needs to be larger than -1 to have always a finite conditional number of particles $N(r, t)$ in any sphere of finite radius. Therefore $\gamma(\alpha) > 0$ if $a/(a + b) < (\alpha + 2)/(d - 1)$. And in particular for $a/(a + b) < 1/(d - 1)$ we have $\gamma(\alpha) > 0$ for all permitted α. This gives a first insight into the difference between attractive and repulsive natural boundary at $r = 0$. In fact for $a/(a + b) > 1/(d - 1)$ (repulsive case) there are initial conditions which are depleted by the dynamics, while for $a/(a + b) > 1/(d - 1)$ (attractive case) all physical initial conditions are amplified. However the fact that any spatial power law with a coefficient satisfying equation (B.1) is always a solution of equation (19) looks unphysical because at any time the average particle density in the infinite volume is not conserved. This unphysical aspect is due to the fact that $\hat{d}(r)$ is divergent on large separations. Therefore our model has to be interpreted as valid up to an upper cutoff L.

In order to better clarify the nature of the solution of equation (19) we therefore study the evolution of $n(r, t)$ with initial condition

$$n(r, 0) = \begin{cases} \gamma r^\alpha & \text{for} \quad r \leqslant r_c \\ 0 & \text{for} \quad r > r_c, \end{cases} \tag{B.2}$$

where $\alpha > -1$ and r_c is an arbitrary finite scale.

Equation (19) can be transformed into a more treatable form performing the change of variable $r = e^u$ and considering the function $m(u, t)$ defined by the relation

$$m(u, t) = n(r = e^u, t) \frac{dr(u)}{du} = n(r = e^u, t) e^u,$$

which conserves the measure. Directly from equation (19) it is simple to show that $m(u, t)$ satisfies the simple Fokker–Planck equation at constant coefficients typical of ordinary Brownian motion with diffusion coefficient $2(a+b)$ and constant drift velocity $-[b+(2-d)a]$,

$$\partial_t m(u, t) = (a + b)\partial_u^2 m(u, t) + [b + (2 - d)a]\partial_u m(u, t). \tag{B.3}$$

The initial condition is given by changing variables in equation (B.2),

$$m(u, 0) = \begin{cases} \gamma \, e^{(\alpha+1)u} & \text{for} \quad u \leqslant u_0 = \log r_c \\ 0 & \text{for} \quad u > u_0 = \log r_c. \end{cases} \tag{B.4}$$

The solution of equation (B.3) is easily found by considering the Fourier transform $\tilde{m}(q, t) = \int_{-\infty}^{+\infty} du \, m(u, t) e^{-iqu}$ which, substituted into equation (B.3), reads

$$\tilde{m}(q, t) = \tilde{m}(q, 0) \exp\{-[(a + b)q^2 - i(b + (2 - d)a)q]t\}. \tag{B.5}$$

From equation (B.4), $\tilde{m}(q, 0)$ is given by

$$\tilde{m}(q, 0) = \gamma \frac{e^{(\alpha+1-iq)u_0}}{\alpha + 1 - iq}.$$

Plugging this expression into equation (B.5) and inverting the Fourier transform one finds

$$m(u, t) = \gamma \exp[(\alpha + 1)(u + c_0 t)] \int_{v_m(u)}^{+\infty} \frac{dv}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}, \tag{B.6}$$

where

$$c_0 = b + (2 - d)a + (\alpha + 1)(a + b)$$

$$v_m(u) = \frac{u - u_0 + [b + (2 - d)a + 2(\alpha + 1)(a + b)]t}{\sqrt{2(a + b)t}}.$$

We now study the asymptotic of equation (B.6) by using the following well-known approximations:

$$\int_{v_m}^{+\infty} \frac{dv}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} \simeq \begin{cases} \dfrac{e^{-\frac{v_m^2}{2}}}{\sqrt{2\pi} \, v_m} & \text{for} \quad v_m \gg 1 \\ 1 & \text{for} \quad v_m \ll -1. \end{cases}$$

Using this and moving back from $m(u, t)$ to $n(r, t)$, we finally find for $r/r_c \gg \exp[-c_2 t + \sqrt{2(a + b)t}]$

$$n(r, t) \simeq \gamma r_c^\alpha \sqrt{\frac{(a + b)t}{\pi}} \left(\frac{r_c}{r}\right) \frac{\exp\left[-\frac{(\log \frac{r}{r_c} + c_1 t)^2}{4(a+b)t}\right]}{\log \frac{r}{r_c} + c_2 t}, \tag{B.7}$$

with

$$\begin{cases} c_1 = b + (2 - d)a \\ c_2 = b + (2 - d)a + 2(\alpha + 1)(a + b), \end{cases}$$

equation (B.7) practically says that, in the region of validity of the approximation, $n(r, t)$ develops a log–normal behavior whose peak drifts with velocity $-c_1 t$.

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Instead for $r/r_c \ll \exp[-c_2 t - \sqrt{2(a+b)t}]$ we have simply

$$n(r,t) \simeq \gamma r^\alpha \exp\{(\alpha+1)c_0 t\} = n(r,0)\exp\{(\alpha+1)c_0 t\}, \qquad \text{(B.8)}$$

i.e., the same amplifying aforementioned behavior for the scale invariant initial condition.

By looking at equation (B.7) we can appreciate better the meaning of 'attractive' or 'repulsive' natural boundary behavior at the singularity $r = 0$.

(i) For $c_1 > 0$, i.e., $\mathcal{P} > d/4$, the peak of the log–normal function shifts to smaller and smaller scales denoting the attractive behavior of the point $r = 0$,
(ii) For $c_1 < 0$, i.e, $\mathcal{P} < d/4$, such peaks moves away from the singularity signing the repulsive behavior of such singularity.

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